An alternative approach to neutrino mixing studies

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Motivation

Matrix theory and singular values:
alternative insight into the problem of the neutrino mixing analysis

- How much space is there for additional neutrinos?
- Can the number of additional neutrinos be distinguished?
- What can be told about the complete mixing?

Development of basic ideas from:
Neutrino mixing, interval matrices and singular values,
K. Bielas, WF, J. Gluza & M. Gluza Phys.Rev. D98 (2018) 053001,
http://arxiv.org/pdf/1708.09196.pdf

Main issues

U_{PMNS}

Mixing matrix

σ_1

Singular values

$$\begin{pmatrix} U_{PMNS} & V_{lh} \\ V_{hl} & V_{hh} \end{pmatrix}$$

Unitary dilation

$\|A\| \leq 1$

Contractions

Neutrino mixing in the Standard Model

$$\nu_\alpha^{(f)} = (U_{PMNS})_{\alpha i} \nu_i^{(m)}$$

Mixing matrix

$$U_{PMNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Experimental values of mixing parameters

$$\theta_{12} \in [31.38^\circ, 35.99^\circ], \quad \theta_{23} \in [38.4^\circ, 53.0^\circ],$$
$$\theta_{13} \in [7.99^\circ, 8.91^\circ], \quad \delta \in [0, 2\pi]$$

Extended mixing - BSM models

Complete mixing

$$\begin{pmatrix} \nu^{(f)} \\ \hat{\nu}^{(f)} \end{pmatrix} = \begin{pmatrix} U_{PMNS} & V_{lh} \\ V_{hl} & V_{hh} \end{pmatrix} \begin{pmatrix} \nu^{(m)} \\ \hat{\nu}^{(m)} \end{pmatrix} \equiv U \begin{pmatrix} \nu^{(m)} \\ \hat{\nu}^{(m)} \end{pmatrix}$$

Observable part

$$\nu_\alpha^{(f)} = \underbrace{(U_{PMNS})_{\alpha i} \nu_i^{(m)}}_{\text{SM part}} + \underbrace{(V_{lh})_{\alpha j} \hat{\nu}_j^{(m)}}_{\text{BSM part}}$$

A standard approach to deviation from unitarity

$$\not{U}_{\mathtt{PMNS}} = (1 - \eta) V,$$
$$\not{U}_{\mathtt{PMNS}} = (1 - \alpha) W.$$

Singular values

Singular values σ_i of a given matrix A are positive square roots of the eigenvalues λ_i of the matrix $AA^\dagger$

$$\sigma_i(A) = \sqrt{\lambda_i(AA^\dagger)}$$

Properties:

- generalization of eigenvalues
- always positive
- stable under perturbations

Unitary matrices

$$UU^\dagger = I = diag(1, 1, ..., 1) \Longrightarrow \text{all singular values equal to 1}$$

Contractions

$$\|A\| \leq 1$$

Operator norm (spectral norm)

$$\|A\| := \sup_{\|x\|=1} \|Ax\| = \sigma_{\max}(A)$$

Contractions as submatrices of the unitary matrix

$$\left\| \begin{pmatrix} U_{PMNS} & V_{lh} \\ V_{hl} & V_{hh} \end{pmatrix} \right\| = 1 \Longrightarrow \|U_{PMNS}\| \leq 1$$

Unitary dilation

BSM?

$$U_{PMNS} \xrightarrow{\text{dilation}} \begin{pmatrix} U_{PMNS} & V_{lh} \\ V_{hl} & V_{hh} \end{pmatrix} \equiv U \rightarrow UU^{\dagger} = I$$

CS decomposition

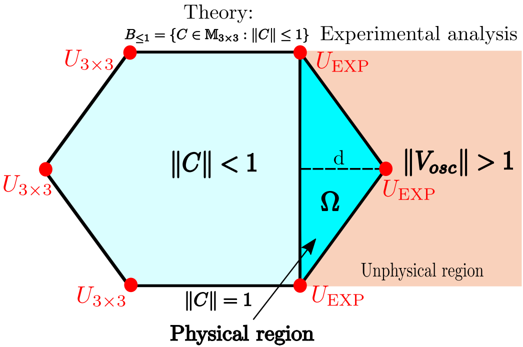
$$U \equiv \begin{pmatrix} U_{PMNS} & V_{lh} \\ V_{hl} & V_{hh} \end{pmatrix} = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} \left(\begin{array}{c|cc} C & -S & 0 \\ \hline S & C & 0 \\ 0 & 0 & I_{m-n} \end{array}\right) \begin{pmatrix} Q_1^{\dagger} & 0 \\ 0 & Q_2^{\dagger} \end{pmatrix}$$

where $C \geq 0$ and $S \geq 0$ are diagonal matrices satisfying $C^2 + S^2 = I_n$
$W_1, Q_1 \in M_{n\times n}$ and $W_2, Q_2 \in M_{m\times m}$ are unitary matrices.

Physical Region

$$\Omega := \mathrm{conv}(U_{PMNS}) = \{\sum_{i=1}^{m} \alpha_i U_i \mid U_i \in U(3), \alpha_1, ..., \alpha_m \geq 0, \sum_{i=1}^{m} \alpha_i = 1,$$

$$\theta_{12}, \theta_{13}, \theta_{23} \text{ and } \delta \text{ given by experimental values}\}$$



Subsets of Ω

Ω is divided into four disjoint subsets

$$V_1 : \sigma_1 = 1, \sigma_2 = 1, \sigma_3 < 1 \longrightarrow \mathbf{3 + 1}$$
$$V_2 : \sigma_1 = 1, \sigma_2 < 1, \sigma_3 < 1 \longrightarrow 3 + 2$$
$$V_3 : \sigma_1 < 1, \sigma_2 < 1, \sigma_3 < 1 \longrightarrow 3 + 3$$
$$U : \sigma_1 = 1, \sigma_2 = 1, \sigma_3 = 1$$

Can we restrict experimental bounds for each subset and distinguish among 3+n scenarios?

α-parametrization and prescribed singular values

$$\not{U}_{\mathtt{PMNS}} = (I - \alpha)W = TW.$$

where W is a unitary matrix and $T = I - \alpha$.

$$T = \begin{pmatrix} t_{11} & 0 & 0 \\ t_{21} & t_{22} & 0 \\ t_{31} & t_{32} & t_{33} \end{pmatrix}, \quad \Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$$

Entry	(I): $m >$ EW	(II): $\Delta m^2 \gtrsim 100\ \mathrm{eV}^2$	(III): $\Delta m^2 \sim 0.1 - 1\ \mathrm{eV}^2$
$T_{11} = 1 - \alpha_{11}$	$0.99870 \div 1$	$0.976 \div 1$	$0.990 \div 1$
$T_{22} = 1 - \alpha_{22}$	$0.99978 \div 1$	$0.978 \div 1$	$0.986 \div 1$
$T_{33} = 1 - \alpha_{33}$	$0.99720 \div 1$	$0.900 \div 1$	$0.900 \div 1$
$T_{21} = \lvert\alpha_{21}\rvert$	$0.0 \div 0.00068$	$0.0 \div 0.025$	$0.0 \div 0.017$
$T_{31} = \lvert\alpha_{31}\rvert$	$0.0 \div 0.00270$	$0.0 \div 0.069$	$0,0 \div 0.045$
$T_{32} = \lvert\alpha_{32}\rvert$	$0.0 \div 0.00120$	$0.0 \div 0.012$	$0.0 \div 0.053$

[M. Blennow et al., 2017]

It is possible to construct lower triangular matrices with prescribed eigenvalues and singular values [C-K. Li and R. Mathias, 2004].

Amount of space for n neutrinos: Analysis

- Construction of matrices with prescribed singular values, e.g., in 3+1 scenario we take $\sigma_1 = 1, \sigma_2 = 1, \sigma_3 < 1$, together with the requirement on the elements to stay within experimental limits.
- Go with the "free" singular values as low as possible, e.g., in the 3+1 scenario we take σ_3 the smallest possible.

Amount of space for n neutrinos: Results

		$3+1$	
			σ_3
$m >$ EW			0.9968
$\Delta m^2 \gtrsim 100$ eV^2			0.900
$\Delta m^2 \sim 0.1 - 1$ eV^2			0.889
		$3+2$	
		σ_2	σ_3
$m >$ EW		0.9987	0.9986
$\Delta m^2 \gtrsim 100$ eV^2		0.976	0.975
$\Delta m^2 \sim 0.1 - 1$ eV^2		0.986	0.985
		$3+3$	
	σ_1	σ_2	σ_3
$m >$ EW	0.9998	0.9996	0.9996
$\Delta m^2 \gtrsim 100$ eV^2	0.979	0.977	0.9773
$\Delta m^2 \sim 0.1 - 1$ eV^2	0.991	0.989	0.989

Error: 0.00003 (follows from Weyl's inequality, slides 21 and 22)

Distinction of the 3+1 scenario: Analysis

$$\sigma_1 = \sigma_2 = 1.$$

- In each massive scenario 10^8 matrices are produced, starting from σ_3 as large as possible and lowering it systematically to the smallest obtained value (previous slide).
- For each value of σ_3 the smallest and the largest values of produced matrix elements are taken.
- Repeating the procedure over possible σ_3 values, the allowed ranges of the 3×3 matrix elements are determined.

Distinction of the 3+1 scenario: Results

	$m >$ EW	$\Delta m^2 \gtrsim 100$ eV^2	$\Delta m^2 \sim 0.1 - 1$ eV^2
$(1,1)$	$0.99885 \div 0.99999$	$0.97641 \div 0.99996$	$0.99020 \div 0.99999$
Exp:	$0.99870 \div 1$	$0.976 \div 1$	$0.990 \div 1$
$(2,2)$	$0.99980 \div 0.99999$	$0.99331 \div 0.99999$	$0.98646 \div 0.99999$
Exp:	$0.99978 \div 1$	$0.978 \div 1$	$0.986 \div 1$
$(3,3)$	$0.99721 \div 0.99996$	$0.90040 \div 0.99985$	$0.90015 \div 0.99958$
Exp:	$0.99720 \div 1$	$0.900 \div 1$	$0.900 \div 1$
$(2,1)$	$0.00001 \div 0.00062$	$0.00031 \div 0.02214$	$0.00014 \div 0.01615$
Exp:	$0.0 \div 0.00068$	$0.0 \div 0.025$	$0.0 \div 0.017$
$(3,1)$	$0.00002 \div 0.00266$	$0.00048 \div 0.06892$	$0.00012 \div 0.04500$
Exp:	$0.0 \div 0.00270$	$0.0 \div 0.069$	$0.0 \div 0.045$
$(3,2)$	$0.00008 \div 0.00113$	$0.00052 - 0.01196$	$0.00024 \div 0.05281$
Exp:	$0.0 \div 0.00120$	$0.0 \div 0.012$	$0.0 \div 0.053$

Similar results for 3+2 and 3+3.

So far no distinction among 3+n scenarios is possible. However,...

(I) Narrowing mixing spreads for individual sing. val.

- Generation of matrices with a prescribed set of singular values and with elements within experimental ranges.
- From the set of these matrices take the smallest and the largest value of each element.

E.g.: $\Delta m^2 \gtrsim 100 \text{ eV}^2, \Sigma = \{1, 1, 0.900\}$:

$$|A_{0.900}| = \begin{pmatrix} 0.999623 \div 0.999999 \ (1.5\%) & 0 & 0 \\ 0.000002 \div 0.000753 \ (3\%) & 0.999623 \div 0.999999 \ (2\%) & 0 \\ 0.000606 \div 0.011919 \ (16\%) & 0.000606 \div 0.011923 \ (94\%) & 0.900002 \div 0.900678 \ (1\%) \end{pmatrix}$$

Values in the brackets represent the percentage of the current experimental bounds.
For the other massive cases these values do not exceed 15%.

(II) Estimation of the "light-heavy" mixing

Estimation of the "light-heavy" mixing via CS decomposition (backup slides 22-23)

- $m > \text{EW}$:

$$|U_{e4}| \leq \mathbf{0.021}, \quad |U_{\mu 4}| \leq \mathbf{0.021}, \quad |U_{\tau 4}| \leq 0.075,$$
$$Others: |U_{e4}| \leq 0.055, \quad |U_{\mu 4}| \leq 0.057, \quad |U_{\tau 4}| \leq 0.079$$

e.g.[*arXiv* : 0803.4008]

- $\Delta m^2 \gtrsim 100 \text{ eV}^2$:

$$|U_{e4}| \leq 0.082, \quad |U_{\mu 4}| \leq 0.099, \quad |U_{\tau 4}| \leq 0.436.$$

- $\Delta m^2 \sim 0.1 - 1 \text{ eV}^2$:

$$|U_{e4}| \leq 0.130, \quad |U_{\mu 4}| \leq 0.167, \quad |U_{\tau 4}| \leq \mathbf{0.436}.$$
$$Others: |U_{e4}| \leq 0.14, \quad |U_{\mu 4}| \leq 0.12, \quad \Delta m^2_{\text{SBL}} = 1.7\ eV^2$$

[PDG 2019]

Summary and Outlook

- Construction of matrices with the prescribed set of singular values gives an opportunity to study important properties of the mixing matrices.
- There is still space for additional neutrinos.
- Singular values allow in principle to distinguish between the number of additional neutrinos.
- Present experimental bounds do not distinguish the number of additional neutrinos, better precision is needed in order to restrict BSM ranges like in α parametrization.
- New estimations of upper bounds for the 3+1 mixings using CS decomposition are given.

Backup slides

Matrix norm

A matrix norm is a function $\|\cdot\|$ from the set of all complex (real matrices) into $\mathbb{R}$ that satisfies the following properties

$$\begin{aligned} &\|A\| \geq 0 \text{ and } \|\mathrm{A}\| = 0 \iff \mathrm{A} = 0, \\ &\|\alpha A\| = |\alpha|\|A\|, \alpha \in C, \\ &\|A + B\| \leq \|A\| + \|B\|, \\ &\|AB\| \leq \|A\|\|B\| \end{aligned}$$

Examples of matrix norms

- spectral norm: $\|A\| = \max_{\|x\|_2=1} \|Ax\|_2 = \sigma_1(A)$
- Frobenius norm: $\|A\|_F = \sqrt{Tr(A^\dagger A)} = \sqrt{\sum_{i,j=1}^n |a_{ij}|^2} = \sqrt{\sum_{i=1}^n \sigma_i^2}$
- maximum absolute column sum norm: $\|A\|_1 = \max_{\|x\|_1=1} \|Ax\|_\infty = \max_j \sum_i |a_{ij}|$
- maximum absolute row sum norm: $\|A\|_\infty = \max_{\|x\|_\infty=1} \|Ax\|_\infty = \max_i \sum_j |a_{ij}|$

Weyl's inequality for singular values

Let A and B be a $m \times n$ matrices and let $q = \min\{m, n\}$. Then

$$\sigma_j(A + B) \leq \sigma_i(A) + \sigma_{j-i+1}(B) \text{ for } i \leq j$$

Error Estimation

Let us assume that the V matrix which realizes some *BSM* scenario include an error matrix E which is of the form $V + E$. Using Weyl inequalities for decreasingly ordered pairs of singular values of V and $V + E$, the following relation takes place

$$|\sigma_i(V + E) - \sigma_i(V)| \leq ||E||.$$

A precision for elements of the A in the $m > \ \mathrm{EW}$ is 10^{-5}. In our analysis we keep the same precision for all massive cases. This do not contradict experimental results since we still work within experimentally established intervals. Thus, all entries of Error matrix can be taken as $E_{ij} \approx 0.00001$. Therefore, uncertainty of the calculated singular values is bounded by $||E|| = 0.00003$.

Algorithm

The following steps lead to a contraction settled by U_{PMNS} and then to its unitary dilation of a minimal dimension

1) Select a finite number of unitary matrices U_i, $i = 1, 2, ...m$, within experimentally allowed range of parameters θ_{13}, θ_{23} and δ.

2) Construct a contraction U_{11} as a convex combination of selected matrices U_i

$$V = \sum_{i=1}^{m} \alpha_i U_i, \quad \alpha_1, ..., \alpha_m \geq 0, \quad \sum_{i=1}^{m} \alpha_i = 1.$$

3) Find singular value decomposition of V, i.e.

$$V = W_1 \Sigma Q_1^\dagger$$

where W_1, Q_1 are unitary, Σ is diagonal, and determine number η of singular values strictly less than 1.

4) Use CS decomposition

$$U = \begin{pmatrix} V & V_{lh} \\ V_{hl} & V_{hh} \end{pmatrix} =$$

$$\begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} \left(\begin{array}{cc|c} I_r & 0 & 0 \\ 0 & C & -S \\ \hline 0 & S & C \end{array}\right) \begin{pmatrix} Q_1^\dagger & 0 \\ 0 & Q_2^\dagger \end{pmatrix}$$

to find the unitary dilation $U \in \mathbb{M}_{(3+\eta)\times(3+\eta)}$ of contraction U_{11}.

3+1 via CS decomposition

Thus we work with the following set of singular values $\sigma_1 = 1, \sigma_2 = 1, \sigma_3 < 1$ and the CS decomposition takes the form

$$\begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & -s \\ \hline 0 & 0 & s & c \end{array}\right) \begin{pmatrix} Q_1^\dagger & 0 \\ 0 & Q_2^\dagger \end{pmatrix}. \tag{1}$$

We are interested in the estimation of the light-heavy mixing sector which is given by

$$U_{12} = W_1 V_{12} Q_2^\dagger, \tag{2}$$

where $W_1 \in \mathbb{C}^{3\times 3}$ is unitary, $V_{12} = (0, 0, -s)^T$ and $Q_2 = e^{i\theta}, \theta \in (0, 2\pi]$. Parametrizing the matrix W_1 as usual by Euler angles we get

$$U_{12} = -(w_{13}, w_{23}, w_{33})^T s e^{-i\theta} = -(-s_{12} e^{-i\theta_{13}}, s_{23} c_{13}, c_{23} c_{13})^T s e^{-i\theta} \tag{3}$$

3+1 via CS decomposition

We can see that if we want estimate just the absolute values for the elements of light-heavy sector we are left only with

$$|s| = |\sqrt{1-c^2}| = |\sqrt{1-\sigma_3^2}|. \tag{4}$$

Thus for each massive scenario we get

$$\begin{aligned} \text{"}m > \text{ EW"} &\equiv m_1 < |0.08359|, \\ \text{"}\Delta m^2 \gtrsim 100 \text{ eV}^2\text{"} &\equiv m_2 < |0.43795|, \\ \text{"}\Delta m^2 \sim 0.1 - 1 \text{ eV}^2\text{"} &\equiv m_3 < |0.43795|. \end{aligned} \tag{5}$$

Results on slide 15 has been obtained by taking exact maximal values of w_{13}, W_{23} and w_{33}, which follow from the singular value decomposition.