Unitarity in multi Higgs production using Schwinger-Dyson equations

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Talk based on the paper [arXiv:1911.04784] by Á. Curkó, and G. Cynolter.

Perturbative Standard Model

What is the scale of new physics, BSM?

- $M_H \simeq 125$ GeV, SM is weakly coupled up to M_{Pl}
- Asymptotic freedom,
  except $U_Y(1)$ and λ_H, Landau pole $\sim exp(1/\lambda)$ beyond M_{Pl}
- Metastability scale of EW vacuum 10^{10-12} GeV
- Problems with perturbative self-interacting scalars at high energies
  - Near kinematic threshold production of 'slow' multi scalars
  - Amplitude $h^* \to n \times h$ grows exponentially
  - $\to$ Unlimited $\sigma(s)$ violates the Froissart bound $(c \times \ln^2(s))$ and unitarity
- Classics 92-94, Brown, Son, Voloshin, Rubakov, Troitski
- Revival, V.V Khoze et al 1411.2925 and 2017-18-

Scalar theory part of the SM

Single, real scalar $\varphi(x)$, $\mathcal{L} = \frac{1}{2}\partial^\mu\varphi(x)\partial_\mu\varphi(x) - \frac{\lambda}{4}\left(\varphi(x)^2 - v^2\right)^2$. After SSB

$h(x) = \varphi(x) - v$, massive, prototype of the SM Higgs in unitary gauge

$$\mathcal{L} = \frac{1}{2}\partial^\mu h(x)\partial_\mu h(x) - \frac{m^2}{2}h^2(x) - \frac{\kappa}{3}h^3(x) - \frac{\lambda}{4}h^4(x), \tag{1}$$

where $m = \sqrt{2\lambda}v$, $\kappa = 3\lambda v$ and $< h >= 0$ perturbative.

- Strange behaviour, decay large number(n) of scalars
- tree-level at kinematic threshold, factorial growth of graphs
- $\langle n|h(x)|0\rangle \sim n!\lambda^{n/2}$
- breakdown of pert. theory at $n \gtrsim 1/\sqrt{\lambda}$, at $E \gtrsim 1/\sqrt{\lambda}m$

Transition rate, $1^* \to n$

We are interested in the decay of excited, energetic $h^* \to nh$ process. Characterized by the Transition rate

$$R_n(p^2) = \int d\Pi_n |\mathcal{M}(1 \to n)|^2, \tag{2}$$

$\lambda n \gg 1$ limit, steepest descent method, semiclassical approach, pioneered by D.T.Son '96

$$R_n(p^2) \propto \exp\left[n\left(\log\left(\frac{\lambda n}{4}\right) - 1 + \frac{3}{2}\left(\log\left(\frac{\varepsilon}{3\pi}\right) + 1\right) - \frac{25}{12}\varepsilon + 0.85\sqrt{\lambda n}\right)\right], \tag{3}$$

Terms are in order, tree-level, $\varepsilon = \frac{E - m\cdot n}{m\cdot n}$ $\varepsilon \neq 0$ tree correction, thin-wall (bubble) q-corr'n ε average kinetic energy of the final-state Higgses. Valid at

- $\frac{1}{\lambda} \to \infty$, $n \to \infty$ $\quad \lambda n = \text{const.} \gg 1$
- $\varepsilon = \text{const} \ll 1$

Still exponential $\leftrightarrow$ Unitarity ?

Transition rate, $1^* \to n$

$$R_n(p^2) \propto \exp\left[n\left(\log\left(\frac{\lambda n}{4}\right) - 1 + \frac{3}{2}\left(\log\left(\frac{\varepsilon}{3\pi}\right) + 1\right) - \frac{25}{12}\varepsilon + 0.85\sqrt{\lambda n}\right)\right],$$

Applied to SM $\lambda_{\mathrm{SM}} \simeq 0.125$, $\sqrt{p^2} = (100 - 200) \cdot m_H$, changing $n \gg 1$
$\lambda n \gg 1$, ε small, but not fixed $\to R_n(p^2) \sim \exp(a\sqrt{p^2})$ with $a > 0$.
Origin of the problem

- limit in saddle point method invalid
- Missing terms
  - tree-level ε not small
  - quantum corrections for $\varepsilon \neq 0$

Use a not perturbative method
Summed up, self-consistent equation - expect unitarity

Calculate $R_n(p^2)$ from Schwinger-Dyson equations

Schwinger-Dyson equations in SSB ϕ^4 model

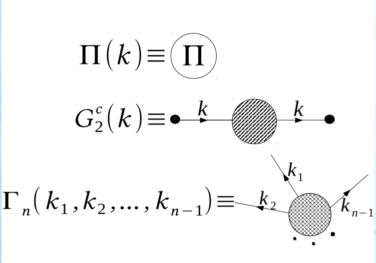
SD system of coupled equations derived from the identity, ϕ^4 V.Sauli(2005)

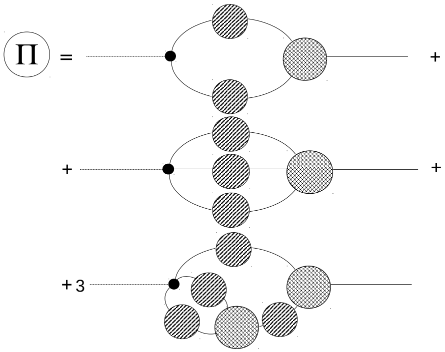
$$\int \mathcal{D}\varphi \frac{\delta\theta[\varphi]}{\delta\varphi(x)} = 0, \text{ where } \theta[\varphi] = \exp\left\{ i \int d^4x \left[\mathcal{L}(\varphi(x)) + J(x)\varphi(x)\right]\right\},$$

$$\Pi(p^2) = \kappa \int \frac{d^4 k_1}{(2\pi)^4} G_2^c(-k_1) G_2^c(k_1+p)\Gamma_3(k_1,p) +$$
$$+\lambda \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} G_2^c(k_1) G_2^c(-k_1-k_2) G_2^c(k_2+p)\Gamma_4(-k_1,k_1+k_2,p) +$$
$$+3\lambda \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} G_2^c(-k_1) G_2^c(-k_2) G_2^c(k_1+k_2+p)\Gamma_3(k_1+p,k_2) G_2^c(k_1+p)\Gamma_3(k_1,p),$$

$\Pi(p^2)$ is the self-energy
$G_2^c(k)$ is the connected two-point function
$\Gamma_n(k_1, k_2, \dots k_{n-1})$ is the n-point vertex function
$m = \sqrt{2\lambda}v$, $\kappa = 3\lambda v$

Schwinger-Dyson equations graphicaly





$\Pi(p^2)$ self-energy

$G_2^c(k)$ connected two-point function

$\Gamma_n(k_1, k_2, \ldots k_{n-1})$ n-point vertex fn

Complete SD system of eq'n, extra eq'ns for vertex function, small λ, κ tree level can be used

Schwinger-Dyson equations simplified

Small λ, κ vertex functions
$$\begin{aligned}\Gamma_3(k_1,k_2) &= -i2\kappa + \ldots, \\ \Gamma_4(k_1,k_2,k_3) &= -i6\lambda + \ldots\end{aligned}$$
Arrive at

$$\begin{aligned}\Pi(p^2) = {} & -i2\kappa^2 \int \frac{d^4k_1}{(2\pi)^4} G_2^c(-k_1) G_2^c(k_1+p) - \\ & -i6\lambda^2 \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} G_2^c(k_1) G_2^c(-k_1-k_2) G_2^c(k_2+p) - \\ & -12\lambda\kappa^2 \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} G_2^c(-k_1) G_2^c(-k_2) G_2^c(k_1+k_2+p) G_2^c(k_1+p),\end{aligned}$$

Bubble+setting sun +... Solve with spectral decomposition method
$\kappa = 3\lambda v$ has mass dimension

Spectral decomposition

Generic decomposition of a scalar two-point function

$$G(p^2) = \underbrace{\frac{iZ}{p^2 - m^2 + i\epsilon}}_{\text{pole}} + \underbrace{\int_{\omega_{th}}^{\infty} d\omega \frac{i\sigma(\omega)}{p^2 - \omega + i\epsilon}}_{\text{continuum}} \qquad \omega_{th} = 4m^2, \quad \omega \sim s$$

Z is the wave-function renormalization, $\sigma(\omega)$ spectral function for the propagator

Divergence structure

bubble $\sim log\Lambda$ logarithmic 1 subtraction

setting sun $\sim \Lambda^2$ quadratic need 2 subtractions

$$\Pi_R(p^2) = \Pi(p^2) - \Pi(m^2) - \frac{d\Pi(p^2)}{dp^2}|_{p^2=m^2}(p^2 - m^2) \text{ Renormalized self energy}$$

$$\Pi_R(p^2 = m^2) = 0, \quad \Pi_R'(p^2 = m^2) = Z - 1 \text{ Renormalization cond., (Nonmin mom. subtr.)}$$

SD equation with Spectral decomposition $\sigma(\omega)$, $\rho(\omega)$

Double subtracted Renormalized self energy

$$\Pi_R(p^2) = \int_{\omega_{th}}^{\infty} d\omega \frac{\rho(\omega)}{p^2 - \omega + i\epsilon} \left(\frac{p^2 - m^2}{\omega - m^2}\right)^2 = \int_{\omega_{th}}^{\infty} d\omega \frac{\tilde{\rho}(\omega, p^2; m^2)}{p^2 - \omega + i\epsilon}$$

Defines $\rho(\omega) = -\mathrm{Im}\Pi_R(p^2)/\pi$ spectral func. of the self energy and double subtracted $\tilde{\rho}(\omega, p^2; m^2)$.

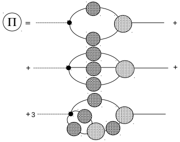
$$\rho(\omega) = \frac{2\kappa^2}{(4\pi)^2}\Big[Z^2 X(m^2, m^2, \omega) + 2Z\int_{\omega_{th}}^{\infty} d\alpha_1 X(\alpha_1, m^2, \omega)\sigma(\alpha_1) +$$
$$+ \int_{\omega_{th}}^{\infty} d\alpha_1 \int_{\omega_{th}}^{\infty} d\alpha_2 X(\alpha_1, \alpha_2, \omega)\sigma(\alpha_1)\sigma(\alpha_2)\Big] +$$
$$+ \frac{6\lambda^2}{(4\pi)^4}\Big[Z^3 Y(m^2, m^2, m^2, \omega) + 3Z^2 \int_{\omega_{th}}^{\infty} d\alpha_1 Y(\alpha_1, m^2, m^2, \omega)\sigma(\alpha_1) + \ldots\Big]$$

Bubble+ Setting sun SDE with Spectral decomp'n*

$$X(\alpha_1,\alpha_2,\omega) = \frac{\sqrt{\lambda(\alpha_1,\alpha_2,\omega)}}{\omega}\Theta(\omega-(\sqrt{\alpha_1}+\sqrt{\alpha_2})^2),\ \lambda^{Kallen}(x,y,z) = x^2-2xy+y^2-\ldots$$

$$\begin{aligned}\rho(\omega) = \frac{2\kappa^2}{(4\pi)^2}\Bigg[& Z^2X(m^2,m^2,\omega)+2Z\int_{\omega_{th}}^{\infty}d\alpha_1 X(\alpha_1,m^2,\omega)\sigma(\alpha_1)+\\ &+\int_{\omega_{th}}^{\infty}d\alpha_1\int_{\omega_{th}}^{\infty}d\alpha_2 X(\alpha_1,\alpha_2,\omega)\sigma(\alpha_1)\sigma(\alpha_2)\Bigg]+\\ &+\frac{6\lambda^2}{(4\pi)^4}\Bigg[Z^3Y(m^2,m^2,m^2,\omega)+3Z^2\int_{\omega_{th}}^{\infty}d\alpha_1 Y(\alpha_1,m^2,m^2,\omega)\sigma(\alpha_1)+\\ &+3Z\int_{\omega_{th}}^{\infty}d\alpha_1\int_{\omega_{th}}^{\infty}d\alpha_2 Y(\alpha_1,\alpha_2,m^2,\omega)\sigma(\alpha_1)\sigma(\alpha_2)+\\ &+\int_{\omega_{th}}^{\infty}d\alpha_1\int_{\omega_{th}}^{\infty}d\alpha_2\int_{\omega_{th}}^{\infty}d\alpha_3 Y(\alpha_1,\alpha_2,\alpha_3,\omega)\sigma(\alpha_1)\sigma(\alpha_2)\sigma(\alpha_3)\Bigg],\end{aligned}$$

Scwinger-Dyson+ extra relation



Third diagram negligible, the prefactor $\frac{12\lambda\kappa^2}{(4\pi)^4} \ll \frac{2\kappa^2}{(4\pi)^2}|_{\text{bubble}}$ by 3 order.

Relation from $1 = G(p^2)G^{-1}(p^2) = \underbrace{G(p^2)}_{\sigma(\omega)}(p^2 - m^2 - \underbrace{\Pi_R(p^2))}_{\rho(\omega)}$

$$\sigma(p^2) = \frac{Z\rho(p^2)}{(p^2-m^2)^2} + \frac{1}{p^2-m^2}\mathrm{P}\cdot\int_{\omega_{th}}^{\infty} d\omega \frac{\sigma(p^2)\tilde{\rho}(\omega, p^2; m^2) + \rho(p^2)\sigma(\omega)}{p^2-\omega}$$

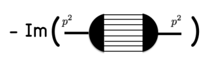
Sum rule can be derived, "particle conservation" pole+ cont.

$$Z + \int_{\omega_{th}}^{\infty} d\omega\sigma(\omega) = 1, \text{ actually } \sigma(\omega) \text{ small} \to Z \simeq 1$$

Transition rate, $\sum_n R_n(p^2) =?$

Use LSZ reduction formula for 2-point functions

$$-Z\Pi(p^2) = \mathcal{M}(p \to p)$$

optical theorem 

$$2\mathrm{Im}\mathcal{M}(p \to p) = \sum_n R_n(p^2)$$ sum over all inner states through a cut

From the renormalized self-energy get the transition rate ($\rho(\omega) = -\mathrm{Im}\Pi_R(p^2)/\pi$)

$$\sum_n R_n(p^2) = 2\pi Z\rho(p^2)$$

Asymptotics, Dominant setting sun, Cutoff

SD eq'ns determine the asymptotic behaviour of the renormalized self energy

$$\rho(\omega) = m^2\lambda f_{\text{bubble}}(\omega, m^2) + \lambda^2 f_{\text{setting sun}}(\omega, m^2) + m^2\lambda^2 f_3(\omega, m^2),$$

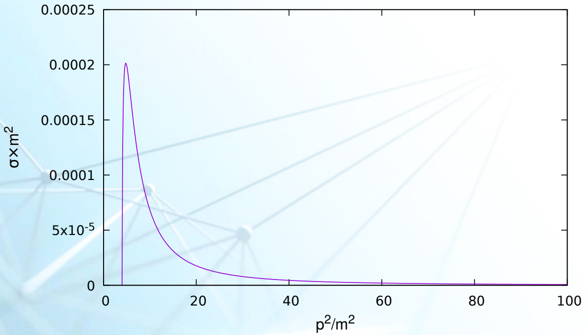
Dimensional analysis , $[\rho] = M^2$, only $f_{settingsun}(\omega, m^2)$ has mass dimension, for $\omega \gg m^2$ $f_{\text{setting sun}}(\omega, m^2) \to \omega$ setting sun is the dominant at high energy.

- ϕ^4 theory is trivial and to satisfy the sum rule $Z + \underbrace{\int d\omega\sigma(\omega)}_{\sim log(\omega)\text{ div.}} = 1$
- Introduce cutoff $\Lambda \gg E_{\text{Higgspersion}} \sim 100 m_H$. (weird, rapid Higgs decay starts here) Choose $\Lambda = (800m)^2$, corresponding to $E_{max} = 100$ TeV.

Numerical iterative solution, SDE+$GG^{-1} = I$, $\sigma(p^2)$

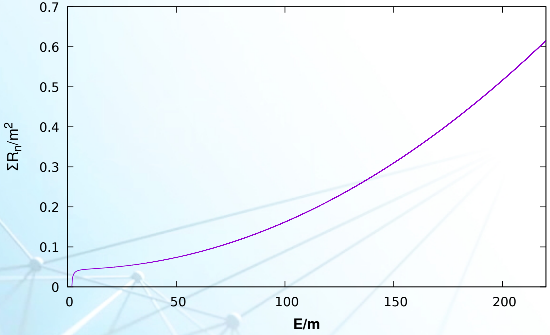
$$Z_0 = 1(\sigma(\omega) = 0) \to \rho_0 \to \sigma_1(\omega) \propto \rho_0 \xrightarrow{sumrule} Z_1 = 1 - \int_{th}^{\Lambda} \sigma_1(\omega) d\omega \xrightarrow[back]{go}$$

Convergence tested by the sum rule (with Z) or $\int_{th}^{\Lambda} d\omega\sigma(\omega) \sim O(10^{-5})$ for $\lambda = 0.12$, also $\lambda = 10$



Numerical solution, Transition Rate

Numerics show $\rho(p^2) \sim p^2$ at $E \gg m$ and dimensional analysis was fine
$\sum_n R_n(p^2) = 2\pi Z \rho(p^2) \sim E^2$ for $E \gg m$, grows with E^2 up to Λ



Exponentially growing $R_n(p^2)$ violates unitarity, here $R_n(p^2) \sim p^2$. Is it safe?

Unitarity, The Higgs propagator

The propagator is the geometric series of the 1PI self energy $\Pi(p^2)$

$$G(p^2) = \frac{i}{p^2 - m^2} \sum_{n=0}^{\infty} \left(-i\Pi(p^2) \frac{i}{p^2 - m^2} \right)^n = \frac{i}{p^2 - m^2 - \Pi(p^2)}.$$

- Can only summed up if the series converges
- Solution of Khoze et al. $\Pi \sim \exp(\sqrt{p^2})$ diverges, it cannot be resummed. [Belyaev]
- SDE solution $\Pi \sim \rho(p^2) \sim p^2$ and G decays with p^2
  Also showed by $\sigma(\omega)$ asymptotically decaying fig.
- SDE provides a well behaving spectral density
- Multi-Higgs production, growth of the σ is cured via 'Higgspersion' mechanism

The Higgspersion, Higgs production

Via the optical theorem $\mathrm{Im}\Pi = \Gamma(s) = \sum_n^{\infty} \Gamma_n(s) \propto \sum_n R_n(s)$ tames the G propagator, upper bound Example h^* production and decay in gg fusion,summed up! self energy

$$\sigma_{gg\to n\times h} \propto |\mathcal{M}_{gg\to h^*}|^2 \times \frac{1}{(p^2 - m^2 - \mathrm{Re}\Pi(p^2))^2 + (\mathrm{Im}\Pi(p^2))^2} \times R_n(p^2)$$

$$\simeq \log^4\left(\frac{m_t}{\sqrt{p^2}}\right) 4Z^2 \frac{R_n(p^2)}{(p^2 + \sum_n R_n(p^2))^2} < \log^4\left(\frac{m_t}{\sqrt{p^2}}\right) 4Z^2 \frac{1}{\sum_n R_n(p^2)} \xrightarrow[p^2\to\infty]{} 0$$

As long as $\sum_n R_n(s) \propto s$ grows faster than $\log^4\left(\frac{m_t}{s}\right)$

- No violation of unitarity
- Multi-Higgs production, growth of the decay rate is cured via 'Higgspersion' mechanism, even faster growing scalar decay width.

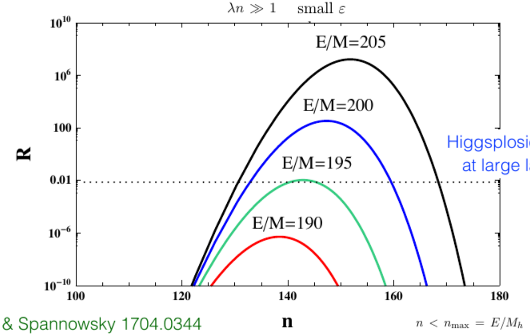
Conclusions

- We have investigated the debated transition rate $h^* \to nh$ in the ϕ^4 theory, SSB 3+1
- Successfully solved by iteration the Schwinger-Dyson equations, calculated the contribution of the dominant setting sun diagram (for $\lambda = 0.125,\ 10$)
- The self energy of the SDE solution can be bubble-resummed, the propagator also well behaved
- No unitarity violation within this approach.
- The transition rate growing with p^2 tamed by the resummed propagator $\sim p^{-2}$ in real physical processes. See no new scale in the SM.

Backup

Transition rate, $1^* \to n$, exponential grow

$$R_n(p^2) \propto \exp\left[n\left(\log\left(\frac{\lambda n}{4}\right) - 1 + \frac{3}{2}\left(\log\left(\frac{\varepsilon}{3\pi}\right) + 1\right) - \frac{25}{12}\varepsilon + 0.85\sqrt{\lambda n}\right)\right],$$



Higgsplosion realised at large lambda n

VVK & Spannowsky 1704.0344

Perturbative limit, simplified equation

Keeping only the leading terms in SD and constraint -pertubative solution, dotted line

$$\rho'(\omega) = \frac{9\lambda m^2}{16\pi^2} Z^2 X(m^2, m^2, \omega) \quad \sigma'(\omega) = \frac{Z\rho'(\omega)}{(\omega - m^2)^2}$$