(Super Light Bose-Einstein Dark Matter)



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In collaboration with Eric Braaten and Abhishek Mahapatra Based on the recent review: Braaten, HZ, arXiv: 1810.11473

Matter To The Deepest, Katowice, Sept. 6th, 2019

Outline

\diamond Axion

- Properties
- Different from other DM

♦ Coherent Bound Configurations

- (Dilute) Axion Star
- Dense Axion Star
- ♦ Observations

Strong CP Problem

• Strong CP-violating term $\mathcal{L}_{\theta} = \theta \frac{\alpha_s}{8\pi} G^a_{\mu\nu} \widetilde{G}^{a\mu\nu}$

Neutron electric dipole moment measurement: $\theta \lesssim 10^{-10}$



Fig. 1. Mechanism responsible for the $O(m_{\pi}^2 \ln m_{\pi}^2)$ contribution to the electric dipole moment of the neutron. The dark blob indicates a *CP* violating π NN interaction induced by the parameter θ .

Figure from Crewther et.al. (1979)

Surprisingly small because

- High-energy physics breaks CP
- $\circ~$ "Anthropic boundary" : $~\theta \lesssim 10^{-3}$

Peccei-Quinn U(1) & Axions

Peccei-Quinn U(1) symmetry solves strong CP problem

Peccei & Quinn (1977)

Introduces a Goldstone boson -- Axion ۲

Weinberg (1978), Wilczek (1978)

Unnaturally small parameter

Dynamical field: Axion.

The potential is **tilted** by quark condensate The axion field **slides** down to $\phi = 0$ **Restore** the CP symmetry



Relativistic Axions

Real pseudoscalar field Energy scale below 1GeV

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \mathcal{V}(\phi)$$

Two models for potential

Attractive force when ϕ is small

• Instanton
$$\mathcal{V}(\phi) = m_a^2 f_a^2 \left[1 - \cos(\phi/f_a)\right] = \frac{1}{2} m_a^2 \phi^2 - \frac{1}{4! f_a^2} \phi^4 + \cdots$$

 m_a : axion mass

$$f_a$$
 : axion decay constant

• Chiral
$$\mathcal{V}(\phi) = m_{\pi}^2 f_{\pi}^2 \left(1 - \left[1 - \frac{4z}{(1+z)^2} \sin^2(\phi/2f_a) \right]^{1/2} \right)$$

 $z = m_u/m_d \approx 0.48$

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Relativistic Axion Potential

Periodic potentials $\mathcal{V}(\phi) = \mathcal{V}(\phi + 2\pi f_a)$



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Parameters & Current Constraints

- Two parameters in relativistic axion Lagrangian: $m_a \ {
 m and} \ f_a$
- Not independent, related for QCD axion

$$m_a^2 f_a^2 = \frac{z}{(1+z)^2} m_\pi^2 f_\pi^2 \longrightarrow m_a f_a = (80 \text{ MeV})^2$$

 $z = m_u/m_d \approx 0.48$

Constraints from astrophysics & cosmology

 $\begin{array}{ll} 10^8 \ {\rm GeV} < f_a < 10^{13} \ {\rm GeV} & \Longrightarrow & 10^{-6} \ {\rm eV} < m_a < 10^{-2} \ {\rm eV} \\ \\ \mbox{Very weak self-interaction} & & & \mbox{Tiny Mass} \ \| \end{array}$

In this talk, I choose $m_a = 10^{-4} \ {\rm eV}$.

(See Braaten, HZ (2018) for formulas for ALPs)

Self-interactions

• Diagrams with loops can be safely ignored.

Each loop is suppressed by $(m_a/f_a)^2 \sim 10^{-48}$ Classical Field Theory!

• Exist vertices with 2n axions $(n \ge 2)$

Instanton model: $\mathcal{V}(\phi) = m_a^2 f_a^2 \left[1 - \cos(\phi/f_a)\right]$



Axion-Photon Coupling

Very weak coupling

$$\mathcal{L}_{\rm em} = \frac{c_{\rm em}\alpha}{16\pi f_a} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}\phi$$

$$c_{\rm em} \sim 1 \quad \text{Model dependent}$$
Suppressed by $f_a \sim 10^{11} \text{GeV}$



• Decay rate into two photons

$$\Gamma_a = \frac{c_{\rm em}^2 \alpha^2 m_a^3}{256\pi^3 f_a^2}.$$
 Axion lifetime ~ 10³⁶ years
Age of Universe ~ 10¹⁰ years

Photon energy: $m_a/2 \sim 10 \text{ GHz}$ Radio frequency

Axion Cosmology

 Cold dark matter axions are produced abundantly at QCD phase transition scale T ~ 1 GeV

Non-thermal axion production mechanism For more details, see Lect. Notes Phys. 741 (2008)

Mostly non-relativistic

 Vacuum misalignment
 Coherent
 Cosmic string decay
 Incoherent
 Coherent
 Preskill, Wise & Wilczek (1983) Abbot & Sikivie (1983) Dine & Fischler (1983)
 Davis (1986) Hararie & Sikivie (1987)

Axion Dark Matter

spin-0 non-relativistic boson

with extremely small mass $6 \times 10^{-6} \text{ eV} \lesssim m_a \lesssim 2 \times 10^{-3} \text{ eV}$ and extremely small self-coupling and coupling to SM particles (suppressed by $3 \times 10^9 \text{ GeV} \lesssim f_a \lesssim 10^{12} \text{ GeV}$)

and lifetime much longer than the age of our universe

Good candidate for dark matter!

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Different from other cold dark matter.

Take $m_a = 10^{-4} \text{ eV}$, de Broglie wave length ~ 2 mm

Use local density 0.4 GeV/ cm^3 , $n \times \lambda_{dB}^3 \sim 3 \times 10^{10}$

Huge occupation number!

In coherence the axions are in BEC!

Gravitational Thermalization

- Axion self-interaction may be too weak to thermalize axions
- Gravitational interaction can thermalize axions

Sikivie & Yang PRL (2009)

- Bring initially incoherent axions into coherence
- Keep the axion field as a Bose-Einstein Condensate as the Universe evolves
- Correlation length

Galactic scale? Sikivie & Yang PRL (2009)

Stellar scale due to attractive self-interaction?

Guth, Hertzberg & Prescod-Weinstein PRD (2015)

Is there a (meta)stable axion star solution?

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Time-dependent Solution

Einstein-Klein-Gordon action

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2} (\partial^{\mu} \phi) (\partial_{\mu} \phi) - V(\phi) \right)$$

 $\phi(t,r)$: axion field $\Phi(t,r)$: grav. potential

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Isotropic solution with weak gravity

$$ds^{2} = [1 + 2\Phi(r)]dt^{2} - [1 - 2\Phi(r)]dr^{2} - r^{2}d\Omega^{2}$$

Derrick's theorem:

no time-independent soliton solution with dimension > 2

Must solve the time-dependent equations

$$\frac{\partial^2 \phi}{\partial t^2} = (1+4\Phi) \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r} \frac{\partial \phi}{\partial r} \right) + 4 \frac{\partial \Phi}{\partial t} \frac{\partial \phi}{\partial t} - (1+2\Phi) \frac{dV(\phi)}{d\phi}$$
$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r} \frac{\partial \Phi}{\partial r} = 4\pi G\rho$$
$$\rho = \rho_{\rm kin} + \rho_{\rm grad} + \rho_{\rm pot} = \frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial r} \right)^2 + V(\phi)$$

Example: 1-d S-G Oscillon (breather)

EOM (1-d, no gravity)

Ablowitz et.al., PRL (1973)

$$\frac{\partial^2}{\partial t^2}\phi(t,x) - \frac{\partial^2}{\partial x^2}\phi(t,x) + m_a^2 f_a \sin\frac{\phi(t,x)}{f_a} = 0$$

Analytic solution ($0 < \omega < 1$ is the frequency)

$$\phi(t,x) = 4f_a \arctan\left[\sqrt{\omega^{-2} - 1}\operatorname{sech}(\sqrt{1 - \omega^2} m_a x)\cos(m_a \omega t)\right]$$

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Features:

- Periodic
- Shape changes slightly
 - --Dominated by ω
 - --Small components with 3ω , 5ω ...
- Exponentially small at infinity (no radiation)
- Stable against perturbation

Non-relativistic EFT (Part I)

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \mathcal{V}(\phi)$$

Real Scalar

Chavanis, PRD (2011), Chavanis, Delfini, PRD (2011) Braaten, Mahapatra, HZ, PRD (2016), PRD(2018) Namjoo, Guth, Kaiser, PRD (2018)

Complex scalar

Naïve non-relativistic reduction

$$\phi(\mathbf{r},t) = \frac{1}{\sqrt{2m_a}} \left[\psi(\mathbf{r},t) e^{-im_a t} + \psi^*(\mathbf{r},t) e^{+im_a t} \right]$$

Ignore all terms with rapid oscillating phase

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Ignore all terms with rapid oscillating phase

$$\begin{split} \mathcal{L}_{\text{eff}} &= \frac{1}{2}i\left(\psi^*\dot{\psi} - \dot{\psi}^*\psi\right) - \frac{1}{2m_a}\nabla\psi^*\cdot\nabla\psi - \mathcal{V}_{\text{eff}} \\ \mathcal{V}_{\text{eff}} &= m_a\psi^*\psi - \frac{1}{16}\frac{(\psi^*\psi)^2}{f_a^2} + \frac{1}{288}\frac{(\psi^*\psi)^3}{m_af_a^4} + \cdots \\ \uparrow & \text{16}\frac{(\psi^*\psi)^2}{m_af_a^4} + \frac{1}{288}\frac{(\psi^*\psi)^3}{m_af_a^4} + \cdots \\ \text{Attractive interaction!} & \text{Expand by } \frac{\psi^*\psi}{m_af_a^2} \\ \end{split}$$

Dilute Axion Stars

Assume: • Truncated potential, dilute axion limit

Newtonian gravity
 Spherically symmetric



M vs R

- Heavier dilute axion stars have smaller radii.
- Critical mass: beyond which the kinetic pressure cannot balance the attractive self-interaction and gravity



Formation of Dilute Axion Stars

- Dilute axion stars can be produced in early universe.
- Vacuum misalignment mechanism produces coherent and non-relativistic axions.
- Spatial fluctuations in the axion field evolve into gravitationally bound "miniclusters" of axions.
- Axion miniclusters first form light and large dilute axion stars.
- Dilute axion stars attract more axions and gradually reaches
 the critical mass (attractor).
 Levkov et.al. PRL (2017)
- Dilute axion stars over the critical mass collapse. In this process, the axions are not dilute at certain time.

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Non-relativistic EFT (Part II)

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}i\left(\psi^*\dot{\psi} - \dot{\psi}^*\psi\right) - \frac{1}{2m_a}\nabla\psi^*\cdot\nabla\psi - \mathcal{V}_{\text{eff}}$$

• Dilute axion field

$$\mathcal{V}_{\text{eff}} = m_a \psi^* \psi - \frac{1}{16} \frac{(\psi^* \psi)^2}{f_a^2} + \frac{1}{288} \frac{(\psi^* \psi)^3}{m_a f_a^4} + \dots \quad \begin{array}{l} \text{Dilute} \\ \text{limit} \end{array}$$

- In dense axion field $(\psi^*\psi)\sim m_a f_a^2$, must keep all orders

Both instanton and chiral potential can be summed to all orders

Instanton potential:

$$\mathcal{V}_{\text{eff}}(\psi^*\psi) = \frac{1}{2}m_a\psi^*\psi + m_a^2 f_a^2 \left[1 - J_0(2\psi^*\psi/m_a f_a^2)\right]$$

Eby, Suranyi, Vaz & Wijewardhana (2015) Braaten, Mahapatra, HZ, PRD (2016), PRD(2018)

Dense Branch

With untruncated potential, a new dense branch is found.

Assume: • NREFT • Newtonian gravity • Isotropic





Quantum pressure balances (gravity + ϕ^4 interaction), Attractive ϕ^4 interaction causes the turning over.



Higher orders in the potential become important. Quantum pressure balances full potential. Gravity can be ignored! Same results are obtained without gravity.



Gravity is important at large mass. Newtonian gravity is not accurate anymore.



Oscillons

• Real scalar field with 3-d isotropic double-well potential



Inside: false vacuum

Bogolubsky & Makhankov (1976)

Outside: true vacuum

• Time evolution?

false

vacuum

V

≻

Time Evolution of Oscillons

Three stages found in some numerical calculation

1. relaxation

Bogolubsky & Makhankov (1976)

100

r

150

200

50

From a given initial profile, radiate a large fraction of energy into outgoing waves

2. oscillon!

Localized oscillating configuration stabe for many oscillations, slowly radiates outgoing waves.

$$\phi(r,t) = \sum_{n=1}^{\infty} \phi_{2n+1}(r) \cos[(2n+1)\omega t] \phi_1$$

$$(\omega \approx m_a)$$

0

3. Sudden collapse

Configuration suddenly become unstable, disappear into outgoing waves.

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Time Evolution of Oscillons

Three stages found in some numerical calculation



disappear into outgoing waves.

Decay of Dense Axion Star

- It is pointed that the decay rate of dense axion star is too large Lifetime t ~ $10^3/m_a$ For $m_a \sim 10^{-4}$ eV, $t \sim 10^{-9}$ s
- The authors truncated the harmonic expansion
 (Gravity can be ignored for dense axion star)
 Visinelli et.al., PLB (2018)

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r} \frac{\partial \phi}{\partial r} - \frac{dV(\phi)}{d\phi}$$
$$\phi(r,t) = \sum_{n=1}^{\infty} \phi_{2n+1}(r) \cos[(2n+1)\omega t] = \phi_1(r) \cos(\omega t) + \phi_3(r) \cos(3\omega t)$$

- Inconsistent: the same method also predicts the analytic stable breather solution has a short lifetime.
- The decay rate of dense axion star is still unclear. (See Braaten, HZ (2018) for previous efforts.)

Formation of Dense Axion Star

- Naively, the dense axion stars are most likely produced as remnants in the collapse of dilute axion stars over critical mass.
- Implied in the simulations. Levkov et.al. PRL (2017)



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Axion Detection

Depends on the tiny axion-photon coupling

$$\mathcal{L}_{em} = \frac{c_{em}\alpha}{16\pi f_a} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}\phi.$$

$$c_{em} \sim 1 \text{ Model dependent}$$
Suppressed by $f_a \sim 10^{11} \text{GeV}$

$$\Gamma_a = \frac{c_{em}^2 \alpha^2 m_a^3}{256\pi^3 f_a^2}.$$
Axion lifetime $\sim 10^{36}$ years
Age of Universe $\sim 10^{10}$ years

Many detection techniques (skip)

Lect. Notes Phys. 741 (2008), A recent review: Kim & Carosi (2010)

Odd-Integer Harmonics



Odd-integer harmonics of the fundamental radio frequency.



Braaten, Mohapatra, HZ, arXiv:1609.05182

Two Types of Sources

- Continuous photon emission
 - Stable axion stars
- Catastrophic phenomenon:
 - a lot of energy released in a short time
 - Collapse of dilute Axion stars
 - Collision of an axion star with a neutron star
 - ≻ ...

Photon Flux Estimate

- Single axion decay to two photons is independent of the configuration.
- Solar system

radius ~ 125,000 AU, DM density ~ 0.3GeV/cm^3 Total DM in Solar system ~ $0.01 M_{\odot}$ 10⁶⁸ axions Photon production rate ~ $10^{22}/\text{sec}$ Energy released: ~ $10^9 \text{GeV/sec} \sim 0.4 \text{ Watt}$

• Milky way:

Total DM: $\sim 10^{12} M_{\odot}$ 10⁸² axions dense axion star close to Schwatzschild radius has ~10⁷⁰ axions $\sim 10^{11} {\rm GeV/sec} \sim 40 {\rm Watt}$

Catastrophic Phenomena

Fast Radio Burst (FRB)

- A ultra-fast (milli-sec) burst of photons in radio frequency.
- Nothing similar observed in optical, X rays and Y rays
- Since 2001, 69 events have been reported.
- Estimated rate ~ 10⁴ sky ⁻¹ day ⁻¹
- Reported frequencies: 0.8, 1.4, 7 GHz (telescope design)
- Extra-galactic sources from dispersion measure
- Energy released up to 10^{40} erg ~ 10^{-14} M_{\odot} (If isotropic)
- Strong linear polarization observed (3 events).

Online database: http://www.astronomy.swin.edu.au/pulsar/frbcat

Are Axion Stars an Explanation?

✓ Observed frequency: 0.8, 1.4, 7 GHz

 $10^{-6} \mathrm{eV} < m_a < 10^{-2} \mathrm{eV}$ 0.2 GHz $< \nu < 2400$ GHz Also explains why such burst is not observed in other bands.

- ✓ Total energy released: up to $\sim 10^{-14} M_{\odot}$
 - Dilute axion star critical mass $6 \times 10^{-14} M_{\odot}$
 - Dense axion star mass $10^{-20} M_{\odot}$ to $2 M_{\odot}$
- ✓ <u>Time duration:</u> ~ 1 ms
 - Dilute axion star critical radius: 200 km
 - Dense axion star radius: 10 m to 10 km
- ✓ Polarized photons

Axions in axion stars are in coherence

Scenarios with Axion Stars

• Collision of a dilute axion star with a neutron star

Coherent electric dipole radiation

From electrons in atmosphere Iwazaki, hep-ph/9908468

From neutrons in outer core Raby, PRD (2016)

Collapse of dilute axion stars above the critical mass

Tkachev, JETP Lett. (2014)

Collision of two axion stars

Eby et.al., arXiv:1701.01476

• Collision of a dense axion star with a neutron star

Observe Odd-integer Harmonics

- One unique feature of axion stars: odd-integer harmonics of the fundamental radio frequency.
- Can we observe the fast radio burst at other frequencies?
 0.8 GHz, 3 × 0.8 GHz, 5 × 0.8 GHz ...
 or

 $1/3 \times 0.8 \text{ GHz}, \quad 0.8 \text{ GHz}, \quad 5/3 \times 0.8 \text{ GHz} \dots$ Many possible combinations

• Need more events in more frequency windows.



- Axion is a very promising dark matter candidate which also solve strong CP problem.
- A large proportion of axions can exist in form of axion stars (dilute or dense).
- The formation and decay of axion stars are still unclear.
- Catastrophic phenomena of axion stars may explain fast radio burst.
- The photons in odd-integer harmonics of a fundamental radio frequency are a unique signature of axions.

Thank you!

Sorry I have to disappear shortly after the session. For more discussions, please email hong.zhang@tum.de



Backup Slides

"Stability" of Dense Branch

Close to the turning point, $E_b > 0$, unstable to large fluctuations. At large M, $E_b < 0$, "stable" to both large and small fluctuations.



Braaten, Mahapatra, HZ (2016), Braaten, HZ (2018) ⁴⁶

"Stability" of Dense Branch

Close to the turning point, $E_b > 0$, unstable to large fluctuations. At large M, $E_b < 0$, "stable" to both large and small fluctuations.



(First) Critical Point

The critical point is because of the attractive interaction.



Simulation of Collapse

Simulation: no stable dense core is produced after the collapse.



Existence of Dense Axion Star

- The collapse of an isolated dilute axion star may not leave a stable dense core behind.
- However, similar dense profiles may exist if there is other object providing additional
 - o gravitational potential
 - o axion source
 - o both (e.g. blackhole superradiance)



Yoshino and H. Kodama, Prog. Theor. Phys. (2012)

Fast Radio Burst



Figure from Nature 530, 453 (2016)

Non-relativistic EFT

► Canonical transformation: $(\phi, \dot{\phi}) \rightarrow (\psi, \psi^*)$ Namjoo, Guth & Kaiser (2018)

$$\phi(\boldsymbol{r},t) = \left[4(m_a^2 - \nabla^2)\right]^{-1/4} \left[\psi(\boldsymbol{r},t)e^{-im_a t} + \psi^*(\boldsymbol{r},t)e^{im_a t}\right]$$

Obtained EFT with real potential

$$\mathcal{L}_{\text{eff}} = \frac{i}{2} \left(\psi^* \dot{\psi} - \dot{\psi}^* \psi \right) - \frac{1}{2m_a} \nabla \psi^* \nabla \psi - \underline{V_{\text{eff}}(\psi^* \psi)} \quad \text{Real potential}$$

U(1) symmetry $\psi \rightarrow e^{i\alpha}\psi$. Particle number conservation?

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U(1) symmetry $\psi \rightarrow e^{i\alpha}\psi$. Particle number conservation?

Particle changing process is included by $Im(V_{eff})$ Added with Optical theorem
Braaten, Mahapatra &

Braaten, Mahapatra & HZ (2017) Braaten, Mahapatra & HZ (2018)



Match Low-power Couplings

• Expand the NR potential

$$\mathcal{V}_{\text{eff}}(\psi^*\psi) = m_a\psi^*\psi + m_a^2 f_a^2 \sum_{n=2}^{\infty} \frac{v_n}{(n!)^2} \left(\frac{\psi^*\psi}{2m_a f_a^2}\right)^n$$

• Check (v_2, v_3, v_4, v_5) for instanton potential

NR reduction: (-1, 1, -1, 1)With matching: (-1, -1.125, -2.25, 1.76)

Deviation: (0, -189%, -56%, -43%)

Contribution of virtual axions is important !

Braaten, Mohapatra, HZ, PRD (2016)