Theory status of the muon $g-2$

Andreas Nyffeler

PRISMA$^+$ Cluster of Excellence, Institut für Kernphysik,
Helmholtz-Institut Mainz
Johannes Gutenberg Universität Mainz, Germany
nyffeler@uni-mainz.de

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Outline

- Basics of the anomalous magnetic moment
- Electron $g-2$, determination of α
- Muon $g-2$: QED, weak interactions, hadronic contributions
- Current status of the muon $g-2$
- Hadronic vacuum polarization (HVP)
- Hadronic light-by-light scattering (HLbL)
- Conclusions and Outlook

Basics of the anomalous magnetic moment

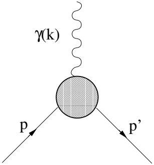
Electrostatic properties of charged particles:
Charge Q, Magnetic moment $\vec{\mu}$, Electric dipole moment $\vec{d}$

For a spin 1/2 particle:

$$\vec{\mu} = g\frac{e}{2m}\vec{s}, \qquad \underbrace{g = 2}_{\text{Dirac}}(1+a), \qquad a = \frac{1}{2}(g-2) : \text{anomalous magnetic moment}$$

Long interplay between experiment and theory: structure of fundamental forces

In Quantum Field Theory (with C,P invariance):



$$= \quad (-ie)\bar{\mathrm{u}}(p') \left[\gamma^\mu \underbrace{F_1(k^2)}_{\text{Dirac}} + \frac{i\sigma^{\mu\nu}k_\nu}{2m} \underbrace{F_2(k^2)}_{\text{Pauli}} \right] \mathrm{u}(p)$$

$$F_1(0) = 1 \quad \text{and} \quad F_2(0) = a$$

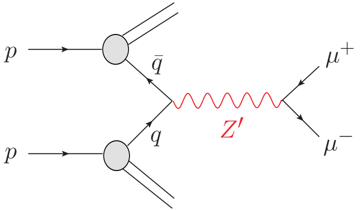
a_e: Test of QED. Precise determination of $\alpha = e^2/4\pi$.
a_μ: Less precisely measured than a_e, but all sectors of Standard Model (SM), i.e. QED, Weak and QCD (hadronic), contribute significantly.
Sensitive to possible contributions from New Physics. Often (but not always !):

$$a_\ell \sim \left(\frac{m_\ell}{m_{\mathrm{NP}}}\right)^2 \Rightarrow \left(\frac{m_\mu}{m_e}\right)^2 \sim 43000 \text{ more sensitive than } a_e \text{ [exp. precision} \to \text{factor 19]}$$

Tests of the Standard Model and search for New Physics

Search for New Physics with two complementary approaches:

1. High Energy Physics:
   e.g. Large Hadron Collider (LHC) at CERN
   Direct production of new particles
   e.g. heavy Z' $\Rightarrow$ resonance peak in invariant mass distribution of $\mu^+\mu^-$ at $M_{Z'}$.

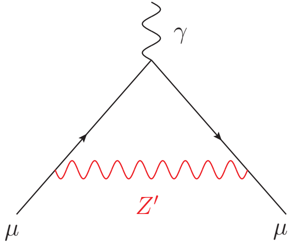


2. Precision physics:
   e.g. anomalous magnetic moments a_e, a_μ
   Indirect effects of virtual particles in quantum corrections
   $\Rightarrow$ Deviations from precise predictions in SM

$$\text{For } M_{Z'} \gg m_\ell : \quad a_\ell \sim \left(\frac{m_\ell}{M_{Z'}}\right)^2$$

Note: there are also non-decoupling contributions of heavy New Physics !
Another example: new light vector meson ("dark photon") with $M_{\gamma'} \sim (10 - 100)$ MeV.

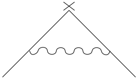
a_e, a_μ allow to exclude some models of New Physics or to constrain their parameter space.



Some theoretical comments on the $g-2$

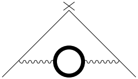
- Anomalous magnetic moment is finite and calculable
  Corresponds to effective interaction Lagrangian of mass dimension 5:
  $$\mathcal{L}_{\rm eff}^{\rm AMM} = -\frac{e_\ell a_\ell}{4m_\ell}\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)F_{\mu\nu}(x)$$
  (mass dimension 6 in SM with $SU(2)_{\rm L}\times U(1)_{\rm Y}$ invariant operator)
  $a_\ell = F_2(0)$ can be calculated unambiguously in renormalizable QFT, since there is no counterterm to absorb potential ultraviolet divergence.

- Anomalous magnetic moments are dimensionless
  To lowest order in perturbation theory in quantum electrodynamics (QED):



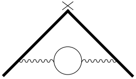
$$= a_e = a_\mu = \frac{\alpha}{2\pi}$$ [Schwinger '48]

- Loops with different masses $\Rightarrow a_e \neq a_\mu$
  - Internal large masses decouple (not always !):



$$= \left[\frac{1}{45}\left(\frac{m_e}{m_\mu}\right)^2 + \mathcal{O}\left(\frac{m_e^4}{m_\mu^4}\ln\frac{m_\mu}{m_e}\right)\right]\left(\frac{\alpha}{\pi}\right)^2$$

  - Internal small masses give rise to large log's of mass ratios:



$$= \left[\frac{1}{3}\ln\frac{m_\mu}{m_e} - \frac{25}{36} + \mathcal{O}\left(\frac{m_e}{m_\mu}\right)\right]\left(\frac{\alpha}{\pi}\right)^2$$

Electron $g-2$

Electron $g-2$: Theory

Main contribution in Standard Model (SM) from mass-independent Feynman diagrams in QED with electrons in internal lines (perturbative series in α):

$$
\begin{aligned}
a_e^{\mathrm{SM}} &= \sum_{n=1}^{5} c_n \left(\frac{\alpha}{\pi}\right)^n \\
&\quad +2.7478(2) \times 10^{-12} \quad [\text{Loops in QED with } \mu, \tau] \\
&\quad +0.0297(5) \times 10^{-12} \quad [\text{weak interactions}] \\
&\quad +1.706(15) \times 10^{-12} \quad [\text{strong interactions / hadrons}]
\end{aligned}
$$

The numbers are from Aoyama et al. '15.

QED: mass-independent contributions to a_e

- α: 1-loop, 1 Feynman diagram; Schwinger '48:

  $c_1 = \frac{1}{2}$

- α^2: 2-loops, 7 Feynman diagrams; Petermann '57, Sommerfield '57:

  $c_2 = \frac{197}{144} + \frac{\pi^2}{12} - \frac{\pi^2}{2}\ln 2 + \frac{3}{4}\zeta(3) = -0.328478965579193 78\ldots$

- α^3: 3-loops, 72 Feynman diagrams; ..., Laporta, Remiddi '96:

$$
\begin{aligned}
c_3 &= \frac{28259}{5184} + \frac{17101}{810}\pi^2 - \frac{298}{9}\pi^2 \ln 2 + \frac{139}{18}\zeta(3) - \frac{239}{2160}\pi^4 \\
&\quad + \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left\{\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{1}{24}\ln^4 2 - \frac{1}{24}\pi^2\ln^2 2\right\} \\
&= 1.181241456587\ldots
\end{aligned}
$$

- α^4: 4-loops, 891 Feynman diagrams; Kinoshita et al. '99, ..., Aoyama et al. '08, '12, '15; Laporta '17 :

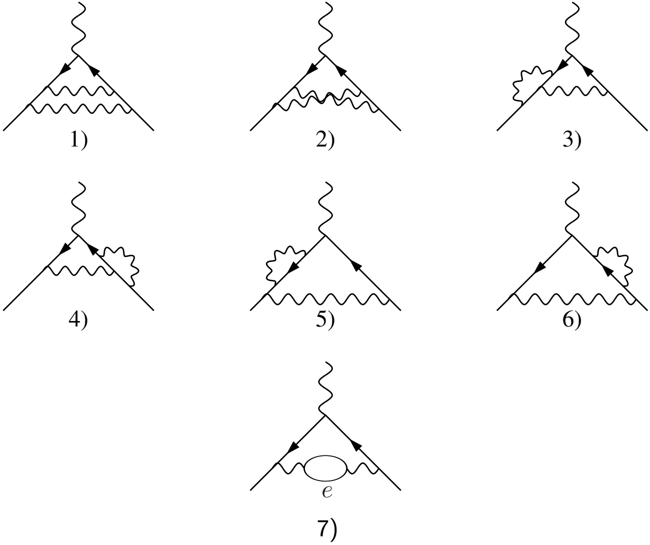
  $c_4 = -1.9122457649\ldots$ (Laporta; semi-analytic calculation)
  $c_4 = -1.91298(84)$ (Aoyama et al.; numerical evaluation)

- α^5: 5-loops, 12672 Feynman diagrams; Aoyama et al. '05, ..., '12, '15:

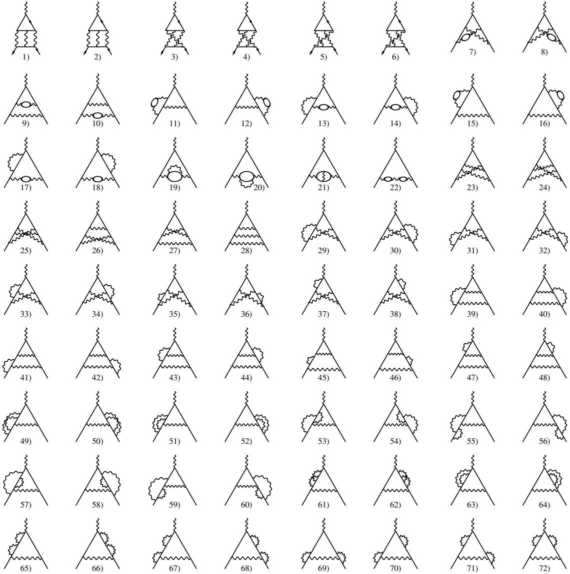
  $c_5 = 7.795(336)$ (numerical evaluation)

  Replaces earlier rough estimate $c_5 = 0.0 \pm 4.6$.
  Result removes biggest theoretical uncertainty in a_e !

Mass-independent 2-loop Feynman diagrams in a_e



Mass-independent 3-loop Feynman diagrams in a_e



4-loop contribution to a_e (mass-independent)

Laporta '17

The first 1100 digits of c_4:

```
-1.912245764926445574152647167439830054060873390658725345171329848006
0384439806517061427608927000036315837558415331473270056378514912854539
1902804327050273822304345578957045562729309941296699760277782211578472
0339064151908166527097970867438115012155147972274322164273431927975958
6074050057837384960701874328314024838025192249460742298558930463506140
4922526634310944240002356356881280620645494013224977594300429288836761
7488992369151808780869897052635785337537769641170245361960134975744943
6126848617516260683238718674730383150596274187801530551487940053697779
8369464278684326918431175889581159743566950433048349073613426586499531
1638781174347538542348836408558444188223721745670687104182330743051744
3055739459611715508589611489952612660612469940731184039274723400234649
6953173548258481799822409737371077365740464513521123091242528111137215
3021544537210148111211598489708842232798797204842014451228284515165852
3656178659459260099173303172130286546721234534050034910470072892448720
0616044261325449069000431915198230047488181494311038495378299406296758
6787538524978194698979313216219797575067670114290489796208505078559
2...
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- “Finalizing a 20-year effort”. Status report at Matter to the Deepest 2003.
- Using dimensional regularization.
- High-precision (few 1000 digits) numerical results for master integrals.
- Method of difference and differential equations for loop integrals.
- Fit to analytic expressions:
  - Usual transcendental constants $\pi, \zeta(3), \zeta(5), \ldots$
  - Harmonic Polylogarithms with arguments $e^{\frac{i\pi}{3}}$, $e^{\frac{2i\pi}{3}}$, $e^{\frac{i\pi}{2}}$.
  - One-dimensional integrals of products of complete elliptic integrals.
  - Six finite parts of master integrals.

Determination of α from $g-2$ of electron and a new discrepancy

- Use a_e^{exp} to determine α from series expansion in QED (contributions from weak and strong interactions under control !). Assume: Standard Model "correct", no New Physics (Laporta '17):

$$\alpha^{-1}(a_e) = 137.035\ 999\ 1596\ \underbrace{(27)}_{c_5}\ \underbrace{(18)}_{\text{EW+had}}\ \underbrace{(331)}_{a_e^{\text{exp}}}\ [333] \quad [0.25\text{ppb}]$$

  The uncertainty from theory has been improved considerably by Aoyama et al. '12, '15 and Laporta '17, the experimental uncertainty in a_e^{exp} is now the limiting factor.

- Recent most precise measurement of α via recoil-velocity of Cesium atoms in atom interferometer (Parker et al. '18) allows test of QED (and SM):

$$\alpha^{-1}(\text{Cs}) = 137.035\ 999\ 046(27) \quad [0.2\text{ppb}]$$

  This leads to (using Aoyama et al. '17, Mohr et al. (CODATA) '16):

$$a_e^{\text{SM}}(\text{Cs}) = 1\ 159\ 652\ 181.61(23) \times 10^{-12} \quad [0.2\text{ppb}]$$

$$\Rightarrow a_e^{\text{exp}} - a_e^{\text{SM}}(\text{Cs}) = -0.88(0.36) \times 10^{-12} \qquad [2.5\sigma \text{ tension ! Note the sign !}]$$

Precision measurements of the fine-structure constant α

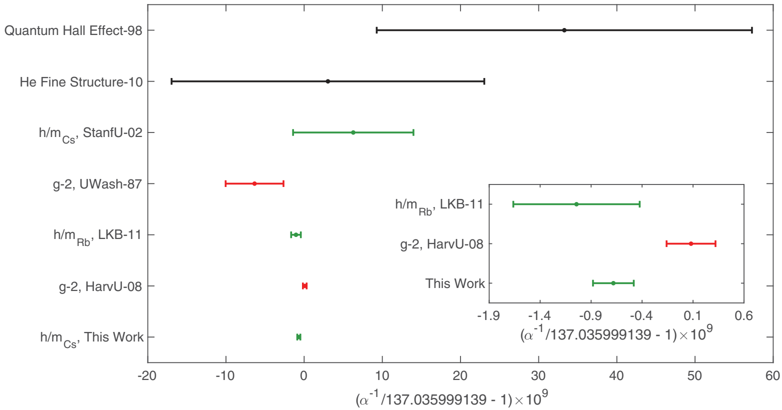


Figure from Parker et al. '18

- "0" = CODATA 2014 value
- Green points: photon recoil experiments
- Red points: from electron $g - 2$ measurements

Muon $g-2$

Milestones in measurements of the muon $g-2$

Authors	Lab	Muon Anomaly
Garwin et al. '60	CERN	0.001 13(14)
Charpak et al. '61	CERN	0.001 145(22)
Charpak et al. '62	CERN	0.001 162(5)
Farley et al. '66	CERN	0.001 165(3)
Bailey et al. '68	CERN	0.001 166 16(31)
Bailey et al. '79	CERN	0.001 165 923 0(84)
Brown et al. '00	BNL	0.001 165 919 1(59) (μ^+)
Brown et al. '01	BNL	0.001 165 920 2(14)(6) (μ^+)
Bennett et al. '02	BNL	0.001 165 920 4(7)(5) (μ^+)
Bennett et al. '04	BNL	0.001 165 921 4(8)(3) (μ^-)

World average experimental value (dominated by $g-2$ Collaboration at BNL, Bennett et al. '06 + CODATA 2008 value for $\lambda = \mu_\mu/\mu_p$):

$$a_\mu^{\text{exp}} = (116\,592\,089 \pm 63) \times 10^{-11} \quad [0.5\text{ppm}]$$

Goal of new planned $g-2$ experiments: $\delta a_\mu = 16 \times 10^{-11}$

Fermilab E989: partly recycled from BNL: moved ring magnet ! First beam in June 2017, should reach BNL precision by end of 2019 and final precision by 2021/22. Talk by Graziano Venanzoni.

J-PARC E34: completely new concept with low-energy muons, not magic γ. Aims in Phase 1 for about $\delta a_\mu = 45 \times 10^{-11}$.

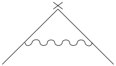
Theory needs to match this precision !

Muon $g-2$: Theory

In Standard Model: $a_\mu^{\text{SM}} = a_\mu^{\text{QED}} + a_\mu^{\text{weak}} + a_\mu^{\text{had}}$

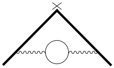
QED contributions

- At 1-loop: Schwinger's result '48 (a_μ dimensionless):



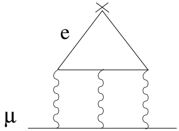
$$= \frac{\alpha}{2\pi}$$

- Diagrams with internal electron loops are enhanced.
  - At 2-loops: vacuum polarization from electron loops enhanced by QED short-distance logarithm



$$= \left[\frac{1}{3}\ln\frac{m_\mu}{m_e} - \frac{25}{36} + \mathcal{O}\left(\frac{m_e}{m_\mu}\right)\right]\left(\frac{\alpha}{\pi}\right)^2$$

  - At 3-loops: light-by-light scattering from electron loops enhanced by QED infrared logarithm [Aldins et al. '69, '70; Laporta, Remiddi '93]



$$+ \ldots \qquad = \left[\frac{2}{3}\pi^2 \ln\frac{m_\mu}{m_e} + \ldots\right]\left(\frac{\alpha}{\pi}\right)^3 = 20.947\ldots\left(\frac{\alpha}{\pi}\right)^3$$

- Loops with tau's suppressed (decoupling)

QED result up to 5 loops

Include contributions from all leptons (Schwinger '48; ...; Aoyama et al. '12):

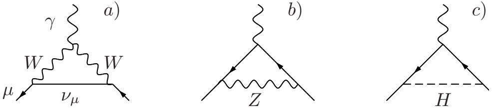
$$
\begin{aligned}
a_\mu^{\mathrm{QED}} &= 0.5 \times \left(\frac{\alpha}{\pi}\right) + 0.765\,857\,425 \underbrace{(17)}_{m_\mu/m_{e,\tau}} \times \left(\frac{\alpha}{\pi}\right)^2 \\
&\quad + 24.050\,509\,96 \underbrace{(32)}_{m_\mu/m_{e,\tau}} \times \left(\frac{\alpha}{\pi}\right)^3 + 130.8796 \underbrace{(63)}_{\text{num. int.}} \times \left(\frac{\alpha}{\pi}\right)^4 \\
&\quad + 753.29 \underbrace{(1.04)}_{\text{num. int.}} \times \left(\frac{\alpha}{\pi}\right)^5 \\
&= 116\,584\,718.853 \underbrace{(9)}_{m_\mu/m_{e,\tau}} \underbrace{(19)}_{c_4} \underbrace{(7)}_{c_5} \underbrace{(29)}_{\alpha(a_e)} [36] \times 10^{-11}
\end{aligned}
$$

- Up to 3-loop analytically known (Laporta, Remiddi '93).
- 4-loop: analytical results for electron and tau-loops (asymptotic expansions) by Kurz et al. '14, '15, '16; Volkov '17
- Earlier evaluation of 5-loop contribution yielded $c_5 = 662(20)$ (Kinoshita, Nio '06, numerical evaluation of 2958 diagrams, known or likely to be enhanced). New value is 4.5σ from this leading log estimate and 20 times more precise.
- Aoyama et al. '12: What about the 6-loop term ? Leading contribution from light-by-light scattering with electron loop and insertions of vacuum-polarization loops of electrons into each photon line $\Rightarrow a_\mu^{\mathrm{QED}}(\text{6-loops}) \sim 0.1 \times 10^{-11}$

Contributions from weak interaction

Numbers from recent reanalysis by Gnendiger et al. '13.

1-loop contributions [Jackiw, Weinberg '72; ...]:



$$a_\mu^{\text{weak, (1)}}(W) = \frac{\sqrt{2} G_\mu m_\mu^2}{16\pi^2} \frac{10}{3} + \mathcal{O}(m_\mu^2/M_W^2) = 388.70 \times 10^{-11}$$

$$a_\mu^{\text{weak, (1)}}(Z) = \frac{\sqrt{2} G_\mu m_\mu^2}{16\pi^2} \frac{(-1 + 4s_W^2)^2 - 5}{3} + \mathcal{O}(m_\mu^2/M_Z^2) = -193.89 \times 10^{-11}$$

Contribution from Higgs negligible: $a_\mu^{\text{weak, (1)}}(H) \leq 5 \times 10^{-14}$ for $m_H = 126$ GeV.

$$a_\mu^{\text{weak, (1)}} = (194.80 \pm 0.01) \times 10^{-11}$$

2-loop contributions (1678 diagrams) [Czarnecki et al. '95, '96; ...]:

$$a_\mu^{\text{weak, (2)}} = (-41.2 \pm 1.0) \times 10^{-11}, \quad \text{large since} \sim G_F m_\mu^2 \frac{\alpha}{\pi} \ln \frac{M_Z}{m_\mu}$$

Total weak contribution:

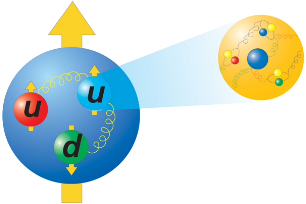
$$a_\mu^{\text{weak}} = (153.6 \pm 1.0) \times 10^{-11}$$

Under control ! With knowledge of $M_H = 125.6 \pm 1.5$ GeV, uncertainty now mostly hadronic $\pm 1.0 \times 10^{-11}$ (Peris et al. '95; Knecht et al. '02; Czarnecki et al. '03, '06). 3-loop effects via RG: $\pm 0.20 \times 10^{-11}$ (Degrassi, Giudice '98; Czarnecki et al. '03).

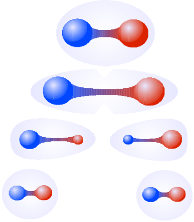
Hadronic contributions to $g-2$

The strong interactions (Quantum Chromodynamics)

- Strong interactions: quantum chromodynamics (QCD) with quarks and gluons
- Observed particles in Nature: Hadrons
  1. Mesons (quark + antiquark: $q\bar{q}$): $\pi, K, \eta, \rho, \ldots$
  2. Baryons (3 quarks: qqq): $p, n, \Lambda, \Sigma, \Delta, \ldots$
- Cannot describe hadrons in series expansion in strong coupling constant of QCD with $\alpha_s(E = m_{\text{proton}}) \approx 0.5$.
  Particularly true for light hadrons which consist of three lightest quarks u, d, s.
  Non-perturbative effects like "confinement" of quarks and gluons inside hadrons.



Source: NIKHEF



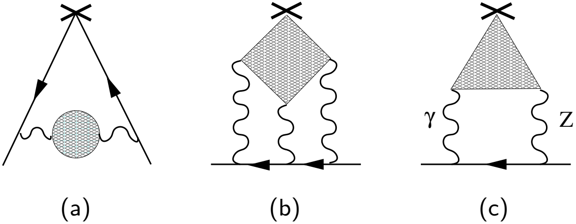
Source: NIC Jülich

- Possible approaches to QCD at low energies:
  1. Lattice QCD: first principle approach, limited applications (Euclidean time), often still limited precision
  2. Effective quantum field theories with hadrons (chiral perturbation theory): based on symmetries of QCD, limited validity in energy
  3. Dispersion relations: extend validity of EFT's, reduce model dependence, often not all the needed input data available
  4. Simplifying hadronic models: model uncertainties not controllable

Hadronic contributions to the muon $g-2$

Largest source of uncertainty in theoretical prediction of a_μ !

Different types of contributions:



Light quark loop not well defined → Hadronic "blob"

(a) Hadronic vacuum polarization (HVP) $\mathcal{O}(\alpha^2), \mathcal{O}(\alpha^3), \mathcal{O}(\alpha^4)$

(b) Hadronic light-by-light scattering (HLbL) $\mathcal{O}(\alpha^3), \mathcal{O}(\alpha^4)$

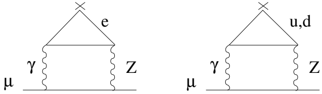
(c) 2-loop electroweak contributions $\mathcal{O}(\alpha G_F m_\mu^2)$

2-Loop EW

Small hadronic uncertainty from triangle diagrams.

Anomaly cancellation within each generation !

Cannot separate leptons and quarks !



Muon $g-2$: current status

Contribution	$a_\mu \times 10^{11}$	Reference
QED (leptons)	$116\,584\,718.853 \pm 0.036$	Aoyama et al. '12
Electroweak	153.6 ± 1.0	Gnendiger et al. '13
HVP: LO	6939 ± 40	Davier et al. '19
NLO	-98.7 ± 0.9	Kurz et al. '14
NNLO	12.4 ± 0.1	Kurz et al. '14
HLbL	86 ± 21	My estimate !
NLO	3 ± 2	Colangelo et al. '14
Theory (SM)	$116\,591\,814 \pm 45$	
Experiment	$116\,592\,089 \pm 63$	Bennett et al. '06
Experiment - Theory	275 ± 78	3.5 σ

My estimate for HLbL: based on work by various authors, who use dispersive and data-driven approaches for $\pi, \eta\,\eta'$ and $\pi\pi$ intermediate states, but still models for heavier states.

Frequently used model estimates for HLbL:

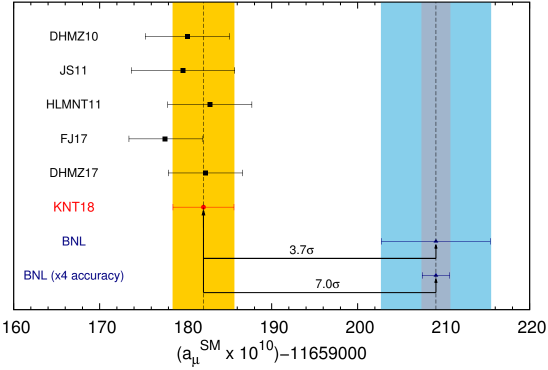
$a_\mu^{\text{HLbL}} = (105 \pm 26) \times 10^{-11}$ (Prades, de Rafael, Vainshtein '09 ("Glasgow consensus"))

$a_\mu^{\text{HLbL}} = (102 \pm 39) \times 10^{-11}$ (Jegerlehner, AN '09; updated in Jegerlehner '15 to take into account smaller axial-vector contribution (Pauk, Vanderhaeghen '14; Jegerlehner '14, '15)).

Discrepancy a sign of New Physics ?

Hadronic uncertainties need to be better controlled in order to fully profit from future $g-2$ experiments at Fermilab (E989) and J-PARC (E34) with four-fold improvement $\delta a_\mu = 16 \times 10^{-11}$.

Muon $g-2$: other recent theoretical evaluations



3.6 σ
3.3 σ
3.3 σ
4.1 σ (includes τ)
3.5 σ

Keshavarzi, Nomura, Teubner '18, arXiv:1802.02995
Note units of 10^{-10} !

Aoyama et al. '12: $a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (249 \pm 87) \times 10^{-11}$ [2.9 σ]

Benayoun et al. '15: $a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (376.8 \pm 75.3) \times 10^{-11}$ [5.0 σ]

New Physics contributions to the muon $g-2$

Define:

$$\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (275 \pm 78) \times 10^{-11}$$

Absolute size of discrepancy is actually unexpectedly large, compared to weak contribution (although there is some cancellation there):

$$\begin{aligned} a_\mu^{\text{weak}} &= a_\mu^{\text{weak, (1)}}(W) + a_\mu^{\text{weak, (1)}}(Z) + a_\mu^{\text{weak, (2)}} \\ &= (389 - 194 - 41) \times 10^{-11} \\ &= 154 \times 10^{-11} \end{aligned}$$

Assume that New Physics contribution with $M_{\text{NP}} \gg m_\mu$ decouples:

$$a_\mu^{\text{NP}} = \mathcal{C} \frac{m_\mu^2}{M_{\text{NP}}^2}$$

where naturally $\mathcal{C} = \frac{\alpha}{\pi}$, like from a one-loop QED diagram, but with new particles. Typical New Physics scales required to satisfy $a_\mu^{\text{NP}} = \Delta a_\mu$:

$\mathcal{C}$	1	$\frac{\alpha}{\pi}$	$\left(\frac{\alpha}{\pi}\right)^2$
M_{NP}	$2.0^{+0.4}_{-0.2}$ TeV	97^{+18}_{-11} GeV	5^{+1}_{-1} GeV

For New Physics models with particles in $250 - 300$ GeV mass range and electroweak-size couplings $\mathcal{O}(\alpha)$, we need some additional enhancement factor, like large $\tan\beta$ in the MSSM, to explain the discrepancy Δa_μ.

Note: general (unconstrained) MSSM can still explain muon $g-2$ discrepancy and evade bounds from LHC. Talk by Dominik Stöckinger.

Muon $g-2$ Theory Initiative

- Steering Commitee:

  Gilberto Colangelo, Michel Davier, Simon Eidelman, Aida El-Khadra, Christoph Lehner, Tsutomu Mibe (J-PARC E34 experiment), Andreas Nyffeler, Lee Roberts (Fermilab E989 experiment), Thomas Teubner

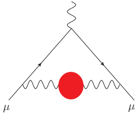
- Tasks:

  1. Workshops to survey and summarize the status of theoretical calculations of hadronic contributions (HVP, HLbL) to muon $g-2$. Encourage participation from all people working on such calculations.
  2. Working groups on different topics (HVP, HLbL) and methods (Dispersive / data-driven, Lattice).
  3. Reports, authored by the participants of the workshops and working groups, on current status of relevant theoretical work. Up-to-date values for HVP and HLbL with reliable uncertainties. Publication coordinated with announcements of new experimental results on muon $g-2$.
     $\Rightarrow$ Muon $g-2$ Whitepaper with theory status planned for end of 2019 !

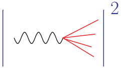
- 1st (plenary) Workshop: June 3-6, 2017 at Fermilab
- HVP WG Workshop: February 12-14, 2018 at KEK
- HLbL WG Workshop: March 12-14, 2018 at University of Connecticut
- 2nd (plenary) Workshop: June 18-22, 2018 at University of Mainz
- 3rd (plenary) Workshop: September 9-13, 2019 at INT Seattle

Hadronic vacuum polarization (HVP)

Hadronic vacuum polarization

$a_\mu^{\text{HVP}} =$ 

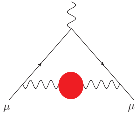
Optical theorem (from unitarity) for hadronic contribution → dispersion relation:

Im  $\sim$  $\sim \quad \sigma(e^+e^- \to \gamma^* \to \text{hadrons})$

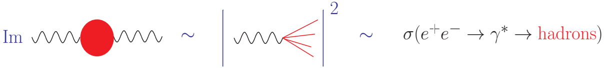
$$a_\mu^{\text{HVP}} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{ds}{s}\, K(s)\, R(s), \qquad R(s) = \frac{\sigma(e^+e^- \to \gamma^* \to \text{hadrons})}{\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-)}$$

[Bouchiat, Michel '61; Durand '62; Brodsky, de Rafael '68; Gourdin, de Rafael '69]
$K(s)$ slowly varying, positive function $\Rightarrow$ a_μ^{HVP} positive. Data for hadronic cross section σ at low center-of-mass energies $\sqrt{s}$ important due to factor $1/s$: $\sim 70\%$ from $\pi\pi$ [$\rho(770)$] channel, $\sim 90\%$ from energy region below 1.8 GeV.

Hadronic vacuum polarization

$a_\mu^{\mathrm{HVP}} =$ 

Optical theorem (from unitarity) for hadronic contribution → dispersion relation:



$$a_\mu^{\mathrm{HVP}} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{ds}{s}\, K(s)\, R(s), \qquad R(s) = \frac{\sigma(e^+e^- \to \gamma^* \to \text{hadrons})}{\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-)}$$

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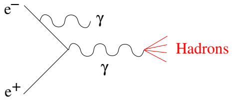
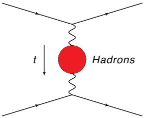
Other method instead of energy scan: Radiative return (initial state radiation) at colliders with fixed center-of-mass energy (DAΦNE, B-Factories, BEPC) [Binner et al. '99; Czyż et al. '00-'03]



Recent new proposal: $\mu - e$ scattering (MUonE), space-like HVP in t-channel [Carloni Calame *et al.* '15 (Bhabha scattering), Abbiendi *et al.* '17 ($\mu - e$ scattering)].
Talk by Graziano Venanzoni.

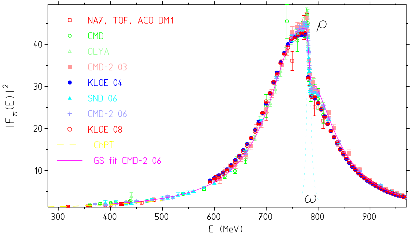


Measured hadronic cross-section

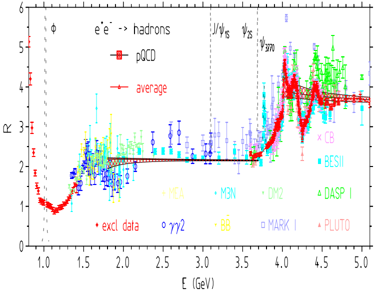
Pion form factor $|F_\pi(E)|^2$
($\pi\pi$-channel)

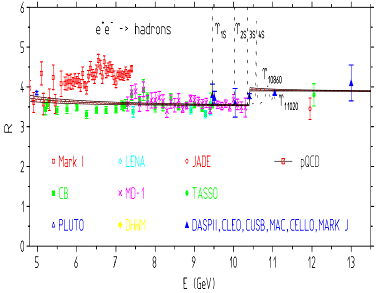
$$R(s) = \frac{1}{4}\left(1 - \frac{4m_\pi^2}{s}\right)^{\frac{3}{2}} |F_\pi(s)|^2$$

$$(4m_\pi^2 < s < 9m_\pi^2)$$



R-ratio:

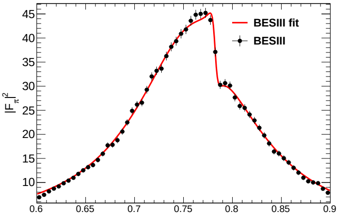




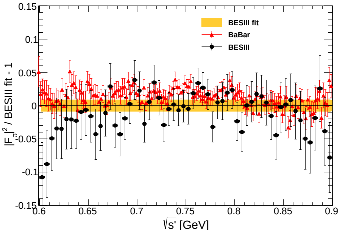
Jegerlehner, AN '09

Recent result on $e^+e^- \to \pi^+\pi^-$ from BESIII

Ablikim et al. (BESIII Collaboration) '16

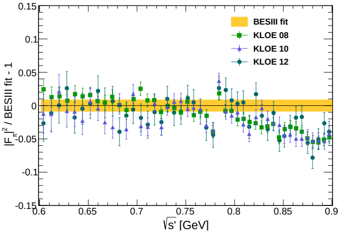


Fit of BESIII data with parametrization of pion form factor by Gounaris-Sakurai. Only statistical errors shown.



BaBar data higher than BESIII below ρ-mass, better agreement above.

Statistical and systematic errors included in data points. Width of BESIII fit band shows systematic uncertainty only.

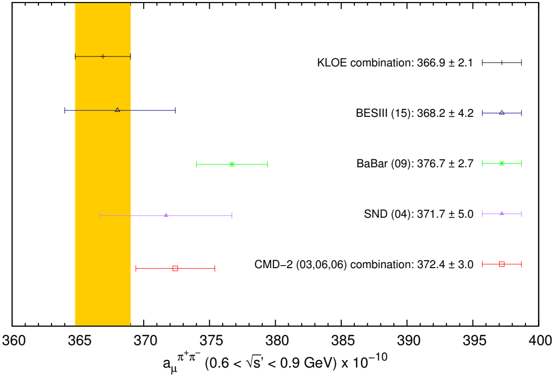


Good agreement with KLOE 08 and KLOE 12 up to mass range of $\rho - \omega$ interference, but disagreement with all three data sets at higher energy.

New: Combination of KLOE data for $e^+e^- \to \pi^+\pi^-$

Anastasi et al. (KLOE-2 collaboration), arXiv:1711.03085 [hep-ex]

Comparison of $a_\mu^{\pi^+\pi^-}(600 - 900 \text{ MeV})$ from experiments using radiative return method (BaBar, BESIII, KLOE) and using energy scan (CMD-2, SND at Novosibirsk):



Combination of KLOE data sets yields even slightly smaller result than BESIII. Compatible with relatively old scan data from Novosibirsk (still to be updated). But further away from BaBar result: differ by $(98 \pm 34) \times 10^{-11}$ ($2.9\,\sigma$) !

Hadronic vacuum polarization: some recent evaluations

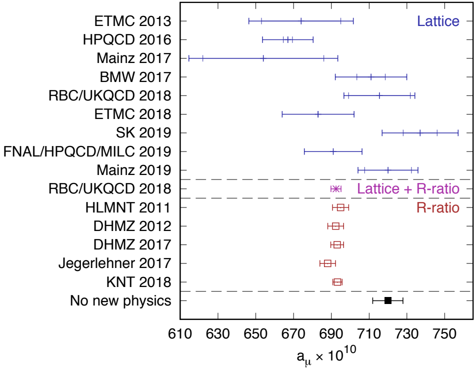
Authors	Contribution to $a_\mu^{\text{HVP}} \times 10^{11}$
Davier et al. '11, '14 (e^+e^-) [+ τ]	6923 ± 42 [7030 ± 44]
Jegerlehner, Szafron '11 (e^+e^-) [+ τ]	6907.5 ± 47.2 [6909.6 ± 46.5]
Hagiwara et al. '11 (e^+e^-)	6949.1 ± 42.7
Benayoun at al. '15 ($e^+e^- + \tau$: BHLS improved)	6818.6 ± 32.0
Jegerlehner '17 (e^+e^-) [+ τ]	6880.7 ± 41.4 [6887.7 ± 33.8]
Davier et al. '17 (e^+e^-)	6931 ± 34
Keshavarzi, Nomura, Teubner '18 (e^+e^-)	6932.6 ± 24.6
Davier et al. '19 (e^+e^-)	6939 ± 40

- Precision: $< 1\%$. Non-trivial because of radiative corrections (radiated photons).
- Even if values for a_μ^{HVP} after integration agree quite well, the systematic differences of a few % in the shape of the spectral functions from different experiments (BABAR, BESIII, CMD-2, KLOE, SND) indicate that we do not yet have a complete understanding.
- Davier et al. '19: for $\pi\pi$ "add as additional systematic uncertainty half of the full difference between the complete integrals without BABAR and KLOE, respectively" $\Rightarrow$ $\pm 28 \times 10^{-11}$.
- Use of τ data: additional sources of isospin violation ? Ghozzi, Jegerlehner '04; Benayoun et al. '08, '09; Wolfe, Maltman '09; Jegerlehner, Szafron '11 ($\rho - \gamma$-mixing), also included in Jegerlehner '17 and in BHLS-approach by Benayoun et al. '15.

HVP from lattice QCD

Lehner et al. (USQCD Whitepaper) arXiv:1904.09479 [hep-lat]

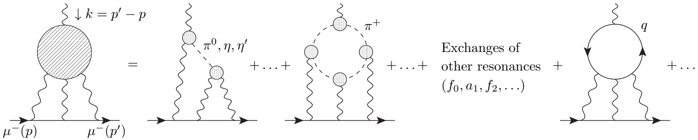
Summary of lattice and R-ratio data estimates:



- Precision: a few percent for most lattice collaborations, compared to less than half a percent from *R*-ratio data. Some collaborations hope to get less than one percent precision in a few years.
- Lattice results do not yet include all effects from strong isospin-breaking (SIB) $m_u \neq m_d$ and QED effects.
- BMW 2017: SIB and QED effects estimated from phenomenology.
- ETMC 2018 (and update 2019), RBC/UKQCD 2018, FNAL/HPQCD/MILC 2019: include some SIB and QED effects. But some parts are taken from other lattice collaborations, are modelled or taken from phenomenology.

Hadronic light-by-light scattering (HLbL)

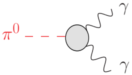
Hadronic light-by-light scattering in the muon $g-2$



de Rafael '94:

Chiral counting:	p^6	p^4	p^8	p^8
N_c-counting:	N_c	1	N_c	N_c

Coupling of photons to hadrons, e.g. π^0, via form factor:



Relevant scales ($\langle VVVV \rangle$ with offshell photons): $0 - 2$ GeV $\gg m_\mu$ (resonance region)

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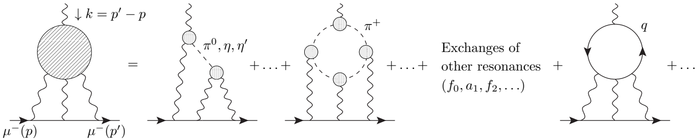

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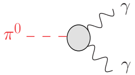

Relevant scales ($\langle VVVV \rangle$ with offshell photons): $0 - 2$ GeV $\gg m_\mu$ (resonance region)

View before 2014: in contrast to HVP, no direct relation to experimental data
$\rightarrow$ size and even sign of contribution to a_μ unknown !

Approach: use hadronic model at low energies with exchanges and loops of resonances and some (dressed) "quark-loop" at high energies.

Constrain models using experimental data (processes of hadrons with photons: decays, form factors, scattering) and theory (ChPT at low energies; short-distance constraints from pQCD / OPE at high momenta).

Problems: Four-point function depends on several invariant momenta $\Rightarrow$ distinction between low and high energies not as easy as for two-point function in HVP.
Mixed regions: one loop momentum Q_1^2 large, the other Q_2^2 small and vice versa.

Summary of selected model evaluations for $a_\mu^{\mathrm{HLbL}} \times 10^{11}$

Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	–	114 ± 13	99 ± 16
axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	–	15 ± 10	22 ± 5
scalars	-6.8 ± 2.0	–	–	–	–	-7 ± 7	-7 ± 2
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	–	-19 ± 19	-19 ± 13
π, K loops +subl. N_C	–	–	–	0 ± 10	–	–	–
quark loops	21 ± 3	9.7 ± 11.1	–	–	–	2.3 (c-quark)	21 ± 3
Total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09 ("Glasgov consensus"); N = AN '09, JN = Jegerlehner, AN '09

- Pseudoscalars π^0, η, η' dominate numerically.
- Other contributions not negligible.
- Cancellation between π, K-loops and quark loops !
- Recent reevaluations of axial vector contribution lead to much smaller estimates than in MV '04: $a_\mu^{\mathrm{HLbL;axial}} = (8 \pm 3) \times 10^{-11}$ (Pauk, Vanderhaeghen '14; Jegerlehner '14, '15). Would shift central values of compilations downwards:

  $a_\mu^{\mathrm{HLbL}} = (98 \pm 26) \times 10^{-11}$ (PdRV) and $a_\mu^{\mathrm{HLbL}} = (102 \pm 39) \times 10^{-11}$ (N, JN).

HLbL in muon $g-2$

- Frequently used estimates:

$$a_\mu^{\mathrm{HLbL}} = (105 \pm 26) \times 10^{-11}$$ (Prades, de Rafael, Vainshtein '09) ("Glasgow consensus")

$$a_\mu^{\mathrm{HLbL}} = (116 \pm 39) \times 10^{-11}$$ (AN '09; Jegerlehner, AN '09)

  Based almost on same input: calculations by various groups using different models for individual contributions. Error estimates are mostly guesses !
- Need much better understanding of complicated hadronic dynamics to get reliable error estimate of $\pm 15 \times 10^{-11}$ (δa_μ(future exp) $= 16 \times 10^{-11}$).
- Proposal in 2014: Colangelo et al. '14, '15; Pauk, Vanderhaeghen '14: use dispersion relations (DR) to connect contribution to HLbL from presumably numerically dominant light pseudoscalars to in principle measurable form factors and cross-sections:

$$\begin{aligned} \gamma^*\gamma^* &\to \pi^0, \eta, \eta' \\ \gamma^*\gamma^* &\to \pi^+\pi^-, \pi^0\pi^0, \pi^0\eta \end{aligned}$$

  Could connect HLbL uncertainty to exp. measurement errors, like HVP.
  Note: no data yet with two off-shell photons !
- HLbL from Lattice QCD (model-independent, first-principle).
  First steps and results: Blum et al. (RBC-UKQCD) '05, ..., '16 - '19
  Work ongoing at Mainz: Green et al. '15; Asmussen et al. '16 - '19, Gérardin et al. '16, '19 (Pion transition form factor, pion-pole in HLbL)

HLbL: dispersive / data-driven approach

Data-driven approach to HLbL using dispersion relations

I: Data-driven evaluation using DR (hopefully numerically dominant):

(1) π^0, η, η' poles

(2) $\pi\pi$ intermediate state

II: Model dependent evaluation (hopefully numerically subdominant):

(1) Axial vectors (3π-intermediate state), ...

(2) Quark-loop, matching with pQCD

Error goals: Part I: 10% precision (data driven), Part II: 30% precision.
To achieve overall error of about 15% ($\delta a_\mu^{\mathrm{HLbL}} = 15 \times 10^{-11}$).

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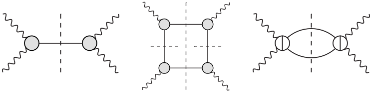
Colangelo et al. '14, '15:
Classify intermediate states in 4-point function. Then project onto $g-2$.



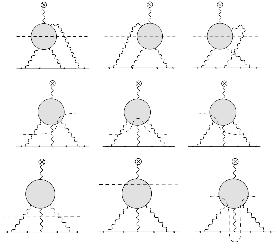
Hoferichter et al. '18:

$$a_\mu^{\pi^0-\mathrm{pole}} = 62.6^{+3.0}_{-2.5} \times 10^{-11}$$

Colangelo et al. '17: pion-box contribution (middle diagram) using precise information on pion vector form factor and S-wave $\pi\pi$-rescattering effects from pion-pole in left-hand cut (LHC) (part of right diagram):

$$a_\mu^{\pi-\mathrm{box}} = -15.9(2) \times 10^{-11}$$
$$a_{\mu,J=0}^{\pi\pi,\pi-\mathrm{pole\ LHC}} = -8(1) \times 10^{-11}$$
$$\text{Sum of the two} = -24(1) \times 10^{-11}$$

Pauk, Vanderhaeghen '14:
DR directly for Pauli FF $F_2(k^2)$.

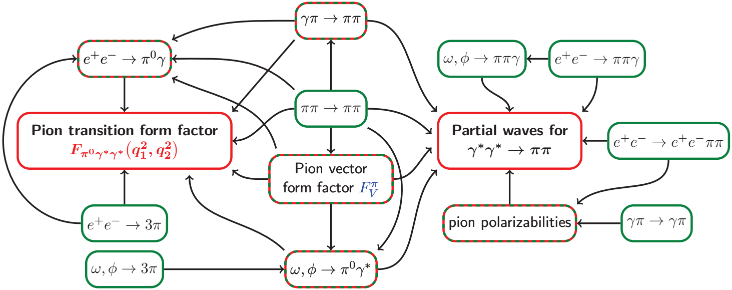


HLbL sum rules (SR) to get constraints from data on models: Pascalutsa, Vanderhaeghen '10; Pascalutsa, Pauk, Vanderhaeghen '12.
DR for $\gamma^{(*)}\gamma \to \pi\pi(\pi\eta)$: Danilkin, (Deineka), Vanderhaeghen '17, ('17) '19 ('19).
Next talk by Oleksandra Deineka.
Schwinger SR: Hagelstein, Pascalutsa '18, '19.

Data-driven approach to HLbL using dispersion relations (continued)

Hadronic light-by-light: a roadmap

GC, Hoferichter, Kubis, Procura, Stoffer `arXiv:1408.2517` (PLB '14)

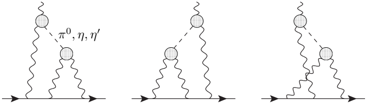


Artwork by M. Hoferichter

A reliable evaluation of the HLbL requires many different contributions by and a collaboration among theorists and experimentalists

From talk by Colangelo at Radio Monte Carlo Meeting, Frascati, May 2016

Pseudoscalar-pole contribution to $a_\mu^{\rm HLbL}$ (DR / data-driven or lattice QCD)



$$a_\mu^{\rm HLbL} = \left(\frac{\alpha}{\pi}\right)^3 \left(a_\mu^{\rm HLbL;P(1)} + a_\mu^{\rm HLbL;P\mathit{P}(2)}\right)$$

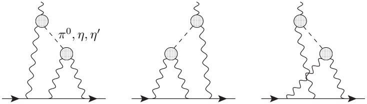
where (Jegerlehner + AN '09)

$$a_\mu^{\rm HLbL;P(1)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_1(Q_1, Q_2, \tau) \; \mathcal{F}_{\rm P\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2) \; \mathcal{F}_{\rm P\gamma^*\gamma^*}(-Q_2^2, 0)$$

$$a_\mu^{\rm HLbL;P(2)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_2(Q_1, Q_2, \tau) \; \mathcal{F}_{\rm P\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) \; \mathcal{F}_{\rm P\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)$$

3-dim. integration over lengths $Q_i = |(Q_i)_\mu|, i = 1, 2$ of the two Euclidean momenta and angle θ between them $Q_1 \cdot Q_2 = Q_1 Q_2 \cos\theta$ with $\tau = \cos\theta$.
$w_{1,2}(Q_1, Q_2, \tau)$: model-independent weight functions, concentrated at small momenta below 1 GeV [AN '16].
Need data-driven description for double-virtual pseudoscalar transition form factors $\mathcal{F}_{\rm P\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$.

Pseudoscalar-pole contribution to a_μ^{HLbL} (DR / data-driven or lattice QCD)



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where (Jegerlehner + AN '09)

$$a_\mu^{\mathrm{HLbL;P(1)}} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_1(Q_1, Q_2, \tau) \; \mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2) \; \mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(-Q_2^2, 0)$$

$$a_\mu^{\mathrm{HLbL;P(2)}} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_2(Q_1, Q_2, \tau) \; \mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) \; \mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)$$

3-dim. integration over lengths $Q_i = |(Q_i)_\mu|, i = 1, 2$ of the two Euclidean momenta and angle θ between them $Q_1 \cdot Q_2 = Q_1 Q_2 \cos\theta$ with $\tau = \cos\theta$.
$w_{1,2}(Q_1, Q_2, \tau)$: model-independent weight functions, concentrated at small momenta below 1 GeV [AN '16].
Need data-driven description for double-virtual pseudoscalar transition form factors $\mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$.

Recent evaluations using DR, Canterbury approximants (CA) (generalization of Pade approximants to 2 variables) and lattice QCD:

Pseudoscalar	$a_\mu^{\mathrm{HLbL};P} \times 10^{11}$	Method to get $\mathcal{F}_{\mathrm{P}\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$	Reference
π^0	$62.6^{+3.0}_{-2.5}$	DR	Hoferichter et al. '18
π^0	63.6 ± 2.7	CA	Masjuan + Sanchez-Puertas '17
π^0	59.7 ± 3.6	Lattice QCD	Gérardin, Meyer, AN '19
π^0	62.3 ± 2.3	Lattice (norm. to $\Gamma(\pi^0 \to \gamma\gamma)_{\mathrm{exp}}$)	Gérardin, Meyer, AN '19
η	16.3 ± 1.4	CA	Masjuan + Sanchez-Puertas '17
η'	14.5 ± 1.9	CA	Masjuan + Sanchez-Puertas '17
Total	93 ± 4	$\pi^0(\mathrm{DR}) + \eta(\mathrm{CA}) + \eta'(\mathrm{CA})$	My average

My estimate for HLbL (largely dispersive, data-driven)

Contribution	PdRV(09)	AN(09) / JN(09)	J(17)	My estimate
π^0, η, η'-poles	114±13	99±16	95.45±12.40	93±4
π, K-loops/boxes	−19±19	−19±13	−20±5	−16.4±0.2
S-wave $\pi\pi$ rescattering	–	–	–	−8±1
Scalars	−7±7	−7±2	−5.98±1.20	−2±2
Tensors	–	–	1.1±0.1	1±1
Axial vectors	15±10	22±5	7.55±2.71	8±3
Quark-loops / short-distance	2.3 (c-quark)	21±3	22.3±5.0	10±10
Total	105±26	116±39	100.4±28.2	86 ± 21

PdRV(09) = Prades, de Rafael, Vainsthein '09 ("Glasgow consensus"); JN(09) = Jegerlehner, AN '09;

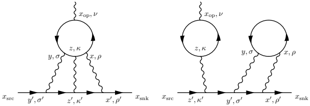
J(17) = Jegerlehner '17

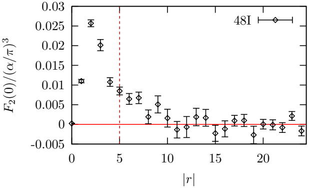
- First three entries in last column based on dispersive / data driven approach to HLbL (Hoferichter et al. '18, Masuan and Sanchez-Puertas '17, Colangelo et al. '17).
- Other entries in last column are based on model-calculations for heavy resonance exchanges ($M_R > 1$ GeV) with hopefully conservative error estimates (Pauk and Vanderhaeghen '14, Jegerlehner '14, '15, Danilkin and Vanderhaeghen '17, Knecht et al. '18) and a recent updated estimate for the short-distance contribution (Bijnens et al. '19).
- Errors added linearly, as done earlier by Bijnens, Pallante, Prades '95, '96, '02, by Bijnens and Prades '07 and by Jegerlehner and AN '09.

HLbL from lattice QCD

HLbL in muon $g-2$ from Lattice QCD: RBC-UKQCD approach

- Blum et al. '05, ..., '15: First attempts: Put QCD + (quenched) QED on the lattice. Subtraction of lower-order in α HVP contribution needed, very noisy.
- Jin et al. '15, '16, '17: Step by step improvement of method to reduce statistical error by one or two orders of magnitude and remove some systematic errors.
- Calculate $a_\mu^{\mathrm{HLbL}} = F_2(q^2=0)$ via moment method in position-space (no extrapolation to $q^2=0$ needed).
- Use exact expression for all photon propagators. Treat $r = x - y$ stochastically by sampling points x, y. Found empirically: short-distance contribution at small $|r| < 0.6$ fm dominates.





Results (for $m_\pi = m_{\pi,\mathrm{phys}}$, lattice spacing $a^{-1} = 1.73$ GeV, $L = 5.5$ fm):

$$a_\mu^{\mathrm{cHLbL}} = (116.0 \pm 9.6) \times 10^{-11} \qquad \text{(quark-connected diagrams)}$$

$$a_\mu^{\mathrm{dHLbL}} = (-62.5 \pm 8.0) \times 10^{-11} \qquad \text{(leading quark-disconnected diagrams)}$$

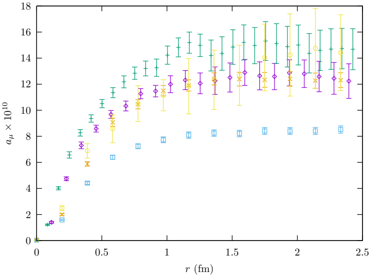
$$a_\mu^{\mathrm{HLbL}} = (53.5 \pm 13.5) \times 10^{-11}$$

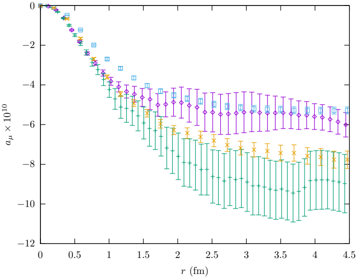
Beware ! Statistical error only ! Missing systematic effects:

- Expect large finite-volume effects from QED $\sim 1/L^2$. Blum et al. '17: use infinite volume, continuum QED (like Mainz approach: Asmussen et al. '16).
- Expect large finite-lattice-spacing effects.
- Omitted subleading quark-disconnected diagrams (10% effect ?).

RBC-UKQCD approach to HLbL: recent update from Moriond 2019

Cumulative contributions to $a_\mu^{\mathrm{HLbL}}(r)$ for several lattice ensembles (connected: left, disconneced: right). r distance between two sampled currents in hadronic loop, the other two currents are summed exactly:





Results with physical mass pions, extrapolated to the continuum and to infinite volume (QED$_L$ formalism):

$$a_\mu^{\mathrm{cHLbL}} = (276.1 \pm 31.4) \times 10^{-11} \qquad \text{(quark-connected diagrams)}$$
$$a_\mu^{\mathrm{dHLbL}} = (-202.0 \pm 56.5) \times 10^{-11} \qquad \text{(quark-disconnected diagrams)}$$
$$a_\mu^{\mathrm{HLbL}} = (74.1 \pm 63.3) \times 10^{-11}$$

Statistical errors dominates.

HLbL in muon $g-2$ from Lattice QCD: Mainz approach

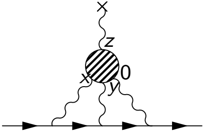
Developed independently from RBC-UKQCD

(Asmussen (Southampton), Green (DESY Zeuthen), Gérardin (DESY Zeuthen), Meyer, AN '15 - '18 and work in progress with Chao and Hudspith)

Master formula in position-space (derivation in Backup slides):

$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} \underbrace{\int d^4y}_{=2\pi^2 \int_0^\infty d|y||y|^3} \underbrace{\int d^4x}_{=4\pi \int_0^\infty d|x||x|^3 \int_0^\pi d\beta \sin^2(\beta)} \underbrace{\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)}_{\mathrm{QED}} \underbrace{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)}_{\mathrm{QCD}}$$

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int d^4z\, z_\rho \left\langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \right\rangle$$



- QCD blob: lattice regularization.
- Everything else: position space perturbation theory in Euclidean formulation.
- QED kernel computed semi-analytically in continuum and in infinite volume. Angular integrations in position space and averaging over direction of muon momentum using Gegenbauer polynomials.
- Lorentz covariance manifest. After summation over Lorentz indices get 3-dimensional integral over $|x|, |y|, x \cdot y$.
- No power law $1/L^2$ finite-volume effects.
- Kernel parametrized by 6 weight functions (and derivatives thereof), calculated on 3D grid in $|x|, |y|, x \cdot y$ to 5 digits precision and stored on disk.
- Test of kernel function: pion-pole contribution to HLbL, lepton loop in QED.
- Challenges: need to calculate QCD four-point function on the lattice, numerical efficiency for physical pion mass not yet shown.

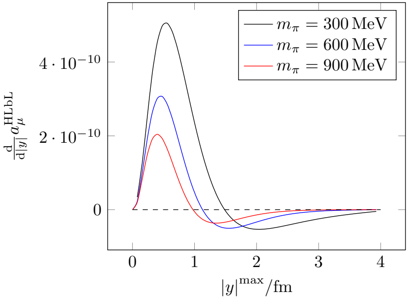
Numerical tests of QED kernel: Pion-pole contribution to a_μ^{HLbL}

Result with VMD model for arbitrary pion mass can easily be obtained from 3-dimensional momentum-space representation (Jegerlehner + AN '09).

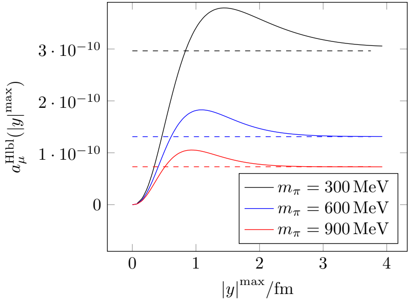
3-dim. integration in position-space:

- $\int_y \to 2\pi^2 \int_0^\infty \mathrm{d}|y||y|^3$
- $\int_x \to 4\pi \int_0^\infty \mathrm{d}|x|\,|x|^3 \int_0^\pi \mathrm{d}\beta \sin^2\beta$ (cutoff for x integration: $|x|^{\max} = 4.05$ fm)

Integrand after integration over $|x|, \beta$:

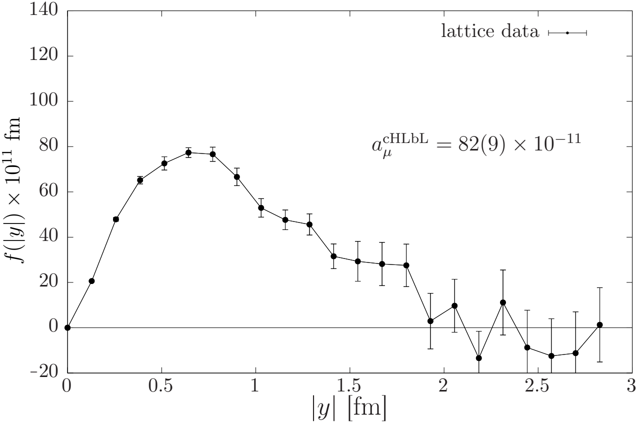


Result for $a_\mu^{\text{HLbL}}(|y|^{\max})$:



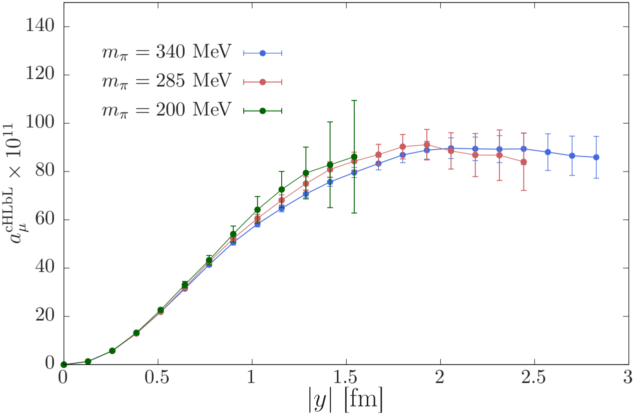
- All 6 weight functions contribute to final result, some only at the percent level. Reproduce known results to better than 1%.
- $|x|^{\max}, |y|^{\max} > 4$ fm needed for $m_\pi < 300$ MeV.
- Reproduce results for lepton loop in QED with $m_l = m_\mu, 2m_\mu$ at percent level.

QCD: Integrand of a_μ^{cHLbL} for $m_\pi = 340$ MeV, $a = 0.064$ fm



- Fully connected contribution only !
- We already observe a good signal.
- Statistical error only.
- Integrand non-zero up to about 2 fm.

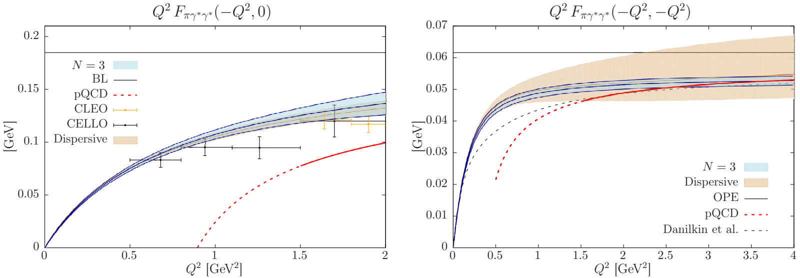
Pion mass dependence of a_μ^{cHLbL}, $a = 0.064\ \mathrm{fm}$



- The results show an upward trend for decreasing pion mass.
- Similarly, at $m_\pi = 285$ MeV, discretization effects from different lattice spacings seem to be small.
- Comparing ensembles with $m_\pi L = 4$ and $m_\pi L = 6$, finite-size effects also seem to be small.

Pion transition form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$ from lattice QCD

Gérardin, Meyer, AN, '16, '19:



- CLS lattice ensembles: 4 lattice spacings $a = (0.050, 0.064, 0.076, 0.086)$ fm and pion masses in range $[200 - 420]$ MeV, uncorrelated global fit with $\chi^2/\mathrm{d.o.f.} = 1.1$.
- Model-independent double z-expansion (conformal mapping) with $N = 3$ to perform extrapolation of lattice data to continuum and physical pion mass.
- Horizontal black lines: predictions from Brodsky-Lepage (BL) (single-virtual) and OPE (double-virtual). Do not impose prefactor as constraint.
- Prediction at large Q^2 with perturbative QCD (pQCD) includes higher twist and NLO corrections and assumes asymptotic pion distribution amplitude.
- Comparison with results within dispersive framework by Hoferichter *et al.* '18 and, for double-virtual case, the model proposed by Danilkin *et al.* '19.

Conclusions and Outlook

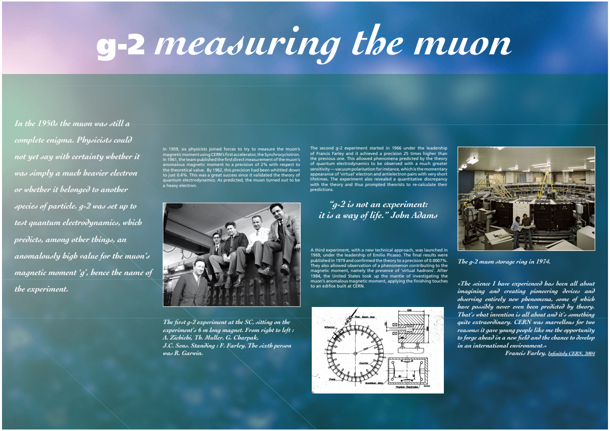
- Over many decades, the (anomalous) magnetic moments of electron and muon have played crucial role in atomic and elementary particle physics.
- Gained important insights into structure of fundamental interactions and matter in the universe (quantum field theory).
- a_μ: Test of Standard Model, potential window to New Physics.
- Current situation:

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (275 \pm 78) \times 10^{-11} \qquad [3.5\ \sigma]$$

  Hadronic effects ? Sign of New Physics ?

- Two new planned $g-2$ experiments at Fermilab (E989) and J-PARC (E34) with goal of $\delta a_\mu^{\text{exp}} = 16 \times 10^{-11}$ (factor 4 improvement)
- Theory needs to match this precision !
- Muon $g-2$ Theory Initiative: concerted effort of experiments (measuring processes with hadrons and photons), phenomenology / theory (data-driven using dispersion relations and modelling) and lattice QCD to improve HVP and HLbL estimates with reliable uncertainties.

And finally:



Source: CERN

"$g-2$ is not an experiment: it is a way of life."

John Adams (Head of the Proton Synchrotron at CERN (1954-61) and Director General of CERN (1960-1961))

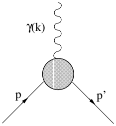
This statement also applies to many theorists working on the $g-2$!

Backup slides

Anomalous magnetic moment in quantum field theory

Quantized spin 1/2 particle interacting with external, classical electromagnetic field

4 form factors in vertex function
(momentum transfer $k = p' - p$, not assuming parity or charge conjugation invariance)

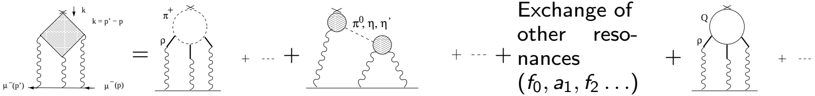


$$\begin{aligned} &\equiv \quad i\langle p', s'|j^\mu(0)|p, s\rangle \\ &= \quad (-ie)\bar{\mathrm{u}}(p', s')\Big[\gamma^\mu \underbrace{F_1(k^2)}_{\text{Dirac}} + \frac{i\sigma^{\mu\nu}k_\nu}{2m}\underbrace{F_2(k^2)}_{\text{Pauli}} \\ &\qquad + \gamma^5 \frac{\sigma^{\mu\nu}k_\nu}{2m} F_3(k^2) + \gamma^5 \left(k^2\gamma^\mu - \not{k}\, k^\mu\right) F_4(k^2)\Big]\mathrm{u}(p, s) \end{aligned}$$

Real form factors for spacelike $k^2 \leq 0$. Non-relativistic, static limit:

$$\begin{aligned} F_1(0) &= 1 \qquad \text{(renormalization of charge } e\text{)} \\ \mu &= \frac{e}{2m}(F_1(0) + F_2(0)) \qquad \text{(magnetic moment)} \\ a &= F_2(0) \qquad \text{(anomalous magnetic moment)} \\ d &= -\frac{e}{2m}F_3(0) \qquad \text{(electric dipole moment, violates P and CP)} \\ F_4(0) &= \text{anapole moment (violates P)} \end{aligned}$$

HLbL in muon $g-2$: summary of selected results (model calculations)



de Rafael '94:

Chiral counting:	p^4	p^6	p^8	p^8
N_C-counting:	1	N_C	N_C	N_C

Contribution to $a_\mu \times 10^{11}$:

BPP:	+83 (32)	-19 (13)	+85 (13)	-4 (3) [f_0, a_1]	+21 (3)
HKS:	+90 (15)	-5 (8)	+83 (6)	+1.7 (1.7) [a_1]	+10 (11)
KN:	+80 (40)		+83 (12)		
MV:	+136 (25)	0 (10)	+114 (10)	+22 (5) [a_1]	0
2007:	+110 (40)				
PdRV:	+105 (26)	-19 (19)	+114 (13)	+8 (12) [f_0, a_1]	+2.3 [c-quark]
N,JN:	+116 (39)	-19 (13)	+99 (16)	+15 (7) [f_0, a_1]	+21 (3)
		ud.: -45	ud.: $+\infty$		ud.: +60

ud. = undressed, i.e. point vertices without form factors

Pseudoscalars: numerically dominant contribution (according to most models !).

Recall (in units of 10^{-11}): $\delta a_\mu(\text{HVP}) \approx 40$; $\delta a_\mu(\text{exp [BNL]}) = 63$; $\delta a_\mu(\text{future exp}) = 16$

BPP = Bijnens, Pallante, Prades '96, '02; HKS = Hayakawa, Kinoshita, Sanda '96, '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; 2007 = Bijnens, Prades; Miller, de Rafael, Roberts; PdRV = Prades, de Rafael, Vainshtein '09 (compilation; "Glasgow consensus"); N,JN = AN '09; Jegerlehner, AN '09 (compilation)

Recent reevaluations of axial vector contribution lead to much smaller estimates than in MV '04: $a_\mu^{\text{HLbL;axial}} = (8 \pm 3) \times 10^{-11}$ (Pauk, Vanderhaeghen '14; Jegerlehner '14, '15). Would shift central values of compilations downwards:

$a_\mu^{\text{HLbL}} = (98 \pm 26) \times 10^{-11}$ (PdRV) and $a_\mu^{\text{HLbL}} = (102 \pm 39) \times 10^{-11}$ (N, JN).

Vertex function for HLbL in momentum space

In Euclidean space ($k = p' - p$):

$$\langle \mu^-(p', s') | j_\rho(0) | \mu^-(p, s) \rangle = -\bar{u}^{s'}(p') \left[\gamma_\rho F_1(k^2) + \frac{\sigma_{\rho\tau} k_\tau}{2m} F_2(k^2) \right] u^s(p)$$

Project on anomalous magnetic moment:

$$a_\mu^{\text{HLbL}} = F_2(0) = \frac{-i}{48m} \mathrm{Tr}\{[\gamma_\rho, \gamma_\sigma](-i\not{p} + m)\Gamma_{\rho\sigma}(p, p)(-i\not{p} + m)\}$$

with on-shell muon momentum $p = im\hat{\epsilon}$ ($p^2 = -m^2$; $\hat{\epsilon}$: unit vector).

$$\begin{aligned}\Gamma_{\rho\sigma}(p', p) = -\, e^6 \int_{q_1, q_2} & \frac{1}{q_1^2 q_2^2 (q_1+q_2-k)^2} \frac{1}{(p'-q_1)^2+m^2} \frac{1}{(p'-q_1-q_2)^2+m^2} \\ & \times \gamma_\mu (i\not{p}' - i\not{q}_1 - m)\gamma_\nu (i\not{p} - i\not{q}_1 - i\not{q}_2 - m)\gamma_\lambda \\ & \times \frac{\partial}{\partial k_\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2)\end{aligned}$$

$$\Pi_{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \int_{x_1, x_2, x_3} e^{-i(q_1 x_1 + q_2 x_2 + q_3 x_3)} \langle j_\mu(x_1) j_\nu(x_2) j_\lambda(x_3) j_\sigma(0) \rangle$$

Where we used the following relation derived from the Ward identities to extract one factor of k to get $F_2(k^2)$ [Kinoshita *et al.* '70]:

$$\Pi_{\mu\nu\lambda\rho}(q_1, q_2, k - q_1 - q_2) = -k_\sigma \frac{\partial}{\partial k_\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2)$$

Notation: $\int_q \equiv \int \frac{\mathrm{d}^4 q}{(2\pi)^4}$, $\int_x \equiv \int \mathrm{d}^4 x$

HLbL in muon $g-2$ in position-space

Vertex function in terms of position-space functions:

$$\Gamma_{\rho\sigma}(p,p) = -\,e^6 \int_{x,y} K_{\mu\nu\lambda}(x,y,p)\,\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)$$

$$K_{\mu\nu\lambda}(x,y,p) = \gamma_\mu(i\not{p} + \not{\partial}^{(x)} - m)\gamma_\nu(i\not{p} + \not{\partial}^{(x)} + \not{\partial}^{(y)} - m)\gamma_\lambda\,\mathcal{I}(\hat{\epsilon},x,y)$$

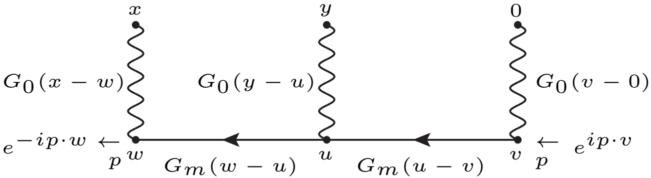
$$\mathcal{I}(\hat{\epsilon},x,y) = \int_{q,k,\text{IR-reg}} \frac{1}{q^2k^2(q+k)^2}\,\frac{1}{(p-q)^2+m^2}\,\frac{1}{(p-q-k)^2+m^2}\,e^{-i(q\cdot x+k\cdot y)}$$

$$\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = \int_z iz_\rho\,\langle j_\mu(x)j_\nu(y)j_\sigma(z)j_\lambda(0)\rangle$$

- $\mathcal{I}$ is logarithmically infrared divergent for $p^2 = -m^2$
  $\Rightarrow$ introduce IR regulator.
- In a_μ^{HLbL} only terms with derivatives remain and $K_{\mu\nu\lambda}$ is infrared finite.

Evaluating $\mathcal{I}(\hat{\epsilon}, x, y)$

Diagrammatic representation of $\mathcal{I}(\hat{\epsilon}, x, y)$ (arrows on muon line are only a reminder of the origin of the diagram, G_m are scalar propagators !):



$$\mathcal{I}(\hat{\epsilon}, x, y) = \int_{u,\text{IR-reg}} G_0(u-y) J(\hat{\epsilon}, u) J(\hat{\epsilon}, x-u)$$

$$J(\hat{\epsilon}, u) = \int_v G_0(v+u) e^{-m\hat{\epsilon}\cdot v} G_m(v) = \sum_{n\geq 0} z_n(u^2) U_n(\hat{\epsilon}\cdot\hat{u})$$

Last expression: expansion in terms of Chebyshev polynomials of the second kind U_n (special case of the Gegenbauer polynomials)

z_n: linear combination of products of two modified Bessel functions K_m and I_k

Scalar propagators in position-space:

$$G_0(x) = \frac{1}{4\pi^2 x^2}$$

$$G_m(x) = \frac{m}{4\pi^2|x|} K_1(m|x|) \quad (K_1 \text{ is a modified Bessel function})$$

Averaging over direction of muon momentum $p = im\hat{\epsilon}$

Evaluating Dirac trace in projector, one obtains an expression of the form

$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} \int_y \int_x \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(\hat{\epsilon}, x, y) i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x, y)$$

Exploit invariance of a_μ under $O(4)$ rotations of the muon momentum and average kernel $\mathcal{L}$ over direction $\hat{\epsilon}$

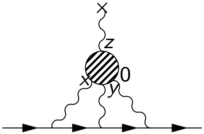
$$\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x, y) = \frac{1}{2\pi^2} \int \mathrm{d}\Omega_{\hat{\epsilon}} \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(\hat{\epsilon}, x, y) \equiv \langle \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(\hat{\epsilon}, x, y) \rangle_{\hat{\epsilon}}$$

Angular average can be performed analytically by using orthogonality property of Chebyshev (Gegenbauer) polynomials that appear in QED kernel $\mathcal{L}$ via $\mathcal{I}$ and J (hyperspherical approach):

$$\langle U_n(\hat{\epsilon} \cdot \hat{x}) U_m(\hat{\epsilon} \cdot \hat{y}) \rangle_{\hat{\epsilon}} = \frac{\delta_{nm}}{n+1} U_n(\hat{x} \cdot \hat{y})$$

HLbL master formula in position-space

$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} \int d^4y \int d^4x \, \underbrace{\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)}_{\mathrm{QED}} \, \underbrace{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)}_{\mathrm{QCD}}$$



After contracting the Lorentz indices the integration reduces to a 3-dimensional integral over $x^2, y^2, x \cdot y = |x||y|\cos\beta$:

$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} \underbrace{\int d^4y}_{=2\pi^2 \int_0^\infty d|y||y|^3} \underbrace{\int d^4x}_{=4\pi \int_0^\infty d|x||x|^3 \int_0^\pi d\beta \sin^2(\beta)} \underbrace{\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)}_{\mathrm{QED}} \, \underbrace{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)}_{\mathrm{QCD}}$$

QCD four-point function (spatial moment):

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int d^4z \, z_\rho \, \langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \rangle$$

QED kernel function $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$

- Computed semi-analytically.
- Weights the QCD four-point function in position-space.
- Tensor decomposition leads to 6 weight functions (and derivatives thereof) that depend on the 3 variables $x^2, y^2, x \cdot y$.
- We have computed these weight functions on a grid to about 5 digits precision, once and for all, and stored on disk.

Tensor decomposition of QED kernel and weight functions

$$\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) = \sum_{A=I,II,III} \mathcal{G}^{A}_{\delta\rho\sigma\mu\alpha\nu\beta\lambda} T^{A}_{\alpha\beta\delta}(x,y)$$

$$\mathcal{G}^{I,II,III}_{\delta\rho\sigma\mu\alpha\nu\beta\lambda} = \text{sums of products of Kronecker deltas (from Dirac trace)}$$

$$T^{I}_{\alpha\beta\delta}(x,y) = \partial^{(x)}_{\alpha}(\partial^{(x)}_{\beta} + \partial^{(y)}_{\beta}) V_{\delta}(x,y)$$

$$T^{II}_{\alpha\beta\delta}(x,y) = m\partial^{(x)}_{\alpha}\left(T_{\beta\delta}(x,y) + \frac{1}{4}\delta_{\beta\delta} S(x,y)\right)$$

$$T^{III}_{\alpha\beta\delta}(x,y) = m(\partial^{(x)}_{\beta} + \partial^{(y)}_{\beta})\left(T_{\alpha\delta}(x,y) + \frac{1}{4}\delta_{\alpha\delta} S(x,y)\right)$$

Scalar: $S(x,y) = \langle \mathcal{I} \rangle_{\hat{\epsilon}}$ (IR regulated)

Vector: $V_{\delta}(x,y) = \langle \hat{\epsilon}_{\delta} \mathcal{I} \rangle_{\hat{\epsilon}}$

Tensor: $T_{\beta\delta}(x,y) = \langle (\hat{\epsilon}_{\beta}\hat{\epsilon}_{\delta} - \frac{1}{4}\delta_{\beta\delta})\mathcal{I} \rangle_{\hat{\epsilon}}$

$$S(x,y) = g^{(0)}$$

$$V_{\delta}(x,y) = x_{\delta} g^{(1)} + y_{\delta} g^{(2)}$$

$$T_{\alpha\beta}(x,y) = \left(x_{\alpha}x_{\beta} - \frac{x^2}{4}\delta_{\alpha\beta}\right) l^{(1)} + \left(y_{\alpha}y_{\beta} - \frac{y^2}{4}\delta_{\alpha\beta}\right) l^{(2)} + \left(x_{\alpha}y_{\beta} + y_{\alpha}x_{\beta} - \frac{x \cdot y}{2}\delta_{\alpha\beta}\right) l^{(3)}$$

where the 6 weight functions depend on $x^2, y^2, x \cdot y$.

Example: Weight function $g^{(2)}(x^2, x \cdot y, y^2)$

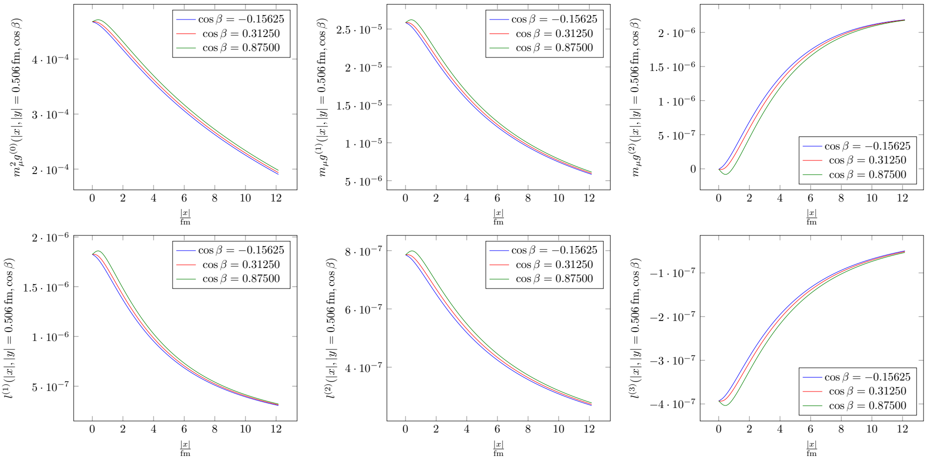
$$
\begin{aligned}
g^{(2)}(x^2, x\cdot y, y^2) &= \frac{1}{8\pi y^2 |x| \sin^3\beta} \int_0^\infty du\, u^2 \int_0^\pi d\phi_1 \\
&\times \left[2\sin\beta + \left(\frac{y^2+u^2}{2|u||y|} - \cos\beta\cos\phi_1 \right) \frac{\log\chi}{\sin\phi_1} \right] \\
&\times \sum_{n=0}^{\infty} \left\{ z_n(|u|) z_{n+1}(|x-u|) \left[|x-u|\cos\phi_1 \frac{U_n}{n+1} + (|u|\cos\phi_1 - |x|) \frac{U_{n+1}}{n+2} \right] \right. \\
&\quad \left. + z_{n+1}(|u|) z_n(|x-u|) \left[(|u|\cos\phi_1 - |x|) \frac{U_n}{n+1} + |x-u|\cos\phi_1 \frac{U_{n+1}}{n+2} \right] \right\}
\end{aligned}
$$

where

$$
\begin{aligned}
x\cdot y =& |x||y|\cos\beta\,, \quad |x-u| = \sqrt{|x|^2 + |u|^2 - 2|x||u|\cos\phi_1} \\
\chi =& \frac{y^2+u^2-2|u||y|\cos(\beta-\phi_1)}{y^2+u^2-2|u||y|\cos(\beta+\phi_1)}\,, \quad U_n = U_n\Big(\frac{|x|\cos\phi_1 - |u|}{|u-x|}\Big) \\
z_n =& \text{linear combination of products of two modified Bessel functions.}
\end{aligned}
$$

Weight functions: $|x|$ dependence

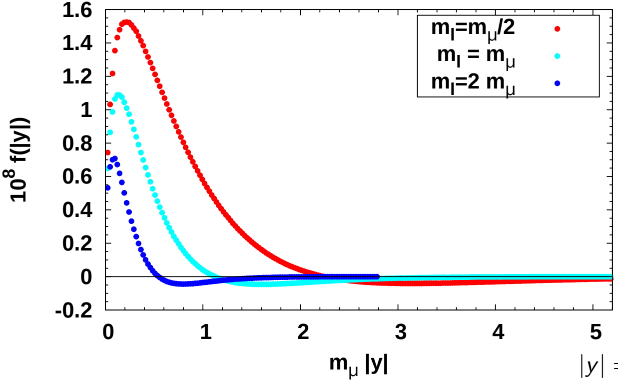
For $|y| = 0.506$ fm:



$g^{(0)}(|x|, x \cdot y, |y|)$ contains an arbitrary additive constant (due to the IR divergence in $\mathcal{I}(\hat{\epsilon}, x, y)$), which does not contribute to $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x, y)$.

Numerical test of QED kernel: Lepton loop contribution a_μ^{LbL} in QED

Integrand of lepton loop contribution a_μ^{LbL}:



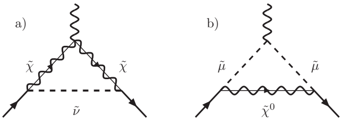
$|y| = 9.3\ \mathrm{fm}$

m_l/m_μ	$a_\mu^{\mathrm{LbL}} \times 10^{11}$ (exact)	$a_\mu^{\mathrm{LbL}} \times 10^{11}$	Precision	Deviation
1/2	1229.07	1257.5(6.2)(2.4)	0.5%	2.3%
1	464.97	470.6(2.3)(2.1)	0.7%	1.2%
2	150.31	150.4(0.7)(1.7)	1.2%	0.06%

1st uncertainty from 3D integration, 2nd uncertainty from extrapolation to small $|y|$.
Behavior for small $|y|$ compatible with $f(|y|) \propto m_\mu |y| \log^2(m_\mu |y|)$.
Analytical results for a_μ^{LbL} with $m_l = m_\mu, 2m_\mu$ reproduced at the percent level.
(Laporta + Remiddi '93, numbers courtesy of Massimo Passera)

a_μ: Supersymmetry

Supersymmetry for large $\tan\beta, \mu > 0$:

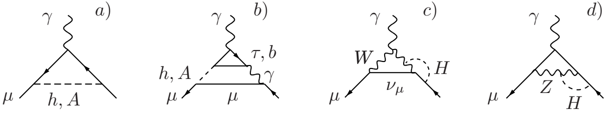


$$a_\mu^{\text{SUSY}} \approx 123 \times 10^{-11} \left(\frac{100\ \text{GeV}}{M_{\text{SUSY}}}\right)^2 \tan\beta$$

(Czarnecki, Marciano, '01)

Explains $\Delta a_\mu = 311 \times 10^{-11}$ if $M_{\text{SUSY}} \approx (89 - 399)$ GeV $\quad (2 < \tan\beta < 40)$.

In some regions of parameter space, large 2-loop contributions (2HDM):



Barr-Zee diagram (b) yields enhanced contribution, which can exceed 1-loop result.
Enhancement factor m_b^2/m_μ^2 compensates suppression by α/π
$((\alpha/\pi) \times (m_b^2/m_\mu^2) \sim 4 > 1)$.

a_μ and Supersymmetry after first LHC run

- LHC so far only sensitive to strongly interacting supersymmetric particles, like squarks and gluinos (ruled out below about 1 TeV).
- Muon $g-2$ and SUSY searches at LHC only lead to tension in constrained MSSM (CMSSM) or NUHM1 / NUHM2 (non-universal contributions to Higgs masses).
- More general SUSY models (e.g. pMSSM10 = phenomenological MSSM with 10 soft SUSY-breaking parameters) with light neutralinos, charginos and sleptons, can still explain muon $g-2$ discrepancy and evade bounds from LHC.

a_e, a_μ: Dark photon

In some dark matter scenarios, there is a relatively light, but massive "dark photon" A'_μ that couples to the SM through mixing with the photon:

$$\mathcal{L}_{\rm mix} = \frac{\varepsilon}{2} F^{\mu\nu} F'_{\mu\nu}$$

$\Rightarrow$ A'_μ couples to ordinary charged particles with strength $\varepsilon \cdot e$.
$\Rightarrow$ additional contribution of dark photon with mass $m_{\gamma'}$ to the $g-2$ of a lepton (electron, muon) (Pospelov '09):

$$\begin{aligned} a_\ell^{\text{dark photon}} &= \frac{\alpha}{2\pi}\varepsilon^2 \int_0^1 dx \frac{2x(1-x)^2}{\left[(1-x)^2 + \frac{m_{\gamma'}^2}{m_\ell^2} x\right]} \\ &= \frac{\alpha}{2\pi}\varepsilon^2 \times \begin{cases} 1 & \text{for } m_\ell \gg m_{\gamma'} \\ \frac{2m_\ell^2}{3m_{\gamma'}^2} & \text{for } m_\ell \ll m_{\gamma'} \end{cases} \end{aligned}$$

For values $\varepsilon \sim (1-2) \times 10^{-3}$ and $m_{\gamma'} \sim (10-100)$ MeV, the dark photon could explain the discrepancy $\Delta a_\mu \sim 300 \times 10^{-11}$.

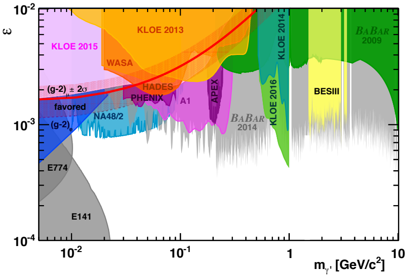
Various searches for the dark photon have been performed, are under way or are planned at many experiments.

For an overview, see: *Dark Sectors and New, Light, Weakly-Coupled Particles* (Snowmass 2013), Essig et al., arXiv:1311.0029 [hep-ph].

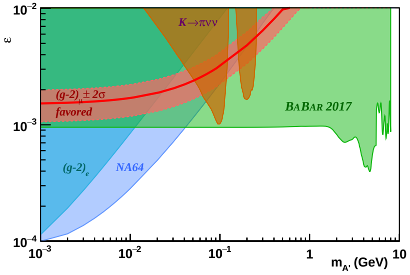
Status of dark photon searches

Essentially all of the parameter space in the $(m_{\gamma'}, \varepsilon)$-plane to explain the muon $g-2$ discrepancy has now been ruled out.

Decay of dark photon into $e^+e^-, \mu^+\mu^-$
e.g. BESIII, arXiv:1705.04265 [hep-ex]



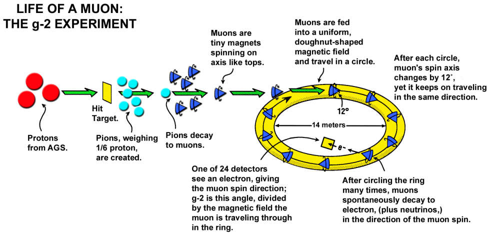
Invisible decay of dark photon
e.g. BABAR, arXiv:1702.03327 [hep-ex]



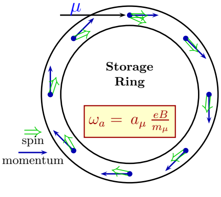
The Brookhaven Muon $g-2$ Experiment

The first measurements of the anomalous magnetic moment of the muon were performed in 1960 at CERN, $a_\mu^{\text{exp}} = 0.00113(14)$ (Garwin et al.) [12% precision] and improved until 1979: $a_\mu^{\text{exp}} = 0.0011659240(85)$ [7 ppm] (Bailey et al.)

In 1997, a new experiment started at the Brookhaven National Laboratory (BNL):



Source: BNL Muon $g-2$ homepage



actual precession $\times$ 2

Angular frequencies for cyclotron precession ω_c and spin precession ω_s:

$$\omega_c = \frac{eB}{m_\mu\,\gamma}, \quad \omega_s = \frac{eB}{m_\mu\,\gamma} + a_\mu \frac{eB}{m_\mu}, \quad \omega_a = a_\mu \frac{eB}{m_\mu}$$

$\gamma = 1/\sqrt{1-(v/c)^2}$. With an electric field to focus the muon beam one gets:

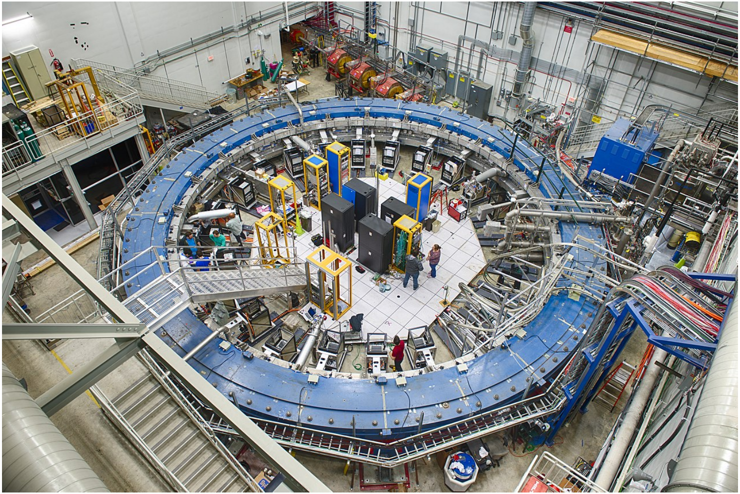
$$\vec{\omega}_a = \frac{e}{m_\mu}\left(a_\mu \vec{B} - \left[a_\mu - \frac{1}{\gamma^2-1}\right]\vec{v}\times\vec{E}\right)$$

Term with $\vec{E}$ drops out, if $\gamma = \sqrt{1+1/a_\mu} = 29.3$: "magic γ" $\rightarrow p_\mu = 3.094$ GeV$/c$

Fermilab Muon $g-2$ Experiment: the big move from BNL in summer 2013

Sources: Independent Seminar Blog, Fermilab Muon $g-2$ homepage

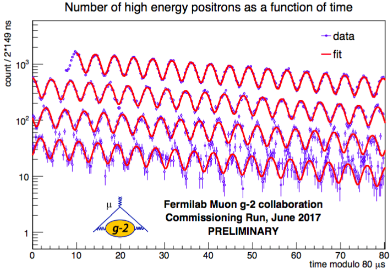
Fermilab Muon $g - 2$ Experiment: storage ring



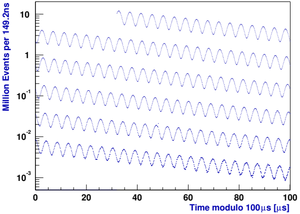
Source: Wikipedia (photo by Glukicov)

Fermilab Muon $g-2$ Experiment: determination of a_μ

First "wiggle plot" from June 2017:



Compare with BNL results, Bennett et al. 2006, 3.6 billion decays of μ^-:



$$N(t) = N_0(E)\,\exp\left(\frac{-t}{\gamma\tau_\mu}\right) \times [1 + A(E)\sin(\omega_a t + \phi(E))]$$

Exponential decay with mean lifetime: $\tau_{\mu,\text{lab}} = \gamma\tau_\mu = 64.378\mu s$ (in lab system).

Oscillations due to angular frequency $\omega_a = a_\mu eB/m_\mu$.

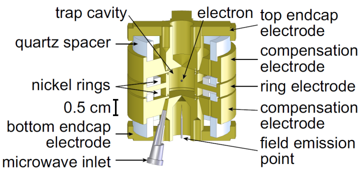
$$a_\mu = \frac{R}{\lambda - R} \text{ where } R = \frac{\omega_a}{\omega_p} \text{ and } \lambda = \frac{\mu_\mu}{\mu_p}$$

Experiment measures ω_a and ω_p (spin precession frequency for proton).

λ from hyperfine splitting of muonium ($\mu^+ e^-$) (external input).

Electron $g-2$: Experiment

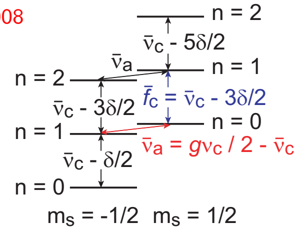
Latest experiment: Hanneke, Fogwell, Gabrielse, 2008



Cylindrical Penning trap for single electron

(1-electron quantum cyclotron)

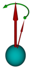
Source: Hanneke et al.



Cyclotron and spin precession levels of electron in Penning trap

Source: Hanneke et al.

$$\frac{g_e}{2} = \frac{\nu_s}{\nu_c} \simeq 1 + \frac{\bar{\nu}_a - \bar{\nu}_z^2/(2\bar{f}_c)}{\bar{f}_c + 3\delta/2 + \bar{\nu}_z^2/(2\bar{f}_c)} + \frac{\Delta g_{cav}}{2}$$



ν_s = spin precession frequency; $\nu_c, \bar{\nu}_c$ = cyclotron frequency: free electron, electron in Penning trap; $\delta/\nu_c = h\nu_c/(m_e c^2) \approx 10^{-9}$ = relativistic correction

4 quantities are measured precisely in experiment:

$\bar{f}_c = \bar{\nu}_c - \frac{3}{2}\delta \approx 149$ GHz; $\bar{\nu}_a = \frac{g}{2}\nu_c - \bar{\nu}_c \approx 173$ MHz;

$\bar{\nu}_z \approx 200$ MHz = oscillation frequency in axial direction;

Δg_{cav} = corrections due to oscillation modes in cavity

$$\Rightarrow a_e^{\exp} = 0.00115965218073(28)$$

[0.24 ppb $\approx$ 1 part in 4 billions]

[Kusch & Foley, 1947/48: 4% precision]

Precision in $g_e/2$ even 0.28 ppt $\approx$ 1 part in 4 trillions !