Soft corrections to inclusive cross sections at four loops and beyond

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Matter To The Deepest: Recent Developments In Physics Of Fundamental Interactions, Katowice, Sep 04, 2019

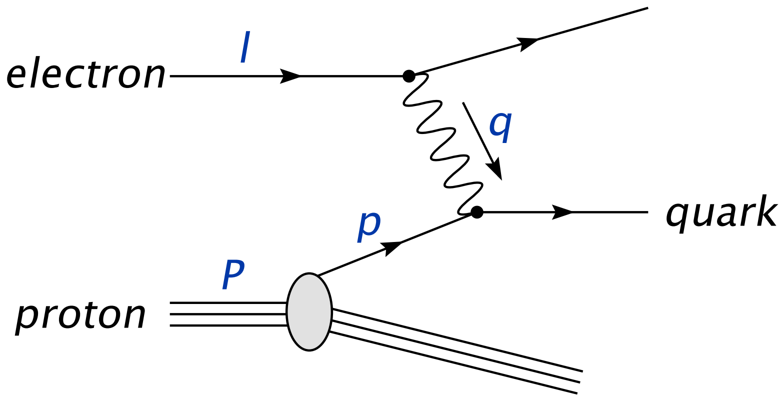
Based on work done in collaboration with:

- *Soft corrections to inclusive DIS at four loops and beyond*
  G. Das, S. M., and A. Vogt arXiv:1908.03071

Plan

- A guided tour through physical quantities in soft and collinear limit
  - coefficient functions in deep-inelastic scattering
  - quark form factor in QCD
  - QCD splitting functions at large-x

Deep-inelastic scattering



Kinematic variables

- momentum transfer $Q^2 = -q^2$
- Bjorken variable $x = Q^2/(2p \cdot q)$

- Structure functions (up to order $\mathcal{O}(1/Q^2)$)

$$F_a\left(x, Q^2\right) = \sum_i \left[C_{a,i}\left(\alpha_s(\mu^2), \mu^2/Q^2\right) \otimes PDF\left(\mu^2\right)\right](x)$$

- Coefficient functions up to N^{3}LO

$$C_{a,i} = \alpha_s^n \left(c_{a,i}^{(0)} + \alpha_s c_{a,i}^{(1)} + \alpha_s^2 c_{a,i}^{(2)} + \alpha_s^3 c_{a,i}^{(3)} + \ldots\right)$$

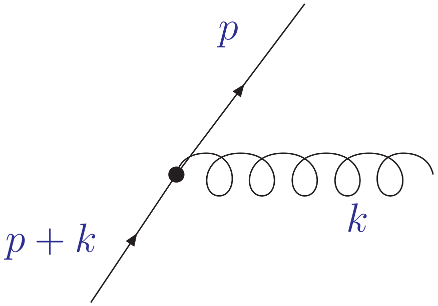
- Evolution equations up to N^{3}LO
  - non-singlet ($2n_f - 1$ scalar) and singlet (2×2 matrix) equations

$$\frac{d}{d \ln \mu^2} PDF(x, \mu^2) = \left[P(\alpha_s(\mu^2)) \otimes PDF(\mu^2)\right](x)$$

  - splitting functions $P_{ij} = \alpha_s P_{ij}^{(0)} + \alpha_s^2 P_{ij}^{(1)} + \alpha_s^3 P_{ij}^{(2)} + \alpha_s^4 P_{ij}^{(3)} + \ldots$

Soft and collinear corrections

- Soft and collinear regions of phase space
  - double logarithms from singular regions in Feynman diagrams
  - propagator vanishes for: $E_g = 0$, soft θ_{qg} = 0 collinear



$$\frac{1}{(p+k)^2} = \frac{1}{2p\cdot k} = \frac{1}{2E_qE_g(1-\cos\theta_{qg})}$$

$$\alpha_s \int d^4k \frac{1}{(p+k)^2} \longrightarrow \alpha_s \int dE_g\, d\sin\theta_{qg}\, \frac{1}{2E_qE_g(1-\cos\theta_{qg})}$$

$$\longrightarrow \alpha_s \ln^2(\dots)$$

- Improved perturbation theory: resum logarithms to all orders
  - long history of resummation Sterman '87; Catani, Trentadue '88; . . .
  - reorganize perturbative expansion $\longrightarrow$ stability
  - generating functional for higher orders of perturbation theory

$$\begin{aligned} \mathcal{O} &= 1 + \alpha\left(\ln^2 + \ln + 1\right) + \alpha^2\left(\ln^4 + \ln^3 + \ln^2 + \ln + 1\right) + \dots \\ &= \left(1 + \alpha\, 1 + \alpha^2\, 1 + \dots\right)\exp\left(\alpha\, \ln^2 + \alpha\, \ln + \alpha^2\, \ln + \dots\right) \end{aligned}$$

Coefficient functions at large N/ large x

- Coefficient function in large x-limit have large logarithms at n^{th}-order

$$\alpha_s^n \, \frac{\ln^{2n-1}(1-x)}{(1-x)_+} \longleftrightarrow \alpha_s^n \, \ln^{2n}(N)$$

- Threshold resummation in Mellin space

$$C^N \;=\; (1 + \alpha_s \, g_{01} + \alpha_s^2 \, g_{02} + \ldots) \cdot \exp \left(G^N\right) \;+\; \mathcal{O}(N^{-1} \ln^n N)$$

- Control over logarithms $\ln(N)$ with $\lambda = \beta_0 \alpha_s \ln(N)$ to N^kLL accuracy

$$G^N \;=\; \ln(N) g_1(\lambda) + g_2(\lambda) + \alpha_s g_3(\lambda) + \alpha_s^2 g_4(\lambda) + \alpha_s^3 g_5(\lambda) + \ldots$$

  - $g_1(\lambda)$: LL Sterman '87; Appell, Mackenzie, Sterman '88
  - $g_2(\lambda)$: NLL Catani Trenatdue '89
  - $g_3(\lambda)$: NNLL or N^2LL Vogt '00; Catani, Grazzini, de Florian, Nason '03
  - $g_4(\lambda)$: N^3LL S.M., Vermaseren, Vogt '05
  - $g_5(\lambda)$: N^4LL Das, S.M., Vogt '19
- Resummed G^N predicts fixed orders in perturbation theory
  - generating functional for towers of large logarithms

Resummation exponent G^N

- Factorization in soft and collinear limit $\longrightarrow$ product of radiative factors

$$G^N \quad = \quad \ln \Delta_{\rm q} + \ln J_{\rm q} + \Delta^{\rm int}$$

- Renormalization group equations for radiative factors $\Delta_{\rm q}$, $J_{\rm q}$ and $\Delta^{\rm int}$

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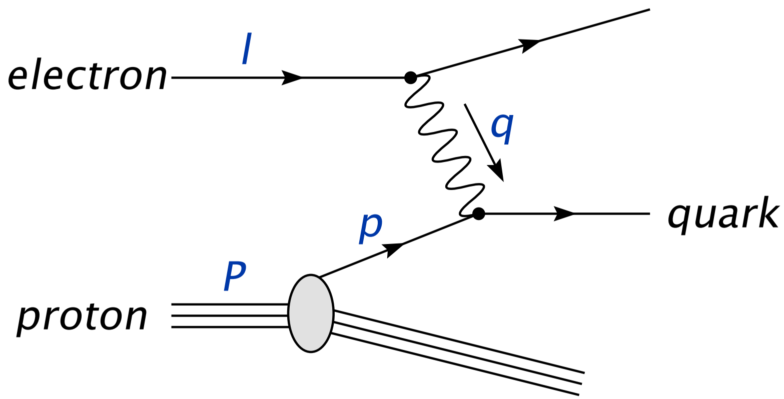
- $\Delta_{\rm p}$: soft collinear radiation off initial state parton p

$$\ln \Delta_{\rm p} \quad = \quad \int\limits_0^1 dz \frac{z^{N-1} - 1}{1-z} \int\limits_{\mu_f^2}^{(1-z)^2 Q^2} \frac{dq^2}{q^2} A^{\rm p}\left(\alpha_s(q^2)\right)$$



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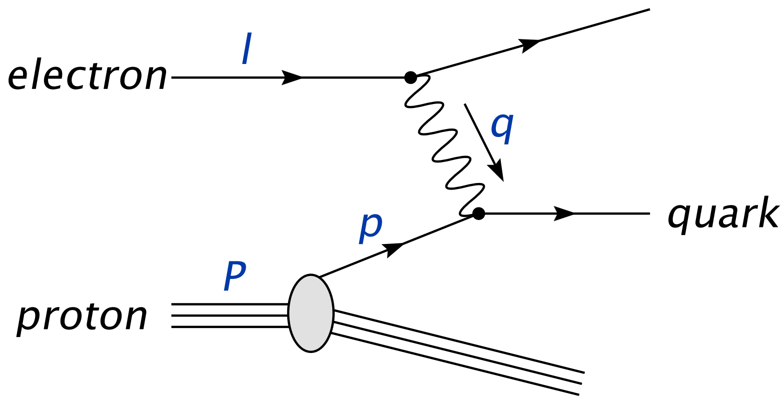
$$\ln J_{\rm p} \quad = \quad \int_0^1 dz \frac{z^{N-1} - 1}{1-z} \left[\int_{(1-z)^2 Q^2}^{(1-z) Q^2} \frac{dq^2}{q^2} A^{\rm p}(\alpha_s(q^2)) + B^J(\alpha_s([1-z]Q^2)) \right]$$



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- $\Delta^{\rm int}$: process dependent gluon emission at large angles

  $\Delta^{\rm int} = 0$ in DIS to all orders Forte, Ridolfi ‘02; Gardi, Roberts ‘02

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Challenge

- Determination of cusp anomalous dimension $A^{\rm p}$ and evolution kernel B^J for $J_{\rm p}$ at four loops

Virtual corretions and real emissions

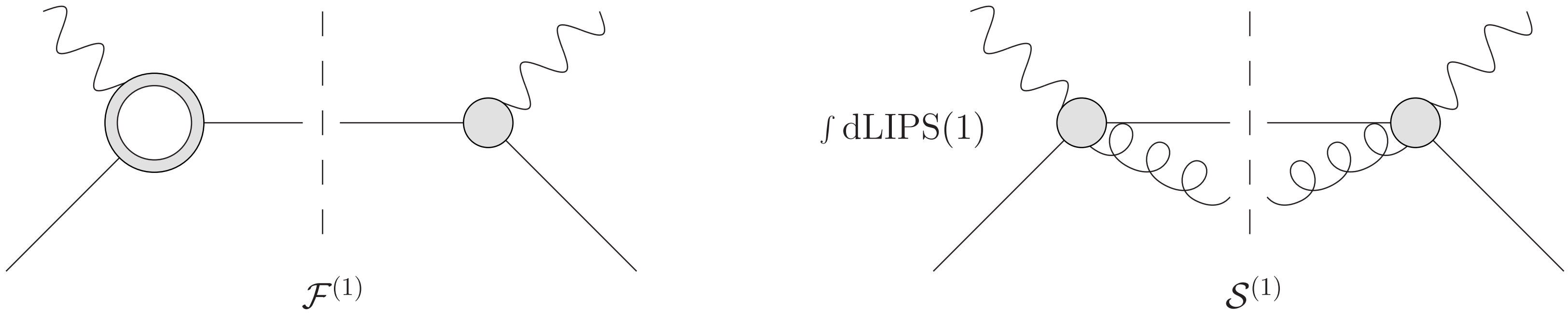
Soft and collinear factorization in $D = 4 - 2\epsilon$-dimensions

- Bare (partonic) structure function $\mathcal{T}_n$ in $D = 4 - 2\epsilon$-dimensions
- $\mathcal{T}_n$ combines
  - virtual corrections $\mathcal{F}_n$ (dependent on $\delta(1-x)$)
  - pure real-emission contributions $\mathcal{S}_n$
    (dependent on D-dimensional +-distributions $f_{k,\epsilon}$)

$$f_{k,\epsilon}(x) \;=\; \epsilon\big[\,(1-x)^{-1-k\epsilon}\,\big]_+ \;=\; -\frac{1}{k}\,\delta(1-x) + \epsilon \sum_{i=0} \frac{(-k\epsilon)^i}{i\,!}\,\frac{\ln^i(1-x)}{(1-x)_+}$$

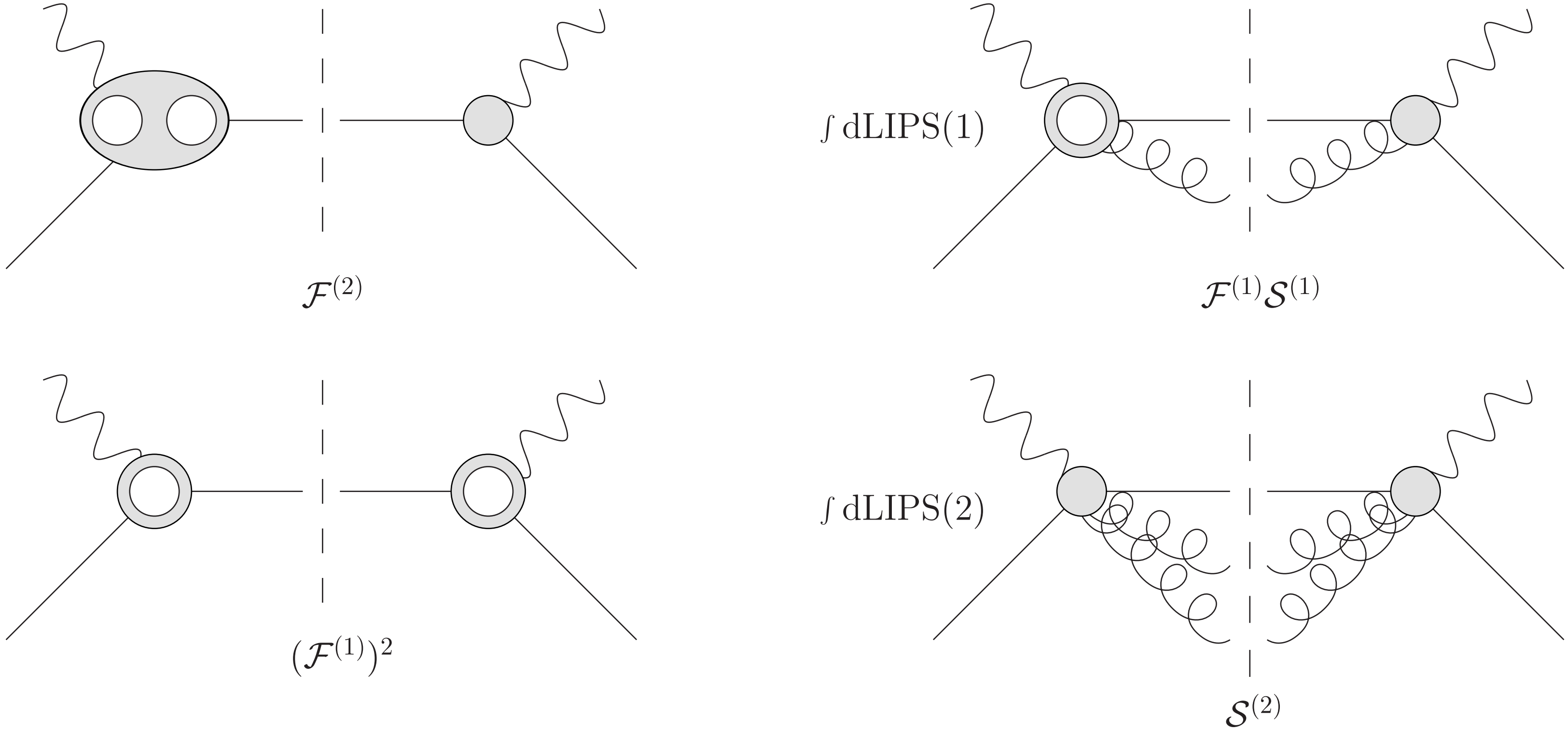
- Laurent-series for $\mathcal{T}_n$ at n$^{\rm th}$-order
  - mass-factorization predicts $\frac{1}{\epsilon^n}$
  - soft and collinear singularities in $\mathcal{F}_n$ and $\mathcal{S}_n$ behave as $\frac{1}{\epsilon^{2n}}$
- Infrared finiteness implies cancellation of poles between $\mathcal{F}_n$ and $\mathcal{S}_n$
  Kinoshita ‘62; Lee, Nauenberg ‘64
- Constructive approach to $\mathcal{F}_n$ and $\mathcal{S}_n$

Factorization of the result (1 loop)



$$\mathcal{T}_1^{\mathrm{b}} \quad = \quad 2\,\mathsf{Re}\,\mathcal{F}_1\,\delta(1-x) + \mathcal{S}_1$$

Factorization of the result (2 loops and higher)



$$\mathcal{T}_2^{\rm b} = (2\,\text{Re}\,\mathcal{F}_2 + |\mathcal{F}_1|^2)\delta(1-x) + 2\,\text{Re}\,\mathcal{F}_1\mathcal{S}_1 + \mathcal{S}_2$$

$$\mathcal{T}_3^{\rm b} = (2\,\text{Re}\,\mathcal{F}_3 + 2\,|\mathcal{F}_1\mathcal{F}_2|)\,\delta(1-x) + (2\,\text{Re}\,\mathcal{F}_2 + |\mathcal{F}_1|^2)\mathcal{S}_1 + 2\,\text{Re}\,\mathcal{F}_1\mathcal{S}_2 + \mathcal{S}_3$$

$$\mathcal{T}_4^{\rm b} = (2\,\text{Re}\,\mathcal{F}_4 + |\mathcal{F}_2|^2 + 2\,|\mathcal{F}_1\mathcal{F}_3|)\,\delta(1-x)$$
$$+(2\,\text{Re}\,\mathcal{F}_3 + 2\,|\mathcal{F}_1\mathcal{F}_2|)\,\mathcal{S}_1 + (2\,\text{Re}\,\mathcal{F}_2 + |\mathcal{F}_1|^2)\mathcal{S}_2 + 2\,\text{Re}\,\mathcal{F}_1\mathcal{S}_3 + \mathcal{S}_4$$

QCD splitting functions at large-x

- Splitting functions (diagonal) in the large-x limit

$$P_{ii}^{(n-1)}(x) \;=\; \frac{A_{n,\mathrm{i}}}{(1-x)_+} + B_{n,\mathrm{i}}\,\delta(1-x) + C_{n,\mathrm{i}}\,\ln(1-x) + D_{n,\mathrm{i}}$$

- Cusp anomalous dimensions related by Casimir scaling up to three loops

$A_{n,\mathrm{g}} = \frac{C_A}{C_F} A_{n,\mathrm{q}}$ for $n \leq 3$

- Casimir scaling at four loops broken due to new color factors
  - $A_{n,\mathrm{q}}$ contains $\frac{d_F^{abcd} d_A^{abcd}}{n_F}$ and $\frac{d_F^{abcd} d_F^{abcd}}{n_F}$
  - $A_{n,\mathrm{g}}$ contains $\frac{d_A^{abcd} d_A^{abcd}}{n_A}$, $\frac{d_F^{abcd} d_A^{abcd}}{n_A}$ and $\frac{d_F^{abcd} d_F^{abcd}}{n_A}$
- Large n_c-limit at four loops restores Casimir scaling — Dixon '17

$$A_{4,\mathrm{g}}\big|_{\mathrm{large-n_c}} = \frac{C_A}{C_F}\, A_{4,\mathrm{q}}\big|_{\mathrm{large-n_c}}$$

Quark and gluon cusp anomalous dimensions

- Large-n_c limit of quark cusp anomalous dimension (Henn, Lee, Smirnov, Smirnov, Steinhauser '16; S. M., Ruijl, Ueda, Vermaseren, Vogt '17)

$$
\begin{aligned}
A_{4,\mathrm{q}}\big|_{\mathrm{large-n_c}} = \; & C_F n_c^3 \left(\frac{84278}{81} - \frac{88832}{81}\,\zeta_2 + \frac{20992}{27}\,\zeta_3 + 1804\zeta_4 - \frac{352}{3}\,\zeta_2\zeta_3 - 352\zeta_5 \right. \\
& \left. -32\zeta_3^2 - 876\,\zeta_6 \right) \\
& - C_F n_c^2\, n_f \left(\frac{39883}{81} - \frac{26692}{81}\,\zeta_2 + \frac{16252}{27}\,\zeta_3 + \frac{440}{3}\,\zeta_4 - \frac{256}{3}\,\zeta_2\zeta_3 - 224\,\zeta_5 \right) \\
& + C_F n_c\, n_f^2 \left(\frac{2119}{81} - \frac{608}{81}\,\zeta_2 + \frac{1280}{27}\,\zeta_3 - \frac{64}{3}\,\zeta_4 \right) - C_F n_f^3 \left(\frac{32}{81} - \frac{64}{27}\,\zeta_3 \right)
\end{aligned}
$$

- Result includes leading-n_c parts of quartic Casimir contributions $\dfrac{d_F^{abcd} d_A^{abcd}}{n_F}$ and $\dfrac{d_F^{abcd} d_F^{abcd}}{n_F}$

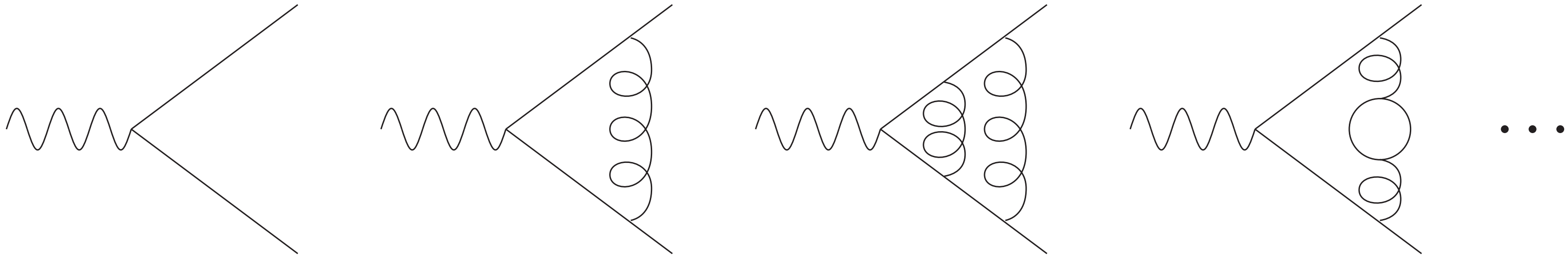
Generalized 'Casimir scaling'

S. M., Ruijl, Ueda, Vermaseren, Vogt '18

quark	gluon	$A_{4,\mathrm{q}}$	$A_{4,\mathrm{g}}$
C_F^4	–	0	–
$C_F^3 C_A$	–	0	–
$C_F^2 C_A^2$	–	0	–
$C_F C_A^3$	C_A^4	610.25 ± 0.1	
$d_{FA}^{(4)}/n_F$	$d_{AA}^{(4)}/n_A$	-507.0 ± 2.0	-507.0 ± 5.0
$n_f\, C_F^3$	$n_f\, C_F^2 C_A$	-31.05543	
$n_f\, C_F^2 C_A$	$n_f\, C_F C_A^2$	38.79538	
$n_f\, C_F C_A^2$	$n_f C_A^3$	-440.6670	
$n_f\, d_{FF}^{(4)}/n_F$	$n_f\, d_{FA}^{(4)}/n_A$	-123.8949	-124.0 ± 0.6
$n_f^2\, C_F^2$	$n_f^2\, C_F C_A$	-21.31439	
$n_f^2\, C_F C_A$	$n_f^2\, C_A^2$	58.36737	
–	$n_f^2\, d_{FF}^{(4)}/n_A$	–	0.0 ± 0.1
$n_f^3\, C_F$	$n_f^3\, C_A$	2.454258	2.454258

- Exact results (rounded to seven digits) for $n_f\, C_F^3$ by Grozin '18 and for $n_f\, C_F^2 C_A$, $n_f\, C_F C_A^2$, $n_f\, d_{FF}^{(4)}/n_F$ by Henn, Peraro, Stahlhofen, Wasser '19 all in agreement with approximations from S.M., Ruijl, Ueda, Vermaseren, Vogt '17

Quark form factor in QCD



- QCD corrections to vertex $\gamma^* q\bar{q}$, i.e. $\Gamma_\mu = ie_{\rm q} \left(\bar{u}\, \gamma_\mu\, u\right) \mathcal{F}_{\rm q}(Q^2, \alpha_{\rm s})$
  - gauge invariant quantity
  - infrared divergent (dimesional regularization $D = 4 - 2\epsilon$)
- Form factor $\mathcal{F}(Q^2, \alpha_s)$ exponentiates Collins '80; Sen '81; Korchemsky '88; Magnea, Sterman '90; Contopanagos, Laenen, Sterman '97; Magnea '00 (long history)

$$Q^2 \frac{\partial}{\partial Q^2} \ln \mathcal{F}\left(Q^2, \alpha_s, \epsilon\right) = \frac{1}{2}\, K(\alpha_s, \epsilon) + \frac{1}{2}\, G\left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon\right) .$$

- Renormalization group equations for functions G and K
  - all Q^2-scale dependence in G (finite in ϵ)
  - pure counter term function K (contains poles in $\frac{1}{\epsilon}$)
- Cusp anomalous dimension A governs evolution for G and K

Solution

- Solution for $\ln \mathcal{F}(Q^2, \alpha_s, \epsilon)$ in D-dimensions
  - boundary condition $\mathcal{F}(0, \alpha_s, \epsilon) = 1$

$$\ln \mathcal{F}\left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon\right) =$$

$$\frac{1}{2} \int\limits_0^{Q^2/\mu^2} \frac{d\xi}{\xi}\left(K(\alpha_s, \epsilon) + G(1, \bar{a}(\xi\mu^2, a_s, \epsilon), \epsilon) + \int\limits_\xi^1 \frac{d\lambda}{\lambda} A(\bar{a}(\lambda\mu^2, \epsilon))\right)\Bigg\}$$

  - use running coupling in D-dimensions from

$$\lambda \frac{\partial}{\partial \lambda} \bar{a}(\lambda, \alpha_s, \epsilon) \quad = \quad -\epsilon\, \bar{a}(\lambda, \alpha_s, \epsilon) \; - \beta_0\, \bar{a}^2(\lambda, \alpha_s, \epsilon) - \ldots$$

  - boundary condition $\bar{a}(1, \alpha_s, \epsilon) = \alpha_s$

Upshot

- Generating functional for Laurent-series in ϵ to all orders

Result

- Result up to four loops in terms of expansion coefficients of A and G

$$
\begin{aligned}
\mathcal{F}_1 &= -\frac{1}{2}\frac{1}{\epsilon^2}A_1 - \frac{1}{2}\frac{1}{\epsilon}G_1 \\
\mathcal{F}_2 &= \frac{1}{8}\frac{1}{\epsilon^4}A_1^2 + \frac{1}{8}\frac{1}{\epsilon^3}A_1(2G_1 - \beta_0) + \frac{1}{8}\frac{1}{\epsilon^2}(G_1^2 - A_2 - 2\beta_0 G_1) - \frac{1}{4}\frac{1}{\epsilon}G_2 \\
\mathcal{F}_3 &= -\frac{1}{48}\frac{1}{\epsilon^6}A_1^3 - \frac{1}{16}\frac{1}{\epsilon^5}A_1^2(G_1 - \beta_0) - \frac{1}{144}\frac{1}{\epsilon^4}A_1(9G_1^2 - 9A_2 - 27\beta_0 G_1 + 8\beta_0^2) \\
&\quad - \frac{1}{144}\frac{1}{\epsilon^3}(3G_1^3 - 9A_2 G_1 - 18A_1 G_2 + 4\beta_1 A_1 - 18\beta_0 G_1^2 + 16\beta_0 A_2 + 24\beta_0^2 G_1) \\
&\quad + \frac{1}{72}\frac{1}{\epsilon^2}(9G_1 G_2 - 4A_3 - 6\beta_1 G_1 - 24\beta_0 G_2) - \frac{1}{6}\frac{1}{\epsilon}G_3 \\
\mathcal{F}_4 &= \ldots
\end{aligned}
$$

- Expansion in terms of bare coupling $a_s^{\mathrm{b}} = \alpha_s^{\mathrm{b}}/(4\pi)$

$$
\mathcal{F}\left(Q^2, \alpha_s^{\mathrm{b}}\right) = 1 + \sum_{l=1} \left(a_s^{\mathrm{b}}\right)^l \left(\frac{Q^2}{\mu^2}\right)^{-l\epsilon} \mathcal{F}_l
$$

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$$\mathcal{F}_1 = -\frac{1}{2}\frac{1}{\epsilon^2}A_1 - \frac{1}{2}\frac{1}{\epsilon}G_1$$

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$$\begin{aligned}\mathcal{F}_3 = & -\frac{1}{48}\frac{1}{\epsilon^6}A_1^3 - \frac{1}{16}\frac{1}{\epsilon^5}A_1^2(G_1 - \beta_0) - \frac{1}{144}\frac{1}{\epsilon^4}A_1(9G_1^2 - 9A_2 - 27\beta_0 G_1 + 8\beta_0^2) \\ & - \frac{1}{144}\frac{1}{\epsilon^3}(3G_1^3 - 9A_2G_1 - 18A_1G_2 + 4\beta_1 A_1 - 18\beta_0 G_1^2 + 16\beta_0 A_2 + 24\beta_0^2 G_1) \\ & + \frac{1}{72}\frac{1}{\epsilon^2}(9G_1G_2 - 4A_3 - 6\beta_1 G_1 - 24\beta_0 G_2) - \frac{1}{6}\frac{1}{\epsilon}G_3\end{aligned}$$

$$\mathcal{F}_4 = \ldots$$

$\mathcal{F}_2$: Hamberg, van Neerven, Matsuura ‘88; Harlander ‘00; Gehrmann, Huber, Maitre ’05; S.M. Vermaseren, Vogt ’05

$\mathcal{F}_3$: S.M. Vermaseren, Vogt ’05; Baikov, Chetyrkin, Smirnov, Smirnov, Steinhauser ‘09; Gehrmann, Glover, Huber, Ikizlerli, Studerus ‘10

$\mathcal{F}_4$: Henn, Smirnov, Smirnov, Steinhauser, Lee ‘16; Lee, Smirnov, Smirnov, Steinhauser ‘17 & ‘19; von Manteuffel, Schabinger ‘19

Universality of subleading infrared poles

- Universal subleading infrared poles in function G Dixon, Magnea, Sterman '08
- Coefficients G_n at n-loops are composed of:
  - twice the $\delta(1-x)$ part $B^{\rm q}$ in parton splitting function
  - single-logarithmic anomalous dimension of *eikonal* form factor
  - terms associated with QCD beta function

$$
\begin{aligned}
G_1 &= 2B_1^{\rm q} + f_1^{\rm q} + \varepsilon f_{01}^{\rm q}\,, \\
G_2 &= 2B_2^{\rm q} + (f_2^{\rm q} + \beta_0 f_{01}^{\rm q}) + \varepsilon f_{02}^{\rm q}\,, \\
G_3 &= 2B_3^{\rm q} + (f_3^{\rm q} + \beta_1 f_{01}^{\rm q} + \beta_0 f_{02}^{\rm q}) + \varepsilon f_{03}^{\rm q}\,, \\
G_4 &= 2B_4^{\rm q} + (f_4^{\rm q} + \beta_2 f_{01}^{\rm q} + \beta_1 f_{02}^{\rm q} + \beta_0 f_{03}^{\rm q}) + \varepsilon f_{04}^{\rm q}
\end{aligned}
$$

- f-function shares maximal non-Abelian property and Casimir scaling with cusp anomalous dimensions

$$
\begin{aligned}
f_1^{\rm q} &= 0\,, \\
f_2^{\rm q} &= C_F\left\{C_A\left(\frac{808}{27} - \frac{22}{3}\zeta_2 - 28\zeta_3\right) + n_f\left(-\frac{112}{27} + \frac{4}{3}\zeta_2\right)\right\}, \\
f_3^{\rm q} &= \ldots
\end{aligned}
$$

$$
\begin{aligned}
f_4^{\mathrm{q}} \;=\;& C_F C_A^3 \Big(\frac{9364079}{6561} - \frac{1186735}{729}\zeta_2 - \frac{837988}{243}\zeta_3 + \frac{115801}{27}\zeta_4 + \frac{11896}{9}\zeta_2\zeta_3 + 3952\zeta_5 \\
& - \frac{4796}{9}\zeta_3^2 - \frac{129547}{54}\zeta_6 - 416\zeta_2\zeta_5 - 720\zeta_3\zeta_4 - 1700\zeta_7 - \frac{1}{24}\, f^{\mathrm{q}}_{4,\, d_{FA}^{(4)}} \Big) \\
& + \frac{d_{FA}^{(4)}}{n_c}\, f^{\mathrm{q}}_{4,\, d_{FA}^{(4)}} + C_F^3 n_f\; f^{\mathrm{q}}_{4,\, n_f C_F^3} + C_F^2 C_A n_f\; f^{\mathrm{q}}_{4,\, n_f C_F^2 C_A} + C_F C_A^2 n_f \Big(- \frac{243859}{432} \\
& + \frac{389228}{729}\zeta_2 + \frac{105193}{243}\zeta_3 - \frac{22667}{18}\zeta_4 - \frac{848}{9}\zeta_2\zeta_3 - \frac{860}{27}\zeta_5 + \frac{2740}{9}\zeta_3^2 + \frac{5179}{9}\zeta_6 \\
& + \frac{1}{24}\, b^{\mathrm{q}}_{4,\, d_{FF}^{(4)}} - \frac{1}{2}\, f^{\mathrm{q}}_{4,\, n_f C_F^2 C_A} - \frac{1}{4}\, f^{\mathrm{q}}_{4,\, n_f C_F^3} \Big) + n_f \frac{d_{FF}^{(4)}}{n_c} \Big(-384 + \frac{4544}{3}\zeta_2 \\
& - \frac{5312}{9}\zeta_3 - \frac{800}{3}\zeta_4 + 128\zeta_2\zeta_3 - \frac{21760}{9}\zeta_5 + \frac{1216}{3}\zeta_3^2 + \frac{1184}{9}\zeta_6 - 2\, b^{\mathrm{q}}_{4,\, d_{FF}^{(4)}} \Big) \\
& + C_F^2 n_f^2 \Big(\frac{16733}{486} - \frac{172}{9}\zeta_2 - \frac{4568}{81}\zeta_3 + \frac{64}{9}\zeta_4 + \frac{32}{3}\zeta_2\zeta_3 + \frac{304}{9}\zeta_5 \Big) \\
& + C_F C_A n_f^2 \Big(\frac{27875}{17496} - \frac{15481}{729}\zeta_2 + \frac{32152}{243}\zeta_3 + \frac{388}{9}\zeta_4 - \frac{224}{9}\zeta_2\zeta_3 - 112\zeta_5 \Big) \\
& + C_F n_f^3 \Big(- \frac{16160}{6561} - \frac{16}{81}\zeta_2 - \frac{400}{243}\zeta_3 + \frac{128}{27}\zeta_4 \Big)
\end{aligned}
$$

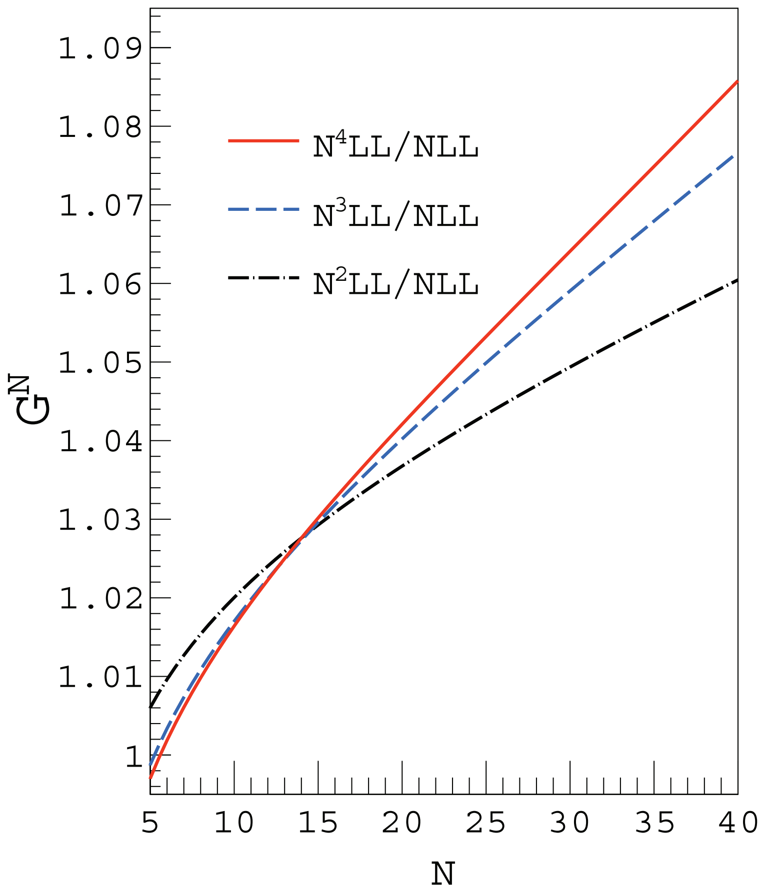
DIS coefficient functions at four loops

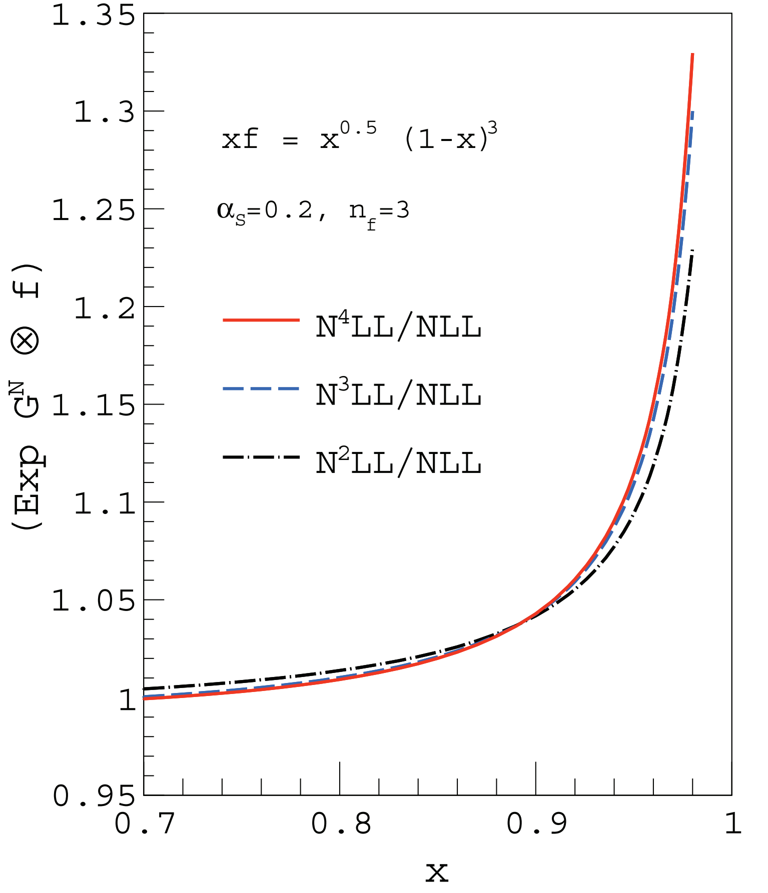
Result

- Four-loop coefficient function $c_{2,\mathrm{q}}^{(4)}$ known $\frac{\ln^7(1-x)}{(1-x)_+}, \dots, \frac{1}{(1-x)_+}$
- New result for $\frac{1}{(1-x)_+}$ term
  - best estimate (using partial large-n_c information)

$$
\begin{aligned}
c_{2,\mathrm{q}}^{(4)}\Big|_{\frac{1}{(1-x)_+},\mathrm{best}} \quad = \quad & (36200. \pm 1900.) + (-34700. \pm 1300.)\, n_f + 2062.715\, n_f^{\,2} \\
& -12.08488\, n_f^{\,3} + 47.55183\, n_f\, fl_{11}
\end{aligned}
$$

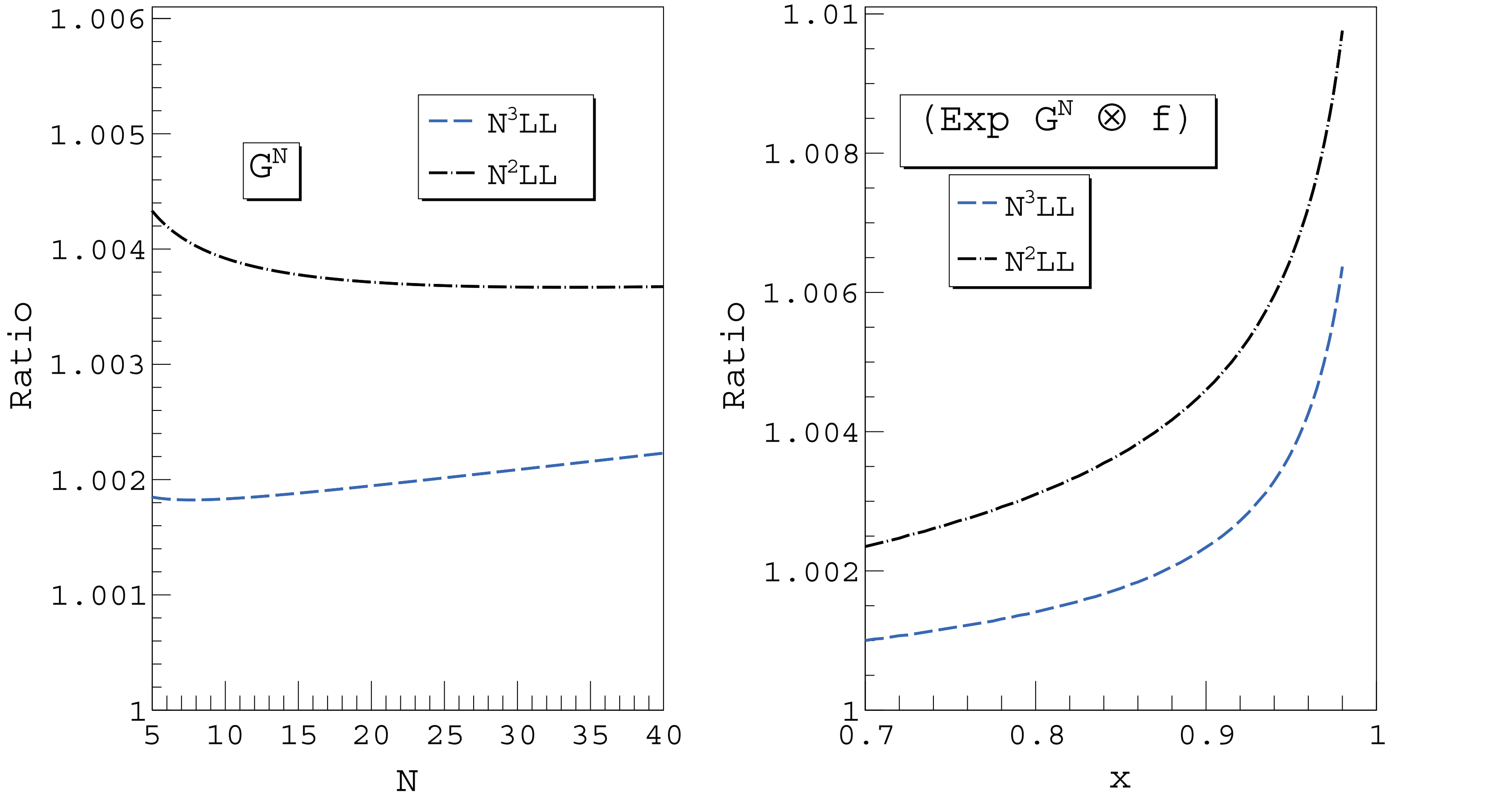
Numerical results for DIS (I)





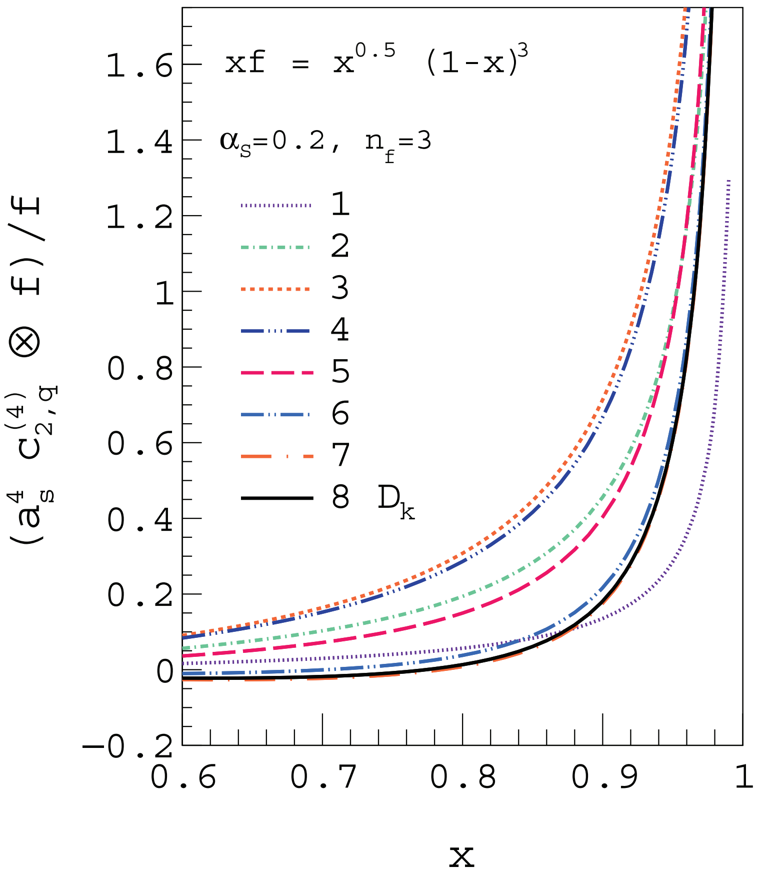
- Left: Resummed exponent G^N normalized to NLL for DIS plotted successively up to N^{4}LL for $\alpha_s = 0.2$ and $n_f = 3$
- Right: Resummed series convoluted with typical shape for a quark distribution $xf = x^{0.5}(1-x)^3$ up to N^{4}LL

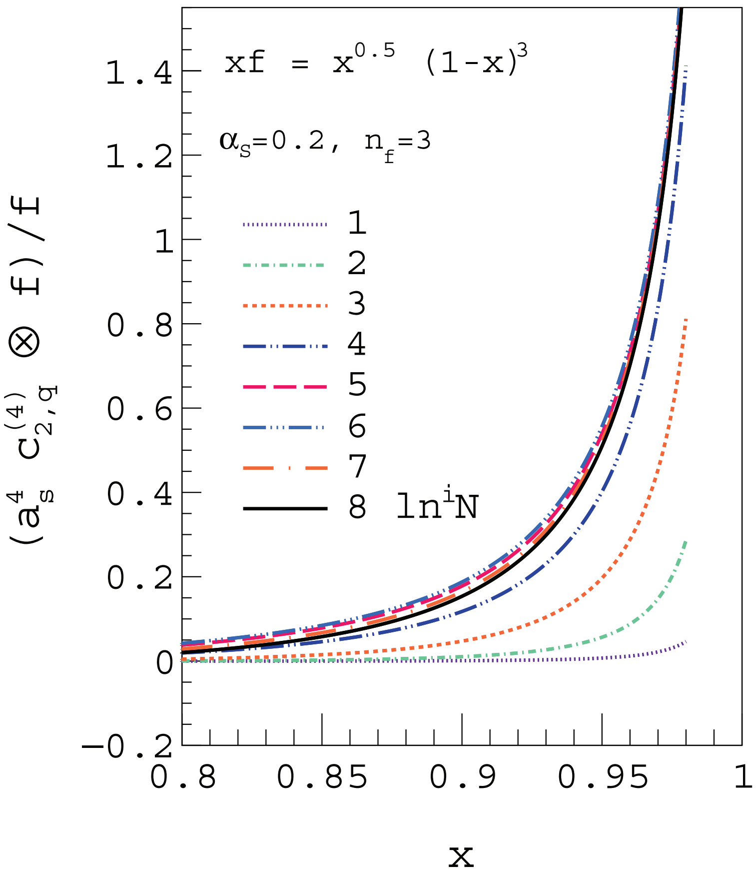
Numerical results for DIS (II)



- Comparison between large-n_c approximation and exact result at each resummed order for $\alpha_s = 0.2$ and $n_f = 3$ light flavors
- Left: Ratio for DIS resummed exponent G^N as function of Mellin-N
- Right: Ratio for resummed series convoluted with typical input shape $xf = x^{0.5}(1-x)^3$ plotted against x

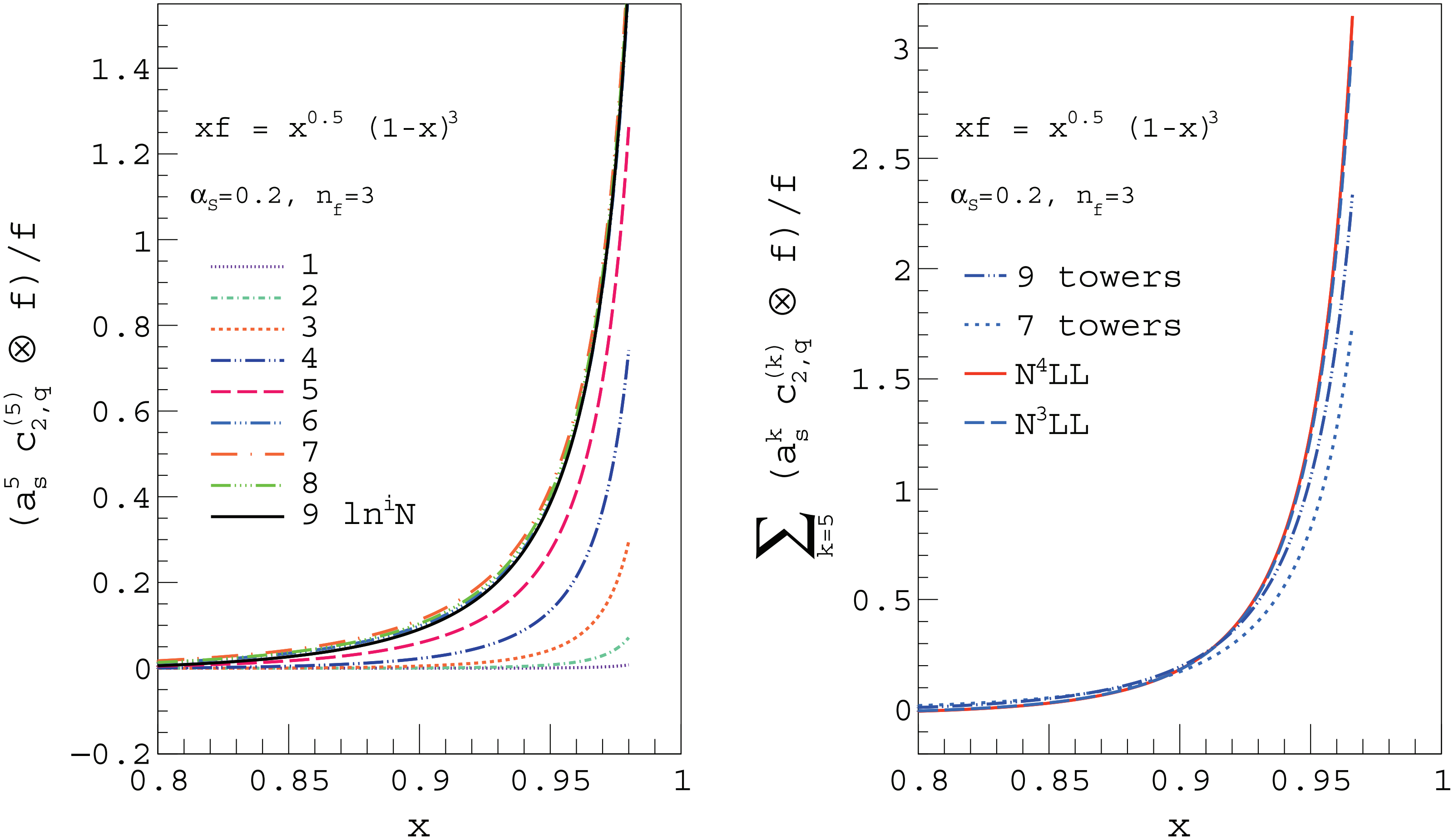
Numerical results for DIS (III)





- Left: DIS Wilson coefficient $c_{2,\mathrm{q}}^{(4)}$ convoluted with input shape xf with successive addition of plus-distributions $\mathcal{D}_k = \ln^k(1-x)/(1-x)_+$ starting from highest term
- Right: Same with the successive addition of the DIS N-space logarithms

Numerical results for DIS (IV)



- Left: Successive approximations of the five-loop coefficient function $c_{2,\mathrm{q}}^{(5)}$ by large-N terms illustrated by convolution with input shape xf
- Right: Corresponding results for effect of higher terms beyond α_s^4 obtained from tower expansion up to nine towers and from exponentiation up to N^{4}LL accuracy

Summary

- QCD radiative corrections known to higher orders
  - wealth of information in perturbation theory
  - factorization in soft and collinear limits
- Long-distance singularities in physical quantities
  - quark form factor in QCD exponentiates $\longrightarrow$ amplitude factorization
  - cancellation of soft and collinear divergences in observables $\longrightarrow$ factorization in D-dimensions
- Phenomenology for DIS
  - new estimate for four-loop coefficient function $c_{2,\mathrm{q}}^{(4)}$ down to $\dfrac{1}{(1-x)_+}$
  - resummation to N^{4}LL accuracy