Optimizing Mellin-Barnes approach to numerical multiloop calculations

Ievgen Dubovyk

Based on collaboration with:

T.Riemann, J.Gluza and J.Usovitsch

Matter To The Deepest

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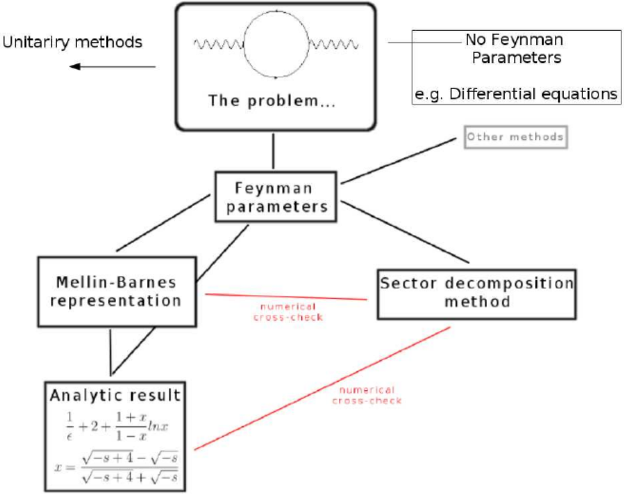
Outline

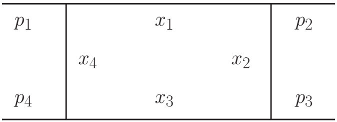
Introduction



Feynman parameters representation

$$G(X) = \frac{(-1)^{N_\nu} \Gamma\left(N_\nu - \frac{d}{2}L\right)}{\prod_{i=1}^{N} \Gamma(n_i)} \int \prod_{j=1}^{N} dx_j \; x_j^{n_j - 1} \delta(1 - \sum_{i=1}^{N} x_i) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

The functions U and F are called graph or Symanzik polynomials.



$$U = x_1 + x_2 + x_3 + x_4$$
$$F_0 = -s x_1 x_3 - t x_2 x_4$$
$$-p_1^2 x_1 x_4 - p_2^2 x_1 x_2 - p_3^2 x_2 x_3 - p_4^2 x_3 x_4$$

$$F = F_0 + U \sum_{i=1}^{N} x_i m_i^2$$

Construction of MB representation

"Om definita integraler", R. H. Mellin, Acta Soc. Sci. Fenn. 20(7), 1 (1895),
"The theory of the gamma function", E. W. Barnes Messenger Math. 29(2), 64 (1900).

General Mellin-Barnes relation:

$$\frac{1}{(A_1 + \ldots + A_n)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{(2\pi i)^{n-1}} \int_{-i\infty}^{i\infty} dz_1 \ldots dz_{n-1} \times \prod_{i=1}^{n-1} A_i^{z_i} \; A_n^{-\lambda - z_1 - \ldots - z_{n-1}} \prod_{i=1}^{n-1} \Gamma(-z_i) \; \Gamma(\lambda + z_1 + \ldots + z_{n-1})$$

Integration over Feynman parameters:

$$\int_0^1 \prod_{i=1}^{N} dx_i \; x_i^{n_i - 1} \; \delta(1 - x_1 - \ldots - x_N) = \frac{\Gamma(n_1) \ldots \Gamma(n_N)}{\Gamma\left(n_1 + \ldots + n_N\right)}$$

Barnes lemmas to improve dimensionality:

$$\int_{-i\infty}^{i\infty} dz\Gamma(a+z)\Gamma(b+z)\Gamma(c-z)\Gamma(d-z) = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}$$

$$\int_{-i\infty}^{i\infty} dz \frac{\Gamma(a+z)\Gamma(b+z)\Gamma(c+z)\Gamma(d-z)\Gamma(e-z)}{\Gamma(a+b+c+d+e+z)} = \frac{\Gamma(a+d)\Gamma(a+e)\Gamma(b+d)\Gamma(b+e)\Gamma(c+d)\Gamma(c+e)}{\Gamma(a+b+d+e)\Gamma(a+c+d+e)\Gamma(b+c+d+e)}.$$

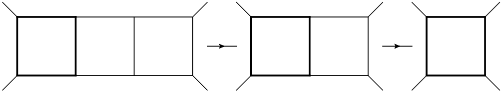
$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma\begin{bmatrix}(a)+s, (b)-s \\ (c)+s, (d)-s\end{bmatrix} \mathrm{d}s$$

$$= \sum_{\mu=1}^{A} \Gamma\begin{bmatrix}(a)-a_\mu, (b)+a_\mu \\ (c)-a_\mu, (d)+a_\mu\end{bmatrix} {}_{B+C}F_{A+D-1}\begin{bmatrix}(b)+a_\mu, 1+a_\mu-(c); \\ 1+a_\mu-(a)', (d)+a_\mu;\end{bmatrix} (-1)^{A+C}\Bigg]$$

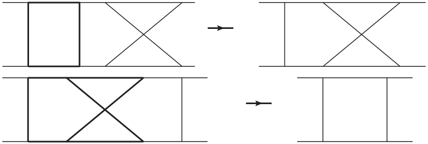
$$= \sum_{\nu=1}^{B} \Gamma\begin{bmatrix}(a)+b_\nu, (b)-b_\nu \\ (c)+b_\nu, (d)-b_\nu\end{bmatrix} {}_{A+D}F_{B+C-1}\begin{bmatrix}(a)+b_\nu, 1+b_\nu-(d); \\ (c)+b_\nu, 1+b_\nu-(b)';\end{bmatrix} (-1)^{B+D}\Bigg], \quad (4.5.1.2)$$

"Generalized Hypergeometric Functions", L.J.Slater, 1966.

- Feynman parameters + MB
  - iteratively to each subloop – loop-by-loop (LA) approach (`AMBREv1.3.1` & `AMBREv2.1.1`)

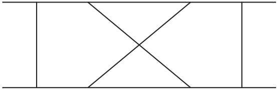


  - in one step to the complete U and F polynomials – global (GA) approach (`AMBREv3.1.1`)
  - combination of the above methods – Hybrid approach



Examples, description, links to basic tools and literature:
http://prac.us.edu.pl/~gluza/ambre/

Complexity of graph polynomials beyond 1-loop



```
U = x[1] x[2] x[4] + x[1] x[3] x[4] + x[2] x[3] x[4] + x[1] x[2] x[5] +
x[1] x[3] x[5] + x[2] x[3] x[5] + x[1] x[4] x[5] + x[2] x[4] x[5] +
x[2] x[4] x[6] + x[3] x[4] x[6] + x[2] x[5] x[6] + x[3] x[5] x[6] +
x[4] x[5] x[6] + x[2] x[4] x[7] + x[3] x[4] x[7] + x[2] x[5] x[7] +
x[3] x[5] x[7] + x[4] x[5] x[7] + x[1] x[2] x[8] + x[1] x[3] x[8] +
x[2] x[3] x[8] + x[1] x[4] x[8] + x[2] x[4] x[8] + x[2] x[6] x[8] +
x[3] x[6] x[8] + x[4] x[6] x[8] + x[2] x[7] x[8] + x[3] x[7] x[8] +
x[4] x[7] x[8] + x[1] x[2] x[9] + x[1] x[3] x[9] + x[2] x[3] x[9] +
x[2] x[4] x[9] + x[3] x[4] x[9] + x[1] x[5] x[9] + x[3] x[5] x[9] +
x[4] x[5] x[9] + x[2] x[6] x[9] + x[3] x[6] x[9] + x[5] x[6] x[9] +
x[2] x[7] x[9] + x[3] x[7] x[9] + x[5] x[7] x[9] + x[1] x[8] x[9] +
x[3] x[8] x[9] + x[4] x[8] x[9] + x[6] x[8] x[9] + x[7] x[8] x[9] +
x[1] x[2] x[10] + x[1] x[3] x[10] + x[2] x[3] x[10] +
x[1] x[4] x[10] + x[2] x[4] x[10] + x[2] x[6] x[10] +
x[3] x[6] x[10] + x[4] x[6] x[10] + x[2] x[7] x[10] +
x[3] x[7] x[10] + x[4] x[7] x[10] + x[1] x[9] x[10] +
x[3] x[9] x[10] + x[4] x[9] x[10] + x[6] x[9] x[10] + x[7] x[9] x[10]
```

Simplification of graph polynomials, factorization and common subexpression, Barnes lemmas

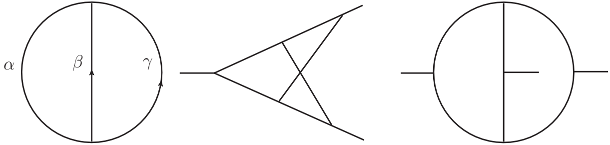
$x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4$	3-dim representation
$(x_1 + x_2)(x_3 + x_4)$	2-dim representation
$(x_1 + x_2)(x_3 + x_4) \to$ $[x_1 \to v_1\xi_{11}, x_2 \to v_1\xi_{12}, \delta(1 - \xi_{11} - \xi_{12});$ $x_3 \to v_2\xi_{21}, \ldots] \to v_1v_2$	0-dim representation
$(x_1 + x_2)(x_3 + x_4)$ + BL	0-dim representation $^{*)}$

$$*)\ (x_1 + x_2)^p \to \int dz_1 x_1^{z_1} x_2^{p-z_1} \Gamma(-z_1)\Gamma(-p + z_1)$$
$$\to \int dz_1 \Gamma(-z_1)\Gamma(-p + z_1)\Gamma(z_1 + 1)\Gamma(p - z_1 + 1)/\Gamma(p + 2)$$

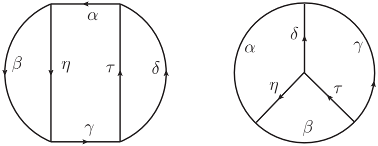
BL can be also applied without factorization, but this requers special transformation of z_i variables.

- random search (method of brackets, Prausa: '17)
- topology based factorization (AMBRE)
  chain diagrams, Kinoshita: '74

2-loop:



3-loop:



transformation of Feynman parameters:

$$\{\vec{x}\}_i: \;\; x_k \to v_i \xi_{ik} \;\; \times \delta\left(1 - \sum_{k=1}^{\eta_i} \xi_{ik}\right),$$

where i denotes chain index and $k \in [1, \eta_i]$, with η_i - number of propagators in chain. δ-function keeps number of variables unchanged.

$$U_{\text{2-loop}} = v_1 v_2 + v_2 v_3 + v_1 v_3$$

$$U_{\text{3-loop(I)}} = v_1 v_2 v_3 + v_1 v_2 v_4 + v_2 v_3 v_4 + v_1 v_2 v_5 + v_1 v_3 v_5 + v_2 v_3 v_5 + v_1 v_4 v_5 + v_3 v_4 v_5$$

$$U_{\text{3-loop(II)}} = v_1 v_2 v_3 + v_1 v_2 v_4 + v_1 v_3 v_4 + v_1 v_2 v_5 + v_1 v_3 v_5 + v_2 v_3 v_5 + v_2 v_4 v_5 + v_3 v_4 v_5$$
$$+ v_1 v_2 v_6 + v_2 v_3 v_6 + v_1 v_4 v_6 + v_2 v_4 v_6 + v_3 v_4 v_6 + v_1 v_5 v_6 + v_3 v_5 v_6 + v_4 v_5 v_6$$

Chang–Wu theorem:
delta function in the feyman parameters representation can be replaced by

$$\delta\left(\sum_{i\in\Omega} x_i - 1\right)$$

where Ω is an arbitrary subset of the lines $1,\ldots,L$, when the integration over the rest of the variables, i.e. for $i \notin \Omega$, is extended to the integration from zero to infinity.

To get minimal dimensionality we do not expand expression for F polynomial

$$F = F_0 + U\sum_{i=1}^{N} x_i m_i^2$$

Do we lose some physics here?

In general form MB representation consists of integrals of the form

$$\frac{1}{(2\pi i)^r} \int\limits_{-i\infty}^{+i\infty} \cdots \int\limits_{-i\infty}^{+i\infty} \prod_i^r dz_i \, \mathbf{F}(Z, S, \epsilon) \frac{\prod\limits_{j=1}^{N_n} \Gamma(\Lambda_j)}{\prod\limits_{k=1}^{N_d} \Gamma(\Lambda_k)}.$$

$\mathbf{F}$ depends on: Z – some subset of integration variables,
S – kinematic parameters and masses;

Λ_i : linear combinations of z_i and ϵ, e.g. $\Lambda_i = \sum_l \alpha_{il} z_l + \gamma_i + \delta_i \epsilon$.

$$\frac{\prod_{j=1}^{N_n} \Gamma(\Lambda_j)}{\prod_{k=1}^{N_d} \Gamma(\Lambda_k)} \Leftrightarrow M_\Gamma Z = \left[\begin{array}{c} \alpha_{ij}(\text{numerator}) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_{ij}(\text{denominator}) \end{array} \right] \begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} .$$

$$M_\Gamma Z = M_\Gamma U^{-1} U Z = M'_\Gamma Z'$$

$$M'_\Gamma = M_\Gamma U^{-1}$$

$$Z' = UZ$$

$$M'_\Gamma(\text{1st BL} \to z_1) = \begin{bmatrix} 1 & \cdots & \cdots \\ 1 & \cdots & \cdots \\ -1 & \cdots & \cdots \\ -1 & \cdots & \cdots \\ 0 & \cdots & \cdots \\ \vdots & \cdots & \cdots \\ 0 & \cdots & \cdots \end{bmatrix}$$

Numerical Integration: transition to Minkowskian region

In general form MB integral can be represented as follows:

$$I = \frac{1}{(2\pi i)^r} \int\limits_{-i\infty+z_{10}}^{+i\infty+z_{10}} \cdots \int\limits_{-i\infty+z_{r0}}^{+i\infty+z_{r0}} \prod_i^r dz_i \, \mathbf{F}(Z,S) \frac{\prod\limits_{j=1}^{N_n} \Gamma(\Lambda_j)}{\prod\limits_{k=1}^{N_d} \Gamma(\Lambda_k)} f_\psi(Z).$$

An example:

$$I^{0h0w}_{5,\epsilon^{-2}} = \frac{1}{2s}\frac{1}{2\pi i} \int\limits_{-i\infty-\frac{1}{2}}^{+i\infty-\frac{1}{2}} dz \left(\frac{M_Z^2}{-s}\right)^z \frac{\Gamma^3(-z)\Gamma(1+z)}{\Gamma^2(1-z)}.$$

Asymptotic behavior:

$$\Gamma(z)_{|z|\to\infty} = \sqrt{2\pi} e^{-z} z^{z-\frac{1}{2}} \left[1 + \frac{1}{12z} + \frac{1}{288z^2} + \ldots\right].$$

- core:

$$\frac{\Gamma^3(-z)\Gamma(1+z)}{\Gamma^2(1-z)} \xrightarrow{|z|\to\infty} e^{z(\ln z - \ln(-z)) + \frac{1}{2}\ln z - \frac{5}{2}\ln(-z)}.$$

$\ln z - \ln(-z) = i\pi\, \mathsf{sign}(\Im m\, z)$
$z = z_0 + i\,t,\;\; t \in (-\infty, \infty),\; |z| \to \infty \Leftrightarrow t \to \pm\infty$

$$\frac{\Gamma^3(-z)\Gamma(1+z)}{\Gamma^2(1-z)} \longrightarrow e^{-\pi|t|}\frac{1}{|t|^2}$$

- kinematics:
  in the Minkowskian case $s \to s + i\delta\,(s > 0)$

$$\left(\frac{M_Z^2}{-s}\right)^z = e^{z\ln(-\frac{M_Z^2}{s} + i\,\delta)} \longrightarrow e^{i\,t\ln\frac{M_Z^3}{s}}\, e^{-\pi t}, s > 0$$

$e^{-\pi|t|}$ and $e^{-\pi t}$ cancel each other when $t \to -\infty$ and oscillations are not damped any more by an exponential factor

Transformation to finite integration region $(-\infty, +\infty) \to [0,1]$:

- $\ln$-type (MB.m)

$$t_i \to \ln\left(\frac{x_i}{1-x_i}\right), \quad dt_i \to \frac{dx_i}{x_i(1-x_i)}.$$

limit $t \to -\infty$ is equivalent to $x \to 0$ and in this limit integrand behaves like

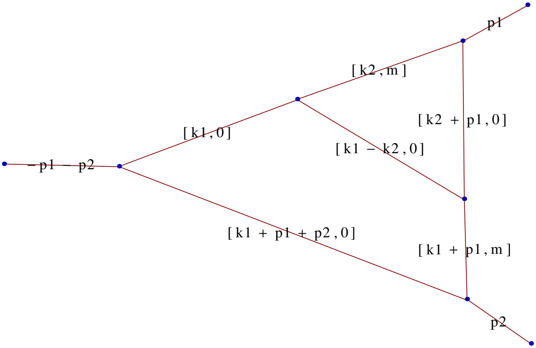
$$\frac{1}{x\ \ln^2 x} \xrightarrow{x\to 0} \infty$$

- tan-type

$$t_i \to \tan\left(\pi(x_i - \frac{1}{2})\right), \quad dt_i \to \frac{\pi dx_i}{\cos^2\left(\pi(x_i - \frac{1}{2})\right)}$$

$$\frac{1}{\sin^2\left(\pi(x_i - \frac{1}{2})\right)} \xrightarrow{x\to 0} 1$$

More complicated example:



$$I_{2,I}^{0h0w} = \frac{1}{s^2}\frac{1}{(2\pi i)^3}\int\limits_{-i\infty-\frac{47}{37}}^{i\infty-\frac{47}{37}} dz_1 \int\limits_{-i\infty-\frac{44}{211}}^{i\infty-\frac{44}{211}} dz_2 \int\limits_{-i\infty-\frac{176}{235}}^{i\infty-\frac{176}{235}} dz_3 \left(-\frac{s}{M_Z^2}\right)^{-z_1}\Gamma(-1-z_1)$$
$$\Gamma(2+z_1)\Gamma(-1-z_{12})\Gamma(-z_2)\Gamma^2(1+z_{12}-z_3)\Gamma(-z_3)\Gamma(1+z_3)$$
$$\Gamma^2(-z_1+z_3)\Gamma(-z_{12}+z_3)/\Gamma(-z_1)\Gamma(1-z_2)\Gamma(1-z_1+z_3)$$

$$t_1 = -t_2 = t,\ t_3 = 0$$

$$I_{2,II}^{0h0w} = \frac{1}{s^2}\frac{1}{(2\pi i)^3}\int\limits_{-i\infty-\frac{47}{37}}^{i\infty-\frac{47}{37}} dz_1 \int\limits_{-i\infty-\frac{139}{94}}^{i\infty-\frac{139}{94}} dz_2 \int\limits_{-i\infty-\frac{176}{235}}^{i\infty-\frac{176}{235}} dz_3 \left(-\frac{s}{M_Z^2}\right)^{-z_1}\Gamma(-1-z_1)$$
$$\Gamma(2+z_1)\Gamma(-1-z_2)\Gamma(z_1-z_2)\Gamma(1+z_2-z_3)^2\Gamma(-z_3)\Gamma(1+z_3)$$
$$\Gamma(-z_1+z_3)^2\Gamma(-z_2+z_3)/\Gamma(-z_1)\Gamma(1+z_1-z_2)\Gamma(1-z_1+z_3).$$

$$t_1 = t,\ t_2 = t_3 = 0$$

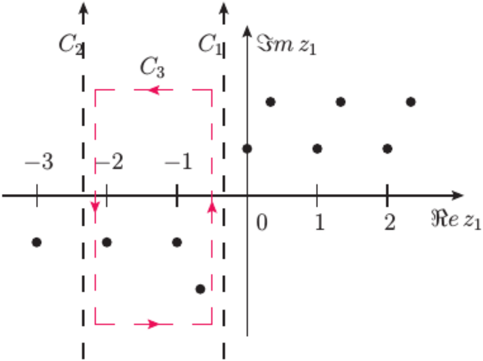
AB	$-1.\mathbf{199526183135} + 5.\mathbf{567365907880}i$	Cuhre, 10^7, 10^{-8}
I, $3\times\tan$	$-1.\mathbf{19952}5259137 + 5.\mathbf{567367}419371i$	Cuhre, 10^7, 10^{-8}
I, $\ln + 2\times\tan$	$-1.\mathbf{19952}4318757 + 5.\mathbf{567365}298565i$	Cuhre, 10^7, 10^{-8}
II, $3\times\tan$	$-1.\mathbf{199526}239547 + 5.\mathbf{567365}843910i$	Cuhre, 10^7, 10^{-8}
II, $2\times\ln+\tan$	$-1.\mathbf{199526183}168 + 5.\mathbf{567365907}904i$	Cuhre, 10^7, 10^{-8}
I, $3\times\ln$	NaN	Cuhre, 10^7
I, $3\times\ln$	$-1.\mathbf{20}4597845834 + 5.\mathbf{567}518701898i$	Vegas, 10^7, 10^{-3}
I, $3\times\tan$	$-1.\mathbf{19951}6455248 + 5.\mathbf{5673}76681167i$	QMC, 10^7, 10^{-5}
I, $3\times\tan$	$-1.\mathbf{19952}7580305 + 5.\mathbf{567367}345229i$	QMC, 10^8, 10^{-6}

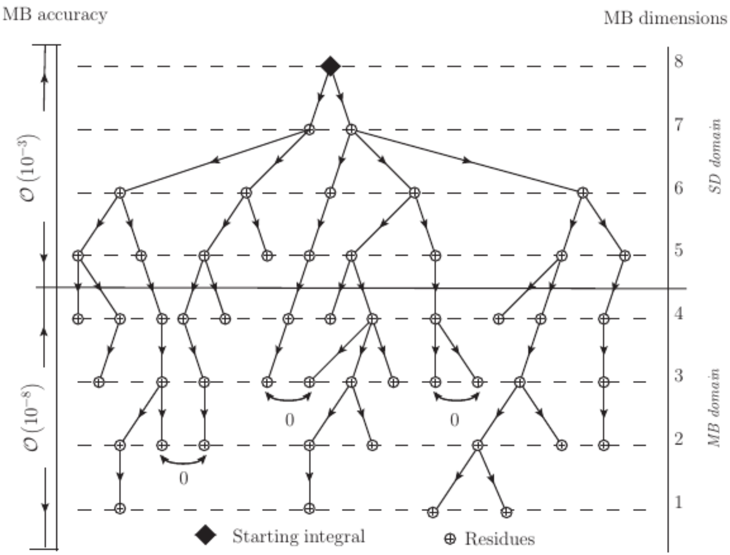
AB – analitical result, Aglietti, Bonciani: '04

QMC – Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk: '18

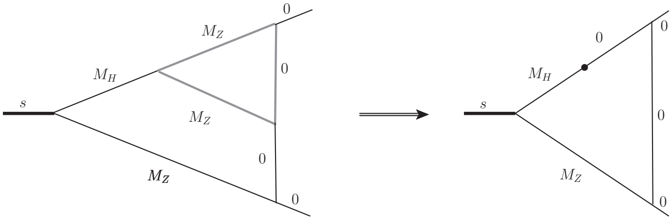
MBnumerics

$$I \to I(z_{01} + n_1, \ldots, z_{0r} + n_r) = I(\vec{n})$$




MB accuracy
MB dimensions
$\mathcal{O}(10^{-3})$
$\mathcal{O}(10^{-8})$
8
7
6
5
4
3
2
1
SD domain
MB domain
0
0
0
Starting integral
Residues

Numerical Integration and Threshold Behaviour

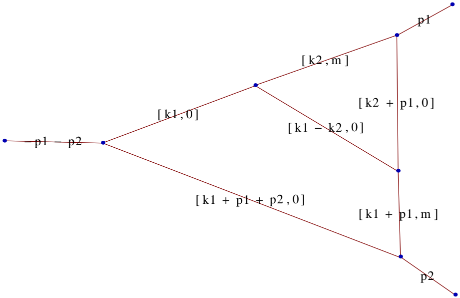


$$F^{(1)}(\vec{x}) = M_Z^2(x_1+x_2) - k_1^2 x_1 x_2 - (k_1+p_1)^2 x_1 x_3$$

$$F_{(2)}(\vec{x}) = M_H^2 x_2 + M_Z^2 x_4 - s(x_1+x_2)x_4$$

$$F_{(2)}(\vec{x}) = M_H^2 x_2 + (M_Z^2 - s)(x_1+x_2)x_4 + M_Z^2(x_3+x_4)x_4$$

$$F_{(2)}(\vec{x}) = M_H^2 x_2(x_1+x_2+x_3) + (M_Z^2 - s)x_1 x_4 + (M_Z^2 + M_H^2 - s)x_2 x_4 + M_Z^2 x_3 x_4 + M_Z^2 x_4^2.$$



$$U = x_1x_2 + x_1x_3 + x_2x_3 + x_2x_4 + x_3x_4 + x_1x_5 + x_2x_5 + x_4x_5 + x_2x_6 + x_3x_6 + x_5x_6$$

$$F = U(M_Z^2x_3 + M_Z^2x_4) - sx_1x_2x_6 - sx_1x_3x_6 - sx_2x_3x_6 - sx_1x_5x_6$$

Conclusions

- For certain classes of Feynman integrals, MB method gives very compact and well integrable representations, but in general, the method is not universal and dimensionality of MB representations strongly depends on topology, number of legs and loops, internal and external masses.
- New QMC integration library opens much more possibilities for application of MB method.
- Application of Barnes lemmas makes from GA universal method of construction of MB representations.
- Proper handling of thresholds allows obtaining "always Euclidean" representation but in cost of higher dimensional representation.
- A concept of minimal possible dimensionality is not always leading to well integrable MB representations.