Power corrections with SCET

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2. SCET formalism beyond LP
   - N-jet operator
   - Soft loops and KSZ theorem

3. Applications
   - Threshold resummation for Drell-Yan
   - $gg \to H$

4. Summary

Power expansion

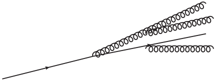
Consider expansion of a cross-section in some threshold variable z

$$\frac{d\sigma}{dz} = \sum_{n=0}^{\infty} \alpha_s^n \left[c_n \delta(1-z) + \sum_{m=0}^{2n-1} \left(c_{nm} \left[\frac{\ln^m(1-z)}{1-z} \right]_+ + d_{nm} \ln^m(1-z) \right) + \ldots \right]$$

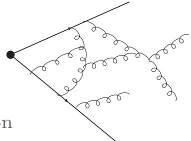
- Leading power

QCD limits

- collinear $p_i \cdot p_j \ll Q^2$



- soft $E_s \ll Q$



QCD singular limits lead to the appearance of large logarithms of a ratio of different scales

Before we discover New Physics, we must be sure that we understand the Standard Model!

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- Leading power
- Next-to-leading power

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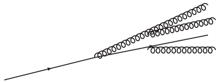

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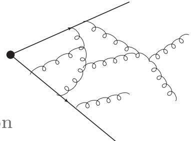

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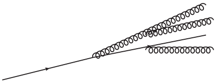
- Leading power
- Next-to-leading power
- Leading Log: $m = 2n-1 \longrightarrow \alpha_s \ln(1-z) + \alpha_s^2 \ln^3(1-z) + \dots$

QCD limits

- collinear $p_i \cdot p_j \ll Q^2$



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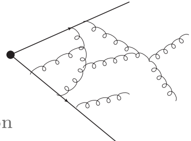

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Factorization at Next-to-Leading power

$$d\sigma = \sum_{a,b} \boxed{C_a C_b^*} \otimes \prod_{i=1}^{N} \boxed{J_a^{(i)} J_b^{(i)}} \otimes \boxed{S_{ab}}$$

- C_a Hard functions; $\sim Q$

λ is a power-counting parameter, e.g. $\lambda = 1 - z$

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- C_a Hard functions; $\sim Q$
- $J_a^{(i)}$ i - collinear (jet) functions; $\sim Q\lambda$
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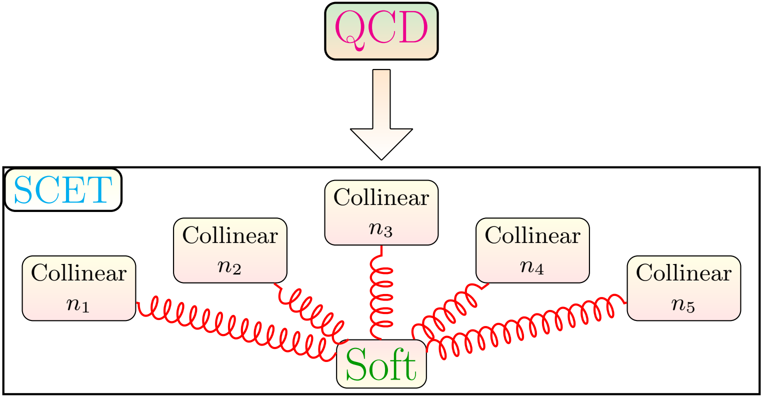
- $\sum_{ab}$ – sum over various functions, related to different sources of power-suppression

λ is a power-counting parameter, e.g. $\lambda = 1 - z$

A bit of SCET formalism

Soft Collinear Effective Field Theory (SCET)

[C. W. Bauer, S. Fleming, D. Pirjol and I. W. Stewart, hep-ph/0011336] What is SCET? Effective field theory used to describe energetic particles.



- Collinear sectors where energetic particles within a single region can interact with each other but not with particle in a different sector.
- Soft sector - mediates interactions between collinear sectors
- Every interaction has well-defined power-counting – allows for systematic expansion

Position space formulation of SCET

[M. Beneke and T. Feldmann, hep-ph/0211358]

$$\mathcal{L}_i^{(0)} = \bar{\xi}_i \left[in_{i-}D + i\not{D}_{\perp i} \frac{1}{in_{i+}D} i\not{D}_{\perp i} \right] \frac{\not{n}_{i+}}{2} \xi_i$$

$$\mathcal{L} = \sum_i \left[\mathcal{L}_i^{(0)} + \mathcal{L}_i^{(1)} + \dots \right]$$

$$\begin{aligned} \xi_i &\sim \lambda \\ n_{i+}D = n_{i+}\partial - ign_{+i}A_i &\sim 1 \\ D^\mu_{\perp i} = \partial^\mu_{\perp i} - ign_{+i}A^\mu_{\perp i} &\sim \lambda \\ n_{i-}D = n_{i-}\partial - ign_{i-}A_i - ign_{i-}A_s(x_{i-}) &\sim \lambda^2 \end{aligned}$$

Soft modes are multipole expanded

$\phi_C(x)\phi_s(x) \to$

$\phi_C(x)\phi_s(x_-) + \dots;\ x_- = \frac{n_-^\mu}{2} n_+ x$

Light-cone coordinates

$$p_i^\mu = (n_{i+}p_i)\frac{n_{i-}^\mu}{2} + p_{i\perp i} + (n_{i-}p_i)\frac{n_{i+}^\mu}{2} \quad p_i p_j \sim Q^2, \quad p_i^2 = 0$$

$$n_{i+}p_i, \sim Q \quad p_{i\perp i} \sim \lambda Q, \quad n_{i-}p_i \sim \lambda^2 Q$$

N-jet operator in SCET

[M. Beneke, M. Garny, R. S. and J. Wang, 1712.04416]

$$O(x) = \int \left(\prod_{i=1}^{N} dt_i \right) C(\{t_i\}) \left(\prod_{i=1}^{N} \psi_i(x + t_i n_{i+}) \right)$$

- $C(\{t_i\})$: hard matching coefficient

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- $C(\{t_i\})$: hard matching coefficient
- ψ_i: collinear field (gauge invariant building blocks)
  - collinear quark $\chi_i \equiv W_i^\dagger \xi_i$
  - collinear gluon $\mathcal{A}^\mu_{\perp i} = W_i^\dagger \left[iD^\mu_{\perp i} W_i\right]$

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Power suppression:

- add derivatives $\partial_\perp \sim \lambda$
- add extra fields in the same direction

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Example 3-jet LP operator:

$$O_3^{A0}(0) = \int dt_1 dt_2 dt_3 C^{A0}(t_1, t_2, t_3) \overline{\chi}_1(t_1 n_{1+}) \gamma_\mu \chi_2(t_2 n_{2+}) \mathcal{A}^\mu_{\perp 3}(t_3 n_{3+})$$

Example 3-jet NLP operators (λ suppressed):

$$O_3^{A1}(0) = \int dt_1 dt_2 dt_3 C^{A1}(t_1, t_2, t_3) \overline{\chi}_1(t_1 n_{1+}) \gamma_\mu \gamma^\nu \partial^\nu_{\perp 2} \chi_2(t_2 n_{2+}) \mathcal{A}^\mu_{\perp 3}(t_3 n_{3+})$$

$$O_3^{B1}(0) = \int dt_1 dt_2 dt_3 C^{B1}(t_1, t_2, t_3) \overline{\chi}_1(t_1 n_{1+}) \gamma_\mu \gamma^\nu \mathcal{A}^\nu_{\perp 2} \chi_2(t_2 n_{2+}) \mathcal{A}^\mu_{\perp 3}(t_3 n_{3+})$$

Leading power anomalous dimension

[T. Becher, M. Neubert, 0901.0722] Simple structure up to two loop:

$$\mathbf{\Gamma} = -\gamma_{\text{cusp}}(\alpha_s) \sum_{i<j} \mathbf{T}_i \cdot \mathbf{T}_j \ln\left(\frac{-s_{ij}}{\mu^2}\right) + \sum_i \gamma_i(\alpha_s)$$

$$s_{ij} = 2p_i \cdot p_j + i0$$

Soft and collinear parts are known at the three loop level [Ø. Almelid, C. Duhr, E. Gardi 1507.00047; S. Moch, J.A.M. Vermaseren, A. Vogt,hep-ph/0507039]

- governs the evolution of the hard functions C^{A0}

$$\frac{d}{d\ln\mu} C_P = \sum_Q \Gamma_{QP} C_Q$$

- QCD: log structure is determined by IR poles
- SCET: turns IR poles of QCD into UV poles of N-jet operator – RG technique can be used

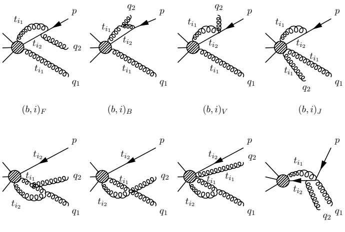
Next-to-Leading power anomalous dimension

$$\Gamma_{PQ}(x,y) = \delta_{PQ}\delta(x-y)\Big[-\gamma_{\rm cusp}(\alpha_s)\sum_{i<j}\sum_{k,l}\mathbf{T}_{i_k}\cdot\mathbf{T}_{j_l}\ln\left(\frac{-s_{ij}x_{i_k}x_{j_l}}{\mu^2}\right) + \sum_i\sum_k\gamma_{i_k}(\alpha_s)\Big] + 2\sum_i \delta^{[i]}(x-y)\,\gamma^i_{PQ}(x,y) + 2\sum_{i<j}\delta(x-y)\,\gamma^{ij}_{PQ}(y)$$

New structures at NLP

- collinear mixing – fields at different positions along the light-cone mix under renormalization

[M. Beneke, M. Garny, R. S. and J. Wang, 1712.04416; M. Beneke, M. Garny, R. S. and J. Wang, 1808.04742]



$(b,i)_F$ $(b,i)_B$ $(b,i)_V$ $(b,i)_J$

$(b,ii)_B$ $(b,ii)_V$ $(b,ii)_J$ $(b,iii)_F$

Next-to-Leading power anomalous dimension

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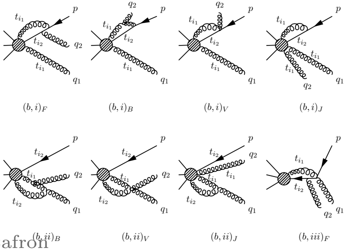
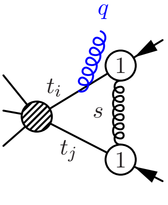
New structures at NLP

- collinear mixing – fields at different positions along the light-cone mix under renormalization
- soft mixing – time-ordered products of NLP Lagrangian with the N-jet operator mix into N-jet operator

[M. Beneke, M. Garny, R. S. and J. Wang, 1712.04416; M. Beneke, M. Garny, R. S. and J. Wang, 1808.04742]





Kluberg-Stern, Zuber theorem in SCET

In a "sensible" field theory, operators proportional to classical equation of motion $\partial_S F(x) \equiv \int d^d y \, \frac{\delta S}{\delta \chi_i(y)} K_i(y,x) F(x)$, can be ignored

- do not contribute to on-shell matrix elements
- do not mix into regular operators

$$\int_x \Gamma^{\text{1PI,div}}_{\partial_S F(x)}(p,q) \propto \frac{p^2}{n_{i+}p} \times \underbrace{\Gamma^{1PI,div}_{F(p)}(q)}_{\propto 1/\epsilon}$$

This in no longer true in SCET

- Double poles lead to non-local divergences $1/\epsilon^2 + 2/\epsilon \times \log \mu^2/p^2$
- NLP Lagrangian contains x-depend terms (due to multipole expansion) which produce momentum derivatives in the Feynman rules

$$\int_x \Gamma^{\text{1PI,div}}_{\partial_S F(x)}(p,q) \propto \frac{p^2}{n_{i+}p} \times \frac{\partial}{\partial p^\mu_{\perp i}} \underbrace{\Gamma^{\text{1PI,div}}_{F^\mu(p)}(q)}_{\propto (p^2)^{-\epsilon}/\epsilon^2}$$

[M. Beneke, M. Garny, R. S. and J. Wang, 1907.05463] SCET is nevertheless "sensible" EFT because off-shell terms are uniquely fixed [M. Beneke, A. Chapovsky, M. Diehl, T. Feldmann, hep-ph/0206152]

Applications

The Drell-Yan process - the leading power threshold factorization

$$A(p_A)B(p_B) \to \gamma^*(Q) + X$$

$$\frac{d\sigma_{\rm DY}}{dQ^2} = \frac{4\pi\alpha_{\rm em}^2}{3N_c Q^4} \sum_{a,b} \int_0^1 dx_a dx_b\, f_{a/A}(x_a) f_{b/B}(x_b)\, \hat{\sigma}_{ab}^{\rm LP}(z)$$

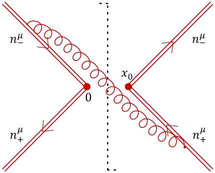
[G. P. Korchemsky, G. Marchesini, 1993]

[T. Becher, M. Neubert, G. Xu, 0710.0680; S. Moch, A. Vogt, hep-ph/0508265]

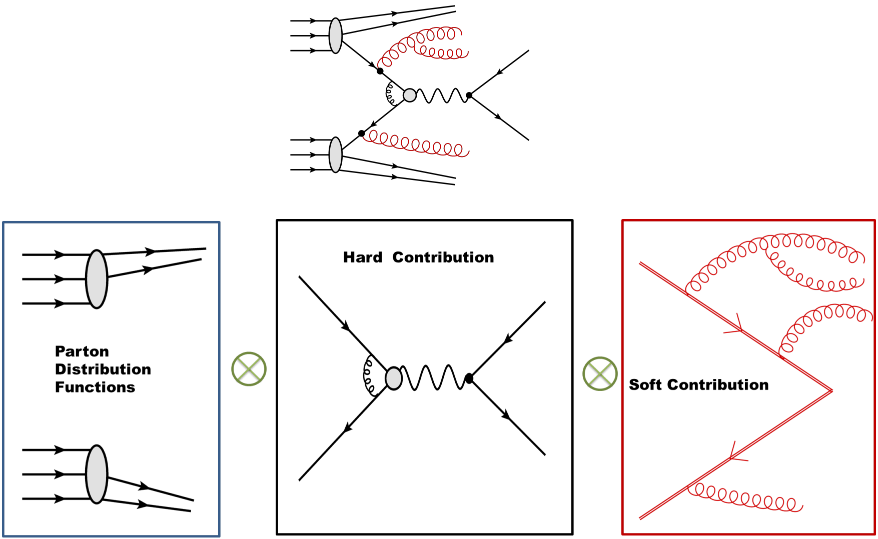
$$\hat{\sigma}^{\,\rm LP}(z) = |C(Q^2)|^2\; Q\; S_{\rm DY}(Q(1-z))$$

$z = Q^2/\hat{s}$ threshold $z \to 1$

$$S_{\rm DY}(\Omega) = \int \frac{dx^0}{4\pi}\, e^{ix^0\Omega/2}\, \frac{1}{N_c}\, {\rm Tr}\, \langle 0|\bar{\mathbf{T}}(Y_+^\dagger(x^0)Y_-(x^0))\, \mathbf{T}(Y_-^\dagger(0)Y_+(0))|0\rangle$$

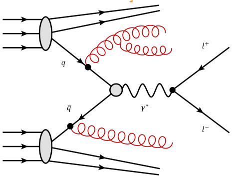


Leading power factorization in SCET



DY cross-section beyond LP

[M. Beneke, A. Broggio, M. Garny, S. Jaskiewicz, R. S., L. Vernazza, J. Wang, 1809.10631]



$z = \frac{Q^2}{\hat{s}}$ threshold $z \to 1$

$\Omega \sim Q(1-z)$

Factorization theorem valid at *LL accuracy*

$$\hat{\sigma}(z) = H(\hat{s}) \times Q^2 \int \frac{d^3\vec{q}}{(2\pi)^3\, 2\sqrt{Q^2+\vec{q}^{\,2}}} \frac{1}{2\pi} \int d^4x\, e^{i(x_a p_A + x_b p_B - q)\cdot x}$$
$$\times \left\{ \widetilde{S}_0(x) + 2 \cdot \frac{1}{2} \int d\omega\, J_{2\xi}^{(O)}(x_a n_+ p_A; \omega)\, \widetilde{S}_{2\xi}(x,\omega) + \bar{c}\text{-term} \right\}$$

Scales:

- hard $\mu_h \sim Q$
- collinear $\mu_c \sim \sqrt{Q\Omega}$ (New object compared to LP)
- soft $\mu_s \sim \Omega$

$Q \gg \Omega$

Factorization of time-ordered products at NLP

We separate the Lagrangian insertions into collinear and soft parts

$$\mathcal{L}_V^{(n)}(z) = \mathcal{L}_c^{(n)}(z) \otimes \mathcal{L}_s^{(n)}(z_-)$$

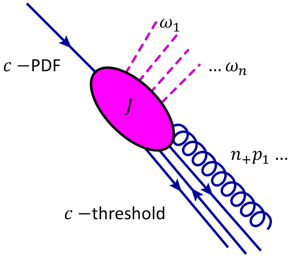
- Soft fields are multipole expanded – convolution variable is one-dimensional
- We perform Fourier transform for each z_-
- We gather all the collinear structures that correspond to a given soft structure

This gives an NLP collinear function

$$i^n \left(\sum_{i=1}^{n} \int d^4 z_j \, e^{i\,\omega_j \, \frac{n_+ z_j}{2}} \right)$$

$$\times \, \mathbf{T} \left[\chi_c(t n_+) \times \mathcal{L}_c^{(n)}(z_1) \times \mathcal{L}_c^{(n)}(z_2) \times \ldots \right]$$

$$= J\left(t; \omega_1, \omega_2, \ldots\right) \chi_c^{\mathrm{PDF}}(t n_+)$$



Collinear function is a non-local object

Next to leading power soft function

Soft operator in position space is a non-local object

$$\widetilde{\mathcal{S}}_{2\xi}\left(x, z_-\right) = \bar{\mathbf{T}}\left[Y_+^\dagger(x) Y_-(x)\right] \mathbf{T}\left[Y_-^\dagger(0) Y_+(0) \frac{i\partial_\perp^\nu}{in_-\partial} \mathcal{B}^+_{\perp\nu}(z_-)\right]$$

with decoupled soft fields

$$\mathcal{B}^\mu_\pm = Y_\pm^\dagger\left[iD_s^\mu Y_\pm\right]$$

Lagrangian is already multipole expanded → soft fields depend only on z_-

$$\mathcal{L}^{(2)}_{2\xi} = \frac{1}{2}\bar{\chi}_c z_\perp^\mu z_\perp^\nu \left[i\partial_\nu in_-\partial\mathcal{B}^+_\mu\right]\frac{\not{n}_+}{2}\chi_c$$

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In the factorization theorem, we need only vacuum matrix element

$$S_{2\xi}(\Omega, \omega) = \int \frac{dx^{0}}{4\pi} \int \frac{d(n_{+}z)}{4\pi} e^{ix^{0}\Omega/2 - i\omega(n_{+}z)/2} \frac{1}{N_{c}} \operatorname{Tr}\langle 0|\widetilde{\mathcal{S}}_{2\xi}(x^{0}, z_{-})|0\rangle$$

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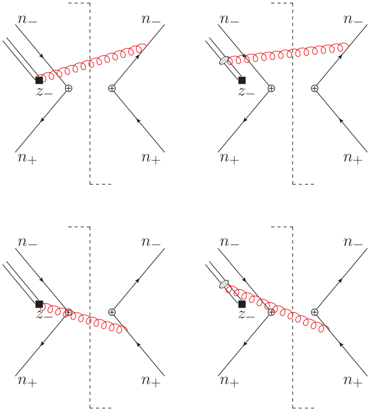

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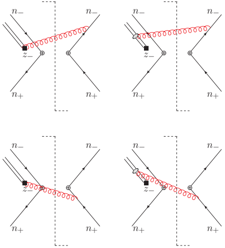

$$S_{2\xi}(\Omega, \omega) = \frac{\alpha_s C_F}{2\pi}\left\{\theta(\Omega)\delta(\omega)\left(-\frac{1}{\epsilon} + \ln\frac{\Omega^2}{\mu^2}\right) + \left[\frac{1}{\omega}\right]_+ \theta(\omega)\theta(\Omega - \omega)\right\}$$

Kinematic corrections

To understand how to renormalize soft function let us analyze other type of corrections and a simple example

Expansion of the kinematic factors leads to

$$Q\int \frac{d^3\vec{q}}{(2\pi)^3\, 2\sqrt{Q^2+\vec{q}^{\,2}}}\,\frac{1}{2\pi}\int d^4x\, e^{i(x_a p_A + x_b p_B - q)\cdot x}\,\widetilde{S}_0(x)$$

$$\to \int \frac{dx^0}{4\pi}\, e^{ix^0\Omega_*/2}\left(1+\frac{ix^0\partial^2_{\vec{x}}}{2Q}+\mathcal{O}\left(\lambda^4\right)\right)\widetilde{S}_0(x^0,\vec{x})_{|\vec{x}=0}$$

$$\to S_{\mathrm{DY}}(Q(1-z))+\frac{1}{Q}S_{K1}(Q(1-z))+\frac{1}{Q}S_{K2}(Q(1-z))+\mathcal{O}(\lambda^4)$$

NLP kinematic soft functions

$$\begin{aligned} S_{K1}(\Omega) &= \frac{\partial}{\partial\Omega}\partial^2_{\vec{x}}S_0\left(\Omega,\vec{x}\right)_{|\vec{x}=0} \\ S_{K2}(\Omega) &= \frac{3}{4}\,\Omega^2\,\frac{\partial}{\partial\Omega}S_0(\Omega,\vec{x})_{|\vec{x}=0} \end{aligned}$$

Example: expansion of the soft function RGE I

In position space, renormalization of the LP soft function is multiplicative

$$\frac{d}{d\ln\mu}\widetilde{S}_0(x) = \left[2\Gamma_{\mathrm{cusp}}L - 2\gamma_W\right]\widetilde{S}_0(x)$$

$$L \equiv \ln\left(-\frac{1}{4}n_-xn_+x\mu^2e^{2\gamma_E}\right)$$

$$\gamma_W = \mathcal{O}(\alpha_s^2)$$

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Expansion of the soft function, $x = (x^0, 0, 0, z)$

$$\widetilde{S}_0(x) = \widetilde{S}_0(x_0) + \ldots + \frac{1}{2}\vec{\partial}_z^{\,2}\widetilde{S}_0(x)_{|\vec{x}=0}z^2 + \ldots$$

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Expansion of the log generates inhomogeneous term

$$L = L_0 - \frac{z^2}{(x^0)^2} + \mathcal{O}\left(\frac{z^4}{(x^0)^4}\right)$$

$$L_0 \equiv \ln\left(-\frac{1}{4}(x^0)^2\mu^2 e^{2\gamma_E}\right)$$

Example: expansion of the soft function RGE II

Coefficient of z^2 gives

$$\frac{d}{d\ln\mu}\,\frac{1}{2}\vec{\partial}_z^2\widetilde{S}_0(x)_{|\vec{x}=0} = \Big[2\Gamma_{\rm cusp}L_0 - 2\gamma_W\Big]\,\frac{1}{2}\vec{\partial}_z^2\widetilde{S}_0(x)_{|\vec{x}=0} - \frac{2}{(x^0)^2}\widetilde{S}_0(x_0)$$

Define soft functions

$$\begin{aligned}\widetilde{S}_3(x_0) &= \frac{ix_0}{2}\vec{\partial}_z^2\widetilde{S}_0(x)_{|\vec{x}=0}\\ \widetilde{S}_{x_0}(x_0) &= \frac{-2i}{x^0-i\varepsilon}\widetilde{S}(x_0)\end{aligned}$$

Example: expansion of the soft function RGE II

Coefficient of z^2 gives

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Example: expansion of the soft function RGE II

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Define soft functions

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Soft functions mix

$$\begin{aligned}\frac{d}{d\ln\mu}\widetilde{S}_3(x_0) &= \left[2\Gamma_{\rm cusp}L_0 - 2\gamma_W\right]\widetilde{S}_3(x_0) + \widetilde{S}_{x_0}(x_0)\\ \frac{d}{d\ln\mu}\widetilde{S}_{x_0}(x_0) &= \left[2\Gamma_{\rm cusp}L_0 - 2\gamma_W\right]\widetilde{S}_{x_0}(x_0)\end{aligned}$$

Note: $\widetilde{S}_3(x_0) = \mathcal{O}\left(\alpha_s L_0\right)$ and $\widetilde{S}_{x_0}(x_0) = 1 + \mathcal{O}\left(\alpha_s L_0^2\right)$

$\widetilde{S}_{x_0}(x_0)$ corresponds to θ-soft function [I. Moult, I. Stewart, G. Vita, H. Xing Zhu,1804.04665]

Soft function renormalization

We can now return to problem of renormalization of the soft function

We assume that renormalization in the momentum space is a convolution in Ω and ω the the divergence is removed through operator mixing

$$S_{2\xi}(\Omega,\omega)_{|\mathrm{ren}} = \int d\Omega' \int d\omega'\, Z_{2\xi,2\xi}(\Omega,\omega;\Omega',\omega')\, S_{2\xi}(\Omega',\omega')_{|\mathrm{bare}} + \int d\Omega'\, Z_{2\xi,x_0}(\Omega,\omega;\Omega')\, S_{x_0}(\Omega')_{|\mathrm{bare}}$$

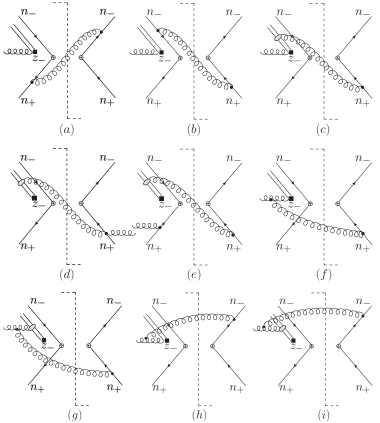
Renormalization through mixing with the same S_{x_0} as in the case of kinematic corrections

$$Z_{2\xi,2\xi}(\Omega,\omega;\Omega,\omega') = \delta(\Omega-\Omega')\delta(\omega-\omega') + \mathcal{O}(\alpha_s)\,,$$

$$Z_{2\xi,x_0}(\Omega,\omega;\Omega') = \frac{\alpha_s C_F}{2\pi}\frac{1}{\epsilon}\delta(\Omega-\Omega')\delta(\omega) + \mathcal{O}(\alpha_s^2)\,.$$

How to determine $Z_{2\xi,2\xi}(\Omega,\omega;\Omega,\omega')$ at one loop?

One loop "real" diagrams

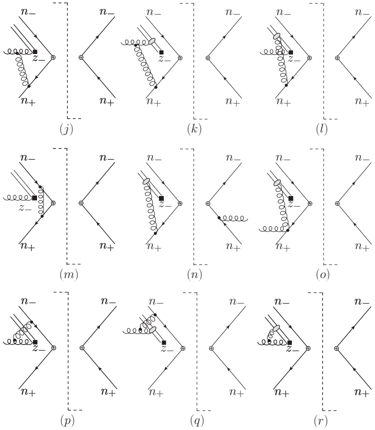


$$\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle^{a)}_{\text{1-loop}} = \left[\frac{\alpha_s}{2\pi}\frac{C_F}{\epsilon^2} + \mathcal{O}\left(\epsilon^{-1}\right)\right]\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle_{\text{tree}}$$

$$\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle^{b)}_{\text{1-loop}} = \left[\frac{\alpha_s}{2\pi}\frac{C_F}{\epsilon^2} + \mathcal{O}\left(\epsilon^{-1}\right)\right]\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle_{\text{tree}}$$

$$\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle^{c)}_{\text{1-loop}} = \left[-\frac{\alpha_s}{4\pi}\frac{C_A}{\epsilon^2} + \mathcal{O}\left(\epsilon^{-1}\right)\right]\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle_{\text{tree}}$$

One loop "virtual" diagrams



$$\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle^{j)+k)}_{\text{1-loop}} =$$
$$\left[\frac{\alpha_s}{4\pi}\frac{C_A}{\epsilon^2} + \mathcal{O}\left(\epsilon^{-1}\right)\right]\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle_{\text{tree}}$$

LL soft function RGE

We checked our result by explicit two-loop computation of the soft function. Both methods lead to the same AD matrix $\to$ non-trivial check of

- the choice of S_{x_0}
- the correctness of our procedure to extract leading poles
- the relation between soft operator and soft function renormalization

At the LL we have

$$\frac{d}{d\ln\mu}\begin{pmatrix} S_{2\xi}(\Omega,\omega) \\ S_{x_0}(\Omega) \end{pmatrix} = \frac{\alpha_s}{\pi}\begin{pmatrix} 4C_F\ln\frac{\mu}{\mu_s} & -C_F\delta(\omega) \\ 0 & 4C_F\ln\frac{\mu}{\mu_s} \end{pmatrix}\begin{pmatrix} S_{2\xi}(\Omega,\omega) \\ S_{x_0}(\Omega) \end{pmatrix}$$

with a solution

$$\begin{aligned} S_{2\xi}^{\mathrm{LL}}(\Omega,\omega,\mu) &= \frac{2C_F}{\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_s)}\exp\left[-4S^{\mathrm{LL}}(\mu_s,\mu)\right]\theta(\Omega)\delta(\omega) \\ &= C_F\frac{\alpha_s}{\pi}\ln\frac{\mu_s}{\mu}\exp\left[-2C_F\frac{\alpha_s}{\pi}\ln^2\frac{\mu_s}{\mu}\right]\theta(\Omega)\delta(\omega) \end{aligned}$$

LL resummation

The resummed collinear function does not contribute to the LL result, we only need tree level result

$$J^{\mu\rho}_{2\xi;\alpha\beta,abde}(n_+p, n_+p'; \omega) = -\frac{g_\perp^{\mu\rho}}{n_+p}\delta(n_+p - n_+p')\delta_{\alpha\beta}\delta_{ad}\delta_{eb} + \mathcal{O}\left(\alpha_s \ln\left(\frac{\mu}{\mu_c}\right)\right)$$

The resummed cross-section is

$$\begin{aligned}\Delta^{\rm LL}(z) &= \Delta^{\rm LL}_{\rm LP}(z) \\ &\quad - \exp\left[4S^{\rm LL}(\mu_h, \mu) - 4S^{\rm LL}(\mu_s, \mu)\right] \times \frac{8C_F}{\beta_0} \ln \frac{\alpha_s(\mu)}{\alpha_s(\mu_s)}\,\theta(1-z)\end{aligned}$$

where at LL accuracy

$$S^{\rm LL}(\mu_1, \mu_2) = -\frac{\alpha_s C_F}{2\pi}\ln^2\frac{\mu_2}{\mu_1} \quad \text{and} \quad \frac{1}{\beta_0}\ln\frac{\alpha_s(\mu_1)}{\alpha_s(\mu_2)} = \frac{\alpha_s}{2\pi}\ln\frac{\mu_2}{\mu_1}$$

Fixed order expanded result

- R. Hamberg, W. L. van Neerven and T. Matsuura, 1991
- D. de Florian, J. Mazzitelli, S. Moch and A. Vogt, 2014

$$
\begin{aligned}
\Delta^{\rm LL}_{\rm NLP}(z,\mu) \quad = \quad & -\theta(1-z)\left\{4C_F\frac{\alpha_s}{\pi}\Big[\ln(1-z)-L_\mu\Big]\right. \\
& +8C_F^2\left(\frac{\alpha_s}{\pi}\right)^2\Big[\ln^3(1-z)-3L_\mu\ln^2(1-z)+2L_\mu^2\ln(1-z)\Big] \\
& +8C_F^3\left(\frac{\alpha_s}{\pi}\right)^3\Big[\ln^5(1-z)-5L_\mu\ln^4(1-z)+8L_\mu^2\ln^3(1-z)-4L_\mu^3\ln^2(1-z)\Big] \\
& +\frac{16}{3}C_F^4\left(\frac{\alpha_s}{\pi}\right)^4\Big[\ln^7(1-z)-7L_\mu\ln^6(1-z)+18L_\mu^2\ln^5(1-z)-20L_\mu^3\ln^4(1-z) \\
& \qquad +8L_\mu^4\ln^3(1-z)\Big] \\
& +\frac{8}{3}C_F^5\left(\frac{\alpha_s}{\pi}\right)^5\Big[\ln^9(1-z)-9L_\mu\ln^8(1-z)+32L_\mu^2\ln^7(1-z)-56L_\mu^3\ln^6(1-z) \\
& \qquad \left. +48L_\mu^4\ln^5(1-z)-16L_\mu^5\ln^4(1-z)\Big]\right\}+\mathcal{O}(\alpha_s^6\times(\log)^{11})\,,
\end{aligned}
$$

$L_\mu = \ln(\mu/Q)$.

Higgs threshold production

$$A(p_A)B(p_B) \to H(q) + X(p_X)$$

Threshold variable

$$z \equiv \frac{m_H^2}{\hat{s}}$$

$$\mathcal{L}_{\text{eff}} = \frac{\alpha_s(\mu)}{3\pi} C_t(m_t, \mu)\ \frac{1}{4} F^A_{\mu\nu} F^{\mu\nu}_A\ \ln\left(1 + \frac{H}{\nu}\right)$$

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$$A(p_A)B(p_B) \to H(q) + X(p_X)$$

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LP current

$$F^A_{\mu\nu} F^{\mu\nu}_A \to 2 g^{\perp}_{\mu\nu} n_- \partial \mathcal{A}^{\nu A}_{\bar{c}\perp} n_+ \partial \mathcal{A}^{\mu A}_{c\perp}$$

The derivation of the factorization is similar like in the DY case, with Wilson lines in the adjoint representation
The result has the same form as Drell-Yan with $C_F \leftrightarrow C_A$

Summary and Conclusions

- Investigation of power corrections with SCET gives us a better understanding of QCD
- Accuracy of QCD resummation is improved, numerical study will appear soon
- Many more applications, see e.g.
  - Improvement in understanding QED corrections in flavor physics and resummation [M. Beneke, C. Bobeth, R.S, 1908.07011]
  - Thrust resummation in $H \to gg$ [I. Moult, I. Stewart, G. Vita, H. Xing Zhu, 1804.04665]
  - N-jettines subtraction [M. Ebert, I. Moult, I. Stewart, F. Tackmann, G. Vita, H. Xing Zhu, 1807.10764]
  - Rapidity divergences and power corrections in q_T (SCET_{II}) [M. Ebert, I. Moult, I. Stewart, F. Tackmann, G. Vita, H. Xing Zhu, 1812.08189]

Auxiliary slide: Hard function running

Well known RGE for two-jet operator

$$\frac{d}{d\ln\mu}H(Q^2,\mu) = \left(2\Gamma_{\text{cusp}}\ln\frac{Q^2}{\mu^2} + 2\gamma\right)H(Q^2,\mu)$$

$$\Gamma_{\text{cusp}} = \frac{\alpha_s}{\pi}C_F + \mathcal{O}(\alpha_s^2), \qquad \gamma = -\frac{3}{2}\frac{\alpha_s}{\pi}C_F + \mathcal{O}(\alpha_s^2),$$

The general solution RGE reads

$$H(Q^2,\mu) = \exp\left[4S(\mu_h,\mu) - 2a_\gamma(\mu_h,\mu)\right]\left(\frac{Q^2}{\mu_h^2}\right)^{-2a_\Gamma(\mu_h,\mu)} H(Q^2,\mu_h)$$

where

$$S(\nu,\mu) = -\int\limits_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha\,\frac{\Gamma_{\text{cusp}}(\alpha)}{\beta(\alpha)}\int\limits_{\alpha_s(\nu)}^{\alpha}\frac{d\alpha'}{\beta(\alpha')},$$

$$a_\Gamma(\nu,\mu) = -\int\limits_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha\,\frac{\Gamma_{\text{cusp}}(\alpha)}{\beta(\alpha)}, \qquad a_\gamma(\nu,\mu) = -\int\limits_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha\,\frac{\gamma(\alpha)}{\beta(\alpha)}$$

Auxiliary slide: Soft function in position space

At the one-loop order in dimensional regularization with $d = 4 - 2\epsilon$, the bare soft function must have a simple dependence

$$\tilde{S}_{0,\text{bare}}(x) = 1 + \frac{\alpha_s}{\pi}\left(-n_- x n_+ x \mu^2\right)^\epsilon f\left(\epsilon, \frac{x^2}{n_+ x n_- x}\right)$$

Explicit evaluation gives

$$\begin{aligned}
\widetilde{S}_{0,\text{bare}}(x) &= 1 + \frac{\alpha_s C_F}{\pi}\frac{\Gamma(1-\epsilon)}{\epsilon^2} e^{-\epsilon\gamma_E} \\
&\quad \times \left(-\frac{1}{4} n_- x n_+ x \mu^2 e^{2\gamma_E}\right)^\epsilon \left(\frac{x^2}{n_- x n_+ x}\right)^{1+\epsilon} {}_2F_1\left(1,1,1-\epsilon;1-\frac{x^2}{n_- x n_+ x}\right) \\
&= 1 + \frac{\alpha_s C_F}{\pi}\left(\frac{1}{\epsilon^2} + \frac{L}{\epsilon} + \frac{L^2}{2} + \frac{\pi^2}{12} + \mathrm{Li}_2\left(1 - \frac{x^2}{n_- x n_+ x}\right) + \mathcal{O}(\epsilon)\right)
\end{aligned}$$

where we defined

$$L \equiv \ln\left(-\frac{1}{4} n_- x n_+ x \mu^2 e^{2\gamma_E}\right).$$

Auxiliary slide: Kinematic soft functions at $\mathcal{O}(\alpha_s)$

Expanding the kinematic factors in the factorization formula we obtain further corrections related to the LP soft function

$$
\begin{aligned}
S_{K1}(\Omega) &= \frac{\alpha_s C_F}{2\pi}\left(\frac{1}{\epsilon}+2\ln\frac{\mu}{\Omega}-2\right)\theta(\Omega)\\
S_{K2}(\Omega) &= \frac{\alpha_s C_F}{2\pi}\left(\frac{3}{\epsilon}+6\ln\frac{\mu}{\Omega}+6\right)\theta(\Omega)\\
S_{K3}(\Omega) &= \frac{\alpha_s C_F}{2\pi}\left(-\frac{4}{\epsilon}-8\ln\frac{\mu}{\Omega}\right)\theta(\Omega)
\end{aligned}
$$

$$\sum_{i=1}^{3} S_{Ki}(\Omega) = 2\,\frac{\alpha_s C_F}{\pi}\,\theta(\Omega)$$

At $\mathcal{O}(\alpha_s)$ no LL kinematic corrections!

Auxiliary slide: Kinematic corrections

At LP we only need the soft function at $x = x_0$ but for now consider the soft function for generic x

$$\widetilde{S}_0(x) = \frac{1}{N_c} \,\mathrm{Tr}\, \langle 0|\bar{\mathbf{T}}(Y_+^\dagger(x)Y_-(x))\, \mathbf{T}(Y_-^\dagger(0)Y_+(0))|0\rangle$$

Use partonic center-of-mass frame $x_a\vec{p}_A + x_b\vec{p}_B = 0$
Momentum $\vec{p}_{X_s}$ of the soft hadronic final state is balanced by the lepton-pair $\vec{q} + \vec{p}_{X_s} = 0$

$$\vec{q} \sim \lambda^2, \quad q^0 = \sqrt{\hat{s}} + \mathcal{O}(\lambda^2)$$

Energy of the soft radiation

$$[x_1 p_1 + x_2 p_2 - q]^0 = p^0_{X_s} = \sqrt{\hat{s}} - \sqrt{Q^2 + \vec{q}^{\,2}} = \frac{\Omega_*}{2} - \frac{\vec{q}^{\,2}}{2Q} + \mathcal{O}\left(\lambda^6\right)$$

with

$$\Omega_* = 2Q\frac{1-\sqrt{z}}{\sqrt{z}} = Q(1-z) + \frac{3}{4}Q(1-z)^2 + \mathcal{O}\left(\lambda^6\right)$$

Auxiliary slide: Soft function renormalization

We assume that renormalization in the momentum space is a convolution in Ω and ω

$$S_{2\xi}(\Omega,\omega)_{|\mathrm{ren}} = \int d\Omega' \int d\omega'\, Z_{2\xi,2\xi}(\Omega,\omega;\Omega',\omega')\, S_{2\xi}(\Omega',\omega')_{|\mathrm{bare}} + \int d\Omega'\, Z_{2\xi,x_0}(\Omega,\omega;\Omega')\, S_{x_0}(\Omega')_{|\mathrm{bare}}$$

Renormalization through mixing

$$Z_{2\xi,2\xi}(\Omega,\omega;\Omega,\omega') = \delta(\Omega-\Omega')\delta(\omega-\omega') + \mathcal{O}(\alpha_s)\,,$$

$$Z_{2\xi,x_0}(\Omega,\omega;\Omega') = \frac{\alpha_s C_F}{2\pi}\frac{1}{\epsilon}\delta(\Omega-\Omega')\delta(\omega) + \mathcal{O}(\alpha_s^2)\,.$$

Auxiliary slide: Soft function renormalization

We assume that renormalization in the momentum space is a convolution in Ω and ω

$$S_{2\xi}(\Omega,\omega)_{|\mathrm{ren}} = \int d\Omega' \int d\omega'\, Z_{2\xi,2\xi}(\Omega,\omega;\Omega',\omega')\, S_{2\xi}(\Omega',\omega')_{|\mathrm{bare}} + \int d\Omega'\, Z_{2\xi,x_0}(\Omega,\omega;\Omega')\, S_{x_0}(\Omega')_{|\mathrm{bare}}$$

Aside:

Is the convolution assumption too strong?

- Dependence of Z on Ω' cannot be uniquely determined - at LP we determine it from the known properties of Wilson loop renormalization in position space – multiplicative renormalization in position space
- Dependence on ω' can by determined under additional assumptions

Auxiliary slide: Soft function renormalization

We assume that renormalization in the momentum space is a convolution in Ω and ω

$$S_{2\xi}(\Omega,\omega)_{|\mathrm{ren}} = \int d\Omega' \int d\omega' \, Z_{2\xi,2\xi}(\Omega,\omega;\Omega',\omega') \, S_{2\xi}(\Omega',\omega')_{|\mathrm{bare}} + \int d\Omega' \, Z_{2\xi,x_0}(\Omega,\omega;\Omega') \, S_{x_0}(\Omega')_{|\mathrm{bare}}$$

Renormalization through mixing

$$Z_{2\xi,2\xi}(\Omega,\omega;\Omega,\omega') = \delta(\Omega-\Omega')\delta(\omega-\omega') + \mathcal{O}(\alpha_s)\,,$$

$$Z_{2\xi,x_0}(\Omega,\omega;\Omega') = \frac{\alpha_s C_F}{2\pi}\frac{1}{\epsilon}\delta(\Omega-\Omega')\delta(\omega) + \mathcal{O}(\alpha_s^2)\,.$$

How to determine $\mathcal{O}\left(\alpha_s\right)$ of the diagonal Z-factor?

Auxiliary slide: Kinematic corrections III

It is more convenient to introduce

$$\Delta_{ab}(z) = \frac{\hat{\sigma}_{ab}(z)}{z}$$

$\Delta_{ab}^{\mathrm{LP}}(z) = \hat{\sigma}_{ab}^{\mathrm{LP}}(z)$ but $\Delta_{ab}^{\mathrm{NLP}}(z)$ receives additional NLP correction

$$(1-z) \times \hat{\sigma}_{\mathrm{LP}}(z)$$

which leads to

$$S_{K3}(\Omega) = \Omega\, S_0(\Omega, \vec{x})_{|\vec{x}=0}$$

Factorization theorem for $\Delta(z) = \Delta_{q\bar{q}}(z)$:

$$\begin{aligned} \Delta(z) &= H(Q^2) \\ &\times Q\left\{ S_{\mathrm{DY}}(Q(1-z)) + \sum_{i=1}^{3} \frac{1}{Q} S_{Ki}(Q(1-z)) \right. \\ &\left. +2 \cdot \frac{1}{2} \int d\omega\, J_{2\xi}^{(O)}(x_a n_+ p_A; \omega)\, \widetilde{S}_{2\xi}(x,\omega) + \bar{c}\text{-term} \right\} \end{aligned}$$

No further expansion in λ is needed!

Auxiliary slide: RGE for kinematic soft functions

Proceeding like in the example we obtain

$$
\begin{aligned}
\frac{d}{d\ln\mu}\,\vec{S}(x^0) &= \Big[2\Gamma_{\mathrm{cusp}}L_0 - 2\gamma_W\Big]\,\mathbf{1}\,\vec{S}(x) \\
&+ \Gamma_{\mathrm{cusp}}\begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & -6 & +3 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}\vec{S}(x^0)
\end{aligned}
$$

with $\vec{S}(x^0) = \left(\widetilde{S}_{K1}, \widetilde{S}_{K2}, \widetilde{S}_{K3}, \widetilde{S}_{x_0}\right)^T$

$$
\frac{d}{d\ln\mu}\,\widetilde{S}_{K1+K2+K3}(x^0) = \Big[2\Gamma_{\mathrm{cusp}}L_0 - 2\gamma_W\Big]\,\widetilde{S}_{K1+K2+K3}(x^0) - 6\,\Gamma_{\mathrm{cusp}}\,\widetilde{S}_{K3}(x^0)\,,
$$

Note: $\widetilde{S}_{K1+K2+K3}(x^0) = \mathcal{O}(\alpha_s)$

No LL kinematic corrections to all orders!

Auxiliary slide: Alternative approach without operator renormalization

Renormalization condition for the two-loop soft function $S^{(2)}_{2\xi}$

$$\begin{aligned} S^{(2)}_{2\xi} + Z^{(1)}_{2\xi\, x_0} S^{(1)}_{x_0} + Z^{(2)}_{2\xi\, x_0} S^{(0)}_{x_0} + Z^{(1)}_{2\xi\, 2\xi} S^{(1)}_{2\xi} &= \text{finite} \\ S^{(1)}_{x_0} + Z^{(1)}_{x_0\, x_0} S^{(0)}_{x_0} &= \text{finite} \\ S^{(1)}_{2\xi} + Z^{(1)}_{2\xi\, x_0} S^{(0)}_{x_0} &= \text{finite} \end{aligned}$$

Following structure

$$\Gamma = \alpha_s(\mu) \begin{pmatrix} \Gamma_{AA} \ln \frac{\mu}{\mu_s} + \gamma_{AA} & \gamma_{AB} \\ \gamma_{BA} & \Gamma_{BB} \ln \frac{\mu}{\mu_s} + \gamma_{BB} \end{pmatrix}$$

implies

$$Z^{(2)}_{AB} = \frac{1}{4} Z^{(1)}_{AB} \left(Z^{(1)}_{AA} + 3 Z^{(1)}_{BB} \right) + \mathcal{O}\left(\frac{1}{\epsilon^2}\right) \quad A \neq B.$$

$$S^{(2)}_{2\xi} - \frac{1}{4} Z^{(1)}_{2\xi\, x_0} \left(3 Z^{(1)}_{2\xi\, 2\xi} + Z^{(1)}_{x_0\, x_0} \right) S^{(0)}_{x_0} = \mathcal{O}\left(\frac{1}{\epsilon^2}\right)$$

Two loop result agrees with one-loop operator renormalization

Auxiliary slide: Soft operator

Let us consider an operator rather than its matrix element

$$\mathcal{S}_{2\xi}(\Omega,\omega) = \int \frac{dx^0}{4\pi} \int \frac{d(n_+z)}{4\pi} e^{i\left(x^0\Omega - n_+z\omega\right)/2}\, \overline{\mathbf{T}}\left[Y_+^\dagger(x_0)\, Y_-(x_0)\right] \times \mathbf{T}\left[Y_-^\dagger(0)\, Y_+(0)\, \frac{i\partial_{\perp\mu}}{in_-\partial}\mathcal{B}_+^\mu(z_-)\right]$$

Generalize renormalization equation to

$$\left[\mathcal{S}_A(\Omega,\omega_i)\right]_{\text{ren}} = \sum_B \int d\Omega' d\omega_j' \mathcal{Z}_{AB}\left(\Omega,\omega_i;\Omega',\omega_j'\right)\left[\mathcal{S}_B\left(\Omega',\omega_j'\right)\right]_{\text{bare}}$$

$$Z_{2\xi\,2\xi} = \frac{1}{N_c}\sum_{a,c}(\mathcal{Z}_{2\xi\,2\xi})_{aa,cc}$$

For the leading $1/\epsilon^2$ pole we find that

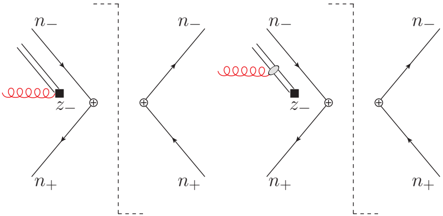
$$(\mathcal{Z}_{2\xi\,2\xi})_{ab,cd} \equiv \delta_{ac}\delta_{bd}\mathcal{Z}_{2\xi\,2\xi} + \mathcal{O}(\epsilon^{-1})$$

hence

$$Z_{2\xi\,2\xi} = \mathcal{Z}_{2\xi\,2\xi} + \mathcal{O}(\epsilon^{-1})$$

Auxiliary slide: Soft matrix elements

Problem of finding Z-factor reduced to operator renormalization



Tree level matrix element is not zero

$$\langle g_A(p)|\mathcal{S}_{2\xi}(\Omega,\omega)|0\rangle_{\text{tree}} = g_s T^A \left(\frac{p_\perp \cdot \epsilon_\perp^*}{n_- p} - \frac{p_\perp^2 n_- \epsilon^*}{(n_- p)^2} \right) \delta(\Omega)\delta(\omega - n_- p).$$

Dependence on the external momentum allows to determine full dependence on ω'

Auxiliary slide: Diagonal part of the anomalous dimension

We find the sum of virtual and real contribution to give a result exactly equal to the corresponding cusp anomalous dimension of the leading power soft function

$$Z^{(1)}_{2\xi\,2\xi}\left(\Omega,\omega;\Omega',\omega'\right) = -\frac{\alpha_s C_F}{\pi}\frac{1}{\epsilon^2}\delta\left(\Omega-\Omega'\right)\delta\left(\omega-\omega'\right)$$

$$\Gamma_{2\xi\,2\xi}\left(\Omega,\omega;\Omega',\omega'\right) = 4\frac{\alpha_s C_F}{\pi}\ln\frac{\mu}{\mu_s}\delta\left(\Omega-\Omega'\right)\delta\left(\omega-\omega'\right)$$

- C_A part cancels!
- leading pole is diagonal in color indices
- result is proportional to the tree level but the dependence on Ω' must be extrapolated from the LP result

Auxiliary slide: Fixed order check

For arbitrary μ we then find

$$\Delta^{\text{LL}}_{\text{NLP}}(z,\mu) \quad = \quad \exp\left[4S^{\text{LL}}(\mu_h,\mu) - 4S^{\text{LL}}(\mu_s,\mu)\right] \times \frac{-8C_F}{\beta_0} \ln \frac{\alpha_s(\mu)}{\alpha_s(\mu_s)}\, \theta(1-z)$$

Note $\Delta^{\text{LL}}_{\text{NLP}}(z,\mu_c)$ has the same form $\rightarrow$ *no LL in collinear function!*

$$S^{\text{LL}}(\mu_1,\mu_2) = -\frac{\alpha_s C_F}{2\pi} \ln^2 \frac{\mu_2}{\mu_1} \quad \text{and} \quad \frac{1}{\beta_0} \ln \frac{\alpha_s(\mu_1)}{\alpha_s(\mu_2)} = \frac{\alpha_s}{2\pi} \ln \frac{\mu_2}{\mu_1}$$

Our result

$$\Delta^{\text{LL}}_{\text{NLP}}(z,\mu) = \frac{\hat{\sigma}^{\text{LL}}_{\text{NLP}}(z,\mu)}{z} \quad = \quad \exp\left[2\frac{\alpha_s C_F}{\pi} \ln^2 \frac{\mu}{\mu_s} - 2\frac{\alpha_s C_F}{\pi} \ln^2 \frac{\mu}{\mu_h}\right] \times(-4)\frac{\alpha_s C_F}{\pi} \ln \frac{\mu_s}{\mu}\, \theta(1-z)$$

agrees with

- R. Hamberg, W. L. van Neerven and T. Matsuura, 1991, full fixed order NNLO computation
- D. de Florian, J. Mazzitelli, S. Moch and A. Vogt, 2014 approximate results for $\mu = \mu_h$ up to N^4LO

Auxiliary slide: RGE for kinematic soft functions – Higgs case

Proceeding like in the example we obtain

$$\begin{aligned} \frac{d}{d\ln\mu}\,\vec{S}(x^0) &= \Big[2\Gamma_{\mathrm{cusp}}L_0 - 2\gamma_W\Big]\,\mathbf{1}\,\vec{S}(x) \\ &+ \Gamma_{\mathrm{cusp}}\begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & -6 & +3 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 0 \end{pmatrix}\vec{S}(x^0) \end{aligned}$$

with $\vec{S}(x^0) = \Big(\widetilde{S}_{K1}, \widetilde{S}_{K2}, \widetilde{S}_{K3}, \widetilde{S}_{x_0}\Big)^T$

$$\begin{aligned} \frac{d}{d\ln\mu}\,\widetilde{S}_{K1+K2+K3}(x^0) =& \Big[2\Gamma_{\mathrm{cusp}}L_0 - 2\gamma_W\Big]\,\widetilde{S}_{K1+K2+K3}(x^0) \\ & -4\Gamma_{\mathrm{cusp}}\widetilde{S}_{x_0}(x^0) - 6\,\Gamma_{\mathrm{cusp}}\,\widetilde{S}_{K3}(x^0)\,, \end{aligned}$$

Note: $\widetilde{S}_{K1+K2+K3}(x^0) = \mathcal{O}\left(\alpha_s \ln\frac{\mu}{\mu_s}\right)$