Mellin-Barnes integrals: current applications and 3-loop prospects

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Matter To The Deepest

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Outline

Introduction

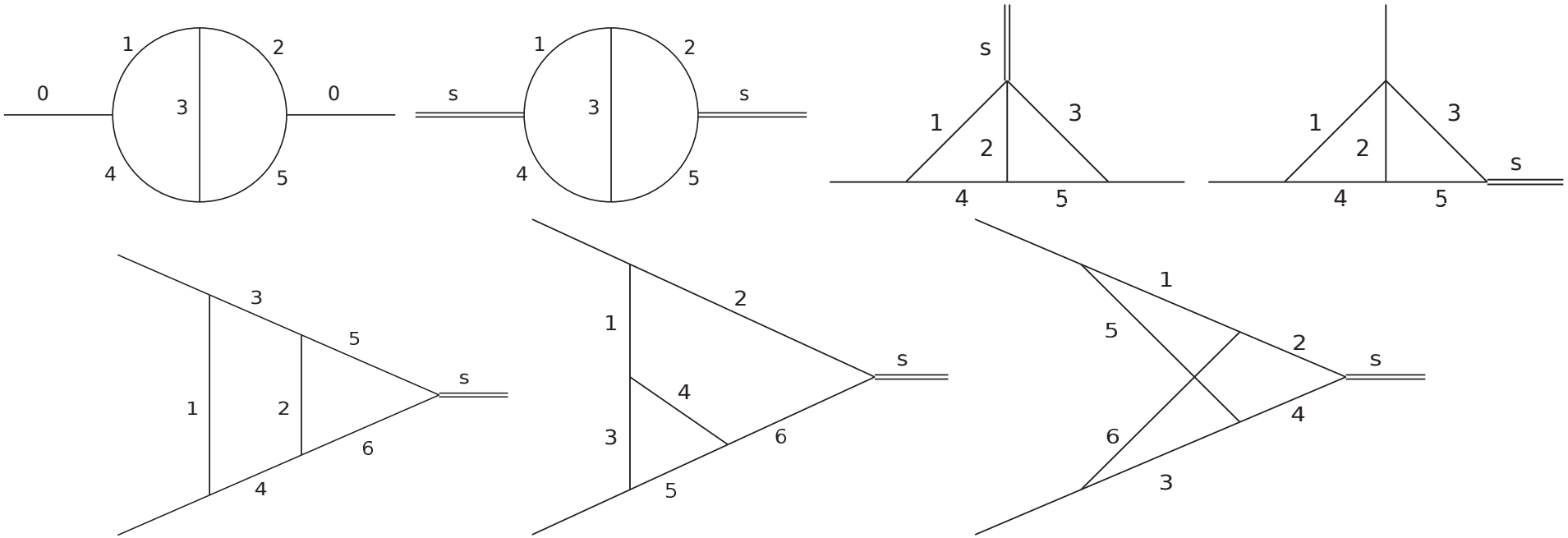
Order	Value $[10^{-4}]$	Order	Value $[10^{-4}]$
α	468.945	$\alpha_t^2\alpha_s$	1.362
$\alpha\alpha_s$	-42.655	α_t^3	0.123
$\alpha_t\alpha_s^2$	-7.074	$\alpha_{\rm ferm}^2$	3.866
$\alpha_t\alpha_s^3$	-1.196	$\alpha_{\rm bos}^2$	**-0.986**

MB method

Table: *Comparison of different orders of radiative corrections to* $\Delta\kappa_{\rm b}$.
Input Parameters: M_Z, Γ_Z, M_W, Γ_W, M_H, m_t, α_s *and* $\Delta\alpha$.

- "The two-loop electroweak bosonic corrections to $\sin^2\theta_{\rm eff}^{\rm b}$"
  Phys.Lett. B762 (2016) 184
- T.Riemann LL16 talk, PoS LL2016 (2016) 075: arXiv:1610.07059;
- J.Gluza LL16 talk, PoS LL2016 (2016) 034: arXiv:1607.07538

The bosonic Zbb topologies

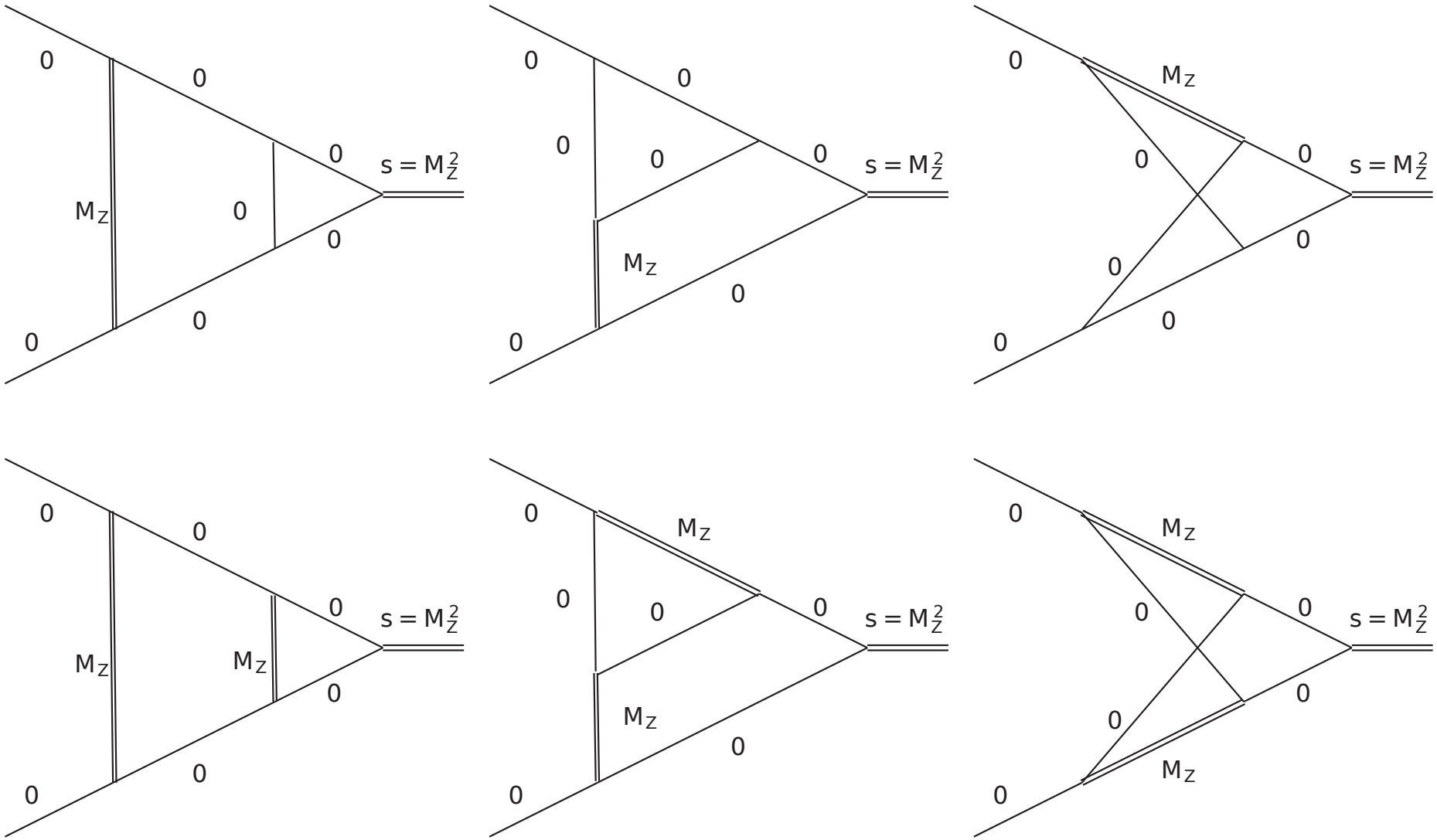


Parameters:
$s = M_Z^2$ and masses: M_Z , M_W , M_H , m_t , while $m_b = 0$;
$\rightarrow$ up to 3 dimensionless scales $\rightarrow$ fully numerical strategy.

Direct numerical integrations

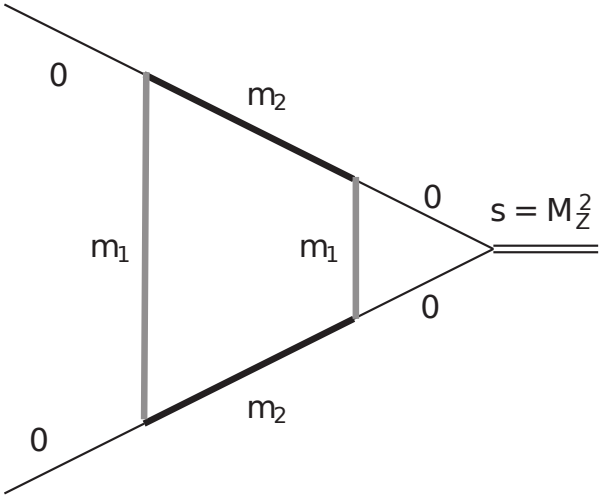
- Sector decomposition (SD)
  - FIESTA 3 [2014], *FIESTA 4 [2016] [A.V.Smirnov, V.A.Smirnov]
  - SecDec 3 [2015], *pySecDec [2017] [S. Borowka, G. Heinrich, et. al.]
- NICODEMOS, ver 2.0 [A. Freitas]
- The Mellin-Barnes (MB) method:
  - AMBRE 2 [J.Gluza, et. al., 2011], AMBRE 3 [I.Dubovyk, et. al., 2015]
  - PlanarityTest [I.Dubovyk, K.Bielas, 2013]
  - MB [M.Czakon, 2006], MBresolve [A.V.Smirnov, V.A.Smirnov, 2009]
  - MBnumerics [J.Usovitsch, I.Dubovyk, T.Riemann, 2015] – Minkowskian kinematics

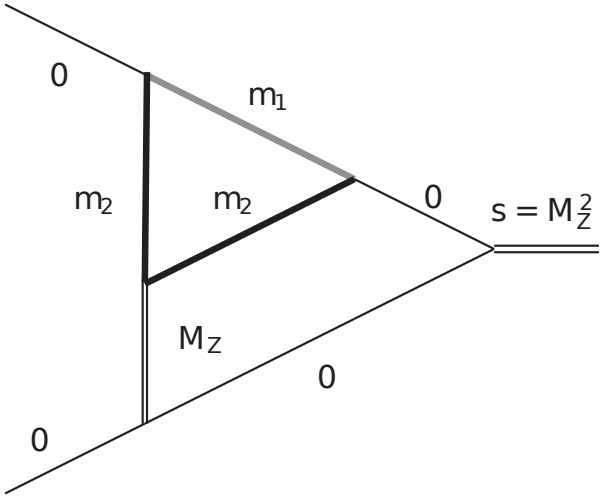
The most difficult cases I

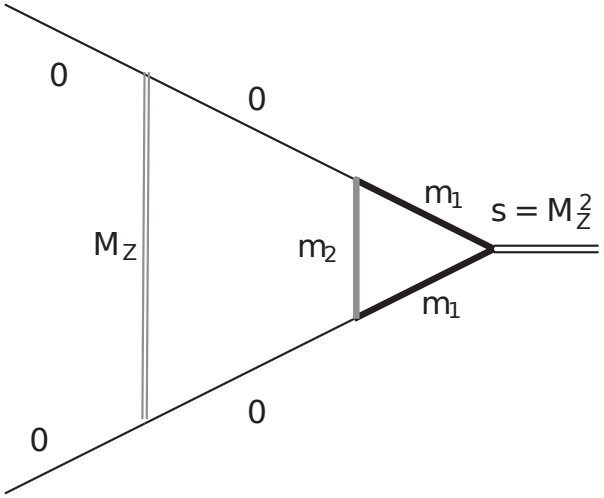


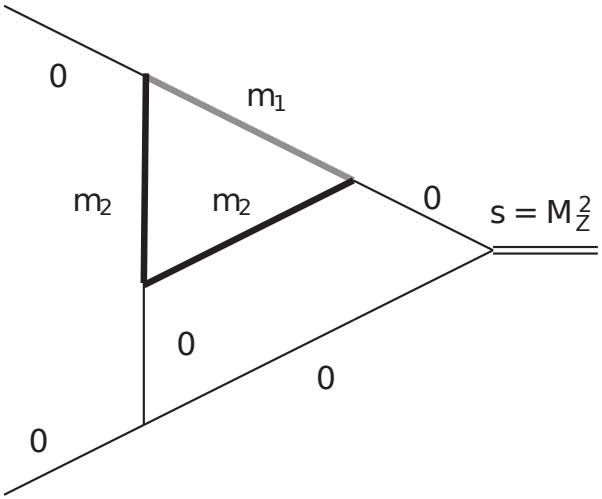
[R.Bonciani, et. al., 2004]

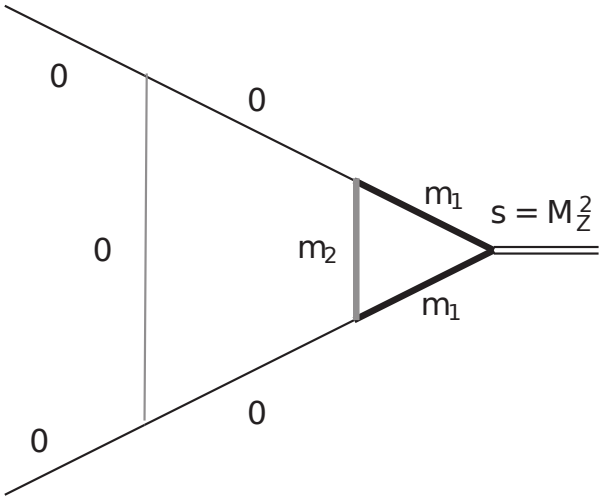
The most difficult cases II











General structure of the MB integrals after expansion in ϵ

$$I = \frac{1}{(2\pi i)^r} \int_{c_1 - i\infty}^{c_1 + i\infty} \cdots \int_{c_r - i\infty}^{c_r + i\infty} \prod_i^r dz_i \; \mathbf{F}(Z, S) \frac{\prod_j \mathbf{G_j}(N_j)}{\prod_k \mathbf{G_k}(N_k)}$$

$\mathbf{F}$ depends on: Z – linear combinations of r complex variables z_i,
$z_i = c_i + it_i$;
S – kinematic parameters and masses;

$\mathbf{G_i}$: Gamma and PolyGamma functions
N_i : linear combinations of z_i, e.g. $N_i = \sum_l \alpha_{il} z_l + \gamma_i$

In practice F is a product of powers of S:

$$\mathbf{F} \sim \prod_k X_k^{\sum_i (\alpha_{ki} z_i + \gamma_k)}$$

$$\alpha_{ij}, \gamma_i \in \text{Integer}, \quad X = \left\{ -\frac{s}{m_1^2}, \frac{m_1^2}{m_2^2}, \frac{s}{t}, \dots \right\}.$$

Asymptotic behavior in generalized spherical coordinates for $r \to \infty$

$$\lim_{r\to\infty} I \sim \frac{e^{-\beta r}}{r^{\alpha}} \qquad \beta = \beta(\vec{\theta}) \geq 0, \quad \alpha = \alpha(c_i) \text{ - arbitrary}$$

Euclidean kinematics: $\beta(\vec{\theta}) > 0 \quad \forall \vec{\theta}$
Minkowskian kinematics: $\beta(\vec{\theta}) = 0$ for some direction $\vec{\theta}'$

MBnumerics

- Contour rotations: $z_i = c_i + (i+\theta)t_i$
  to restore exponential damping factor
- Contour shifts: $z_i = c_i + n_i + it_i$
  to increase α and/or minimize integral
- Mappings: $t_i \to ln\left(\frac{x_i}{1-x_i}\right)$ vs $t_i \to tg\left(\pi(x_i - \frac{1}{2})\right)$
  to improve numerical convergence

Feynman parameters representation:

$$G(X) = \frac{(-1)^{N_\nu}\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\prod_{i=1}^{N}\Gamma(n_i)} \int \prod_{j=1}^{N} dx_j \; x_j^{n_j-1} \delta(1 - \sum_{i=1}^{N} x_i) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

General Mellin-Barnes relation:

$$\frac{1}{(A_1 + \ldots + A_n)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{(2\pi i)^{n-1}} \int_{-i\infty}^{i\infty} dz_1 \ldots dz_{n-1}$$
$$\times \prod_{i=1}^{n-1} A_i^{z_i} \; A_n^{-\lambda - z_1 - \ldots - z_{n-1}} \prod_{i=1}^{n-1} \Gamma(-z_i) \; \Gamma(\lambda + z_1 + \ldots + z_{n-1})$$

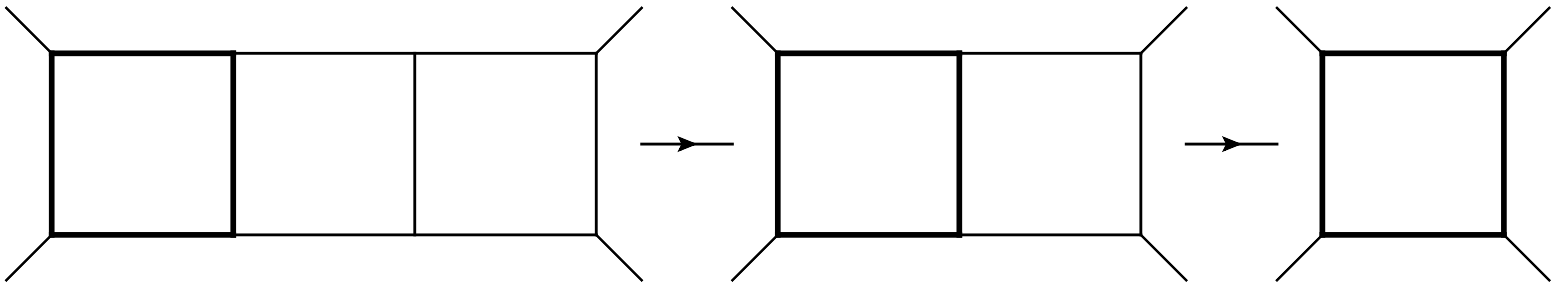
There are several ways to apply them to Feynman integrals:

- iteratively to each subloop – loop-by-loop (LA) approach (`AMBREv1.3.1` & `AMBREv2.1.1`)
- in one step to the complete U and F polynomials – global (GA) approach (`AMBREv3.1.1`)

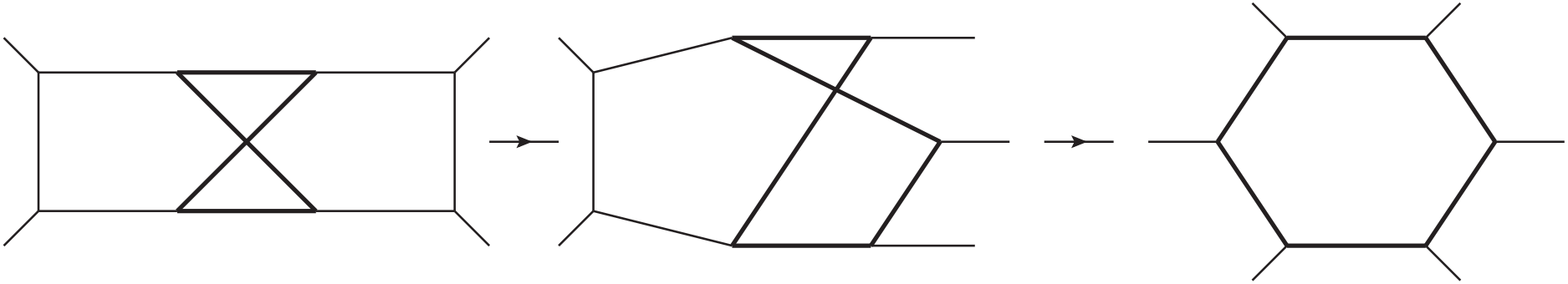
Examples, description, links to basic tools and literature:
http://us.edu.pl/~gluza/ambre/

Limitations of LA approach

Planar case:



Non-planar case:



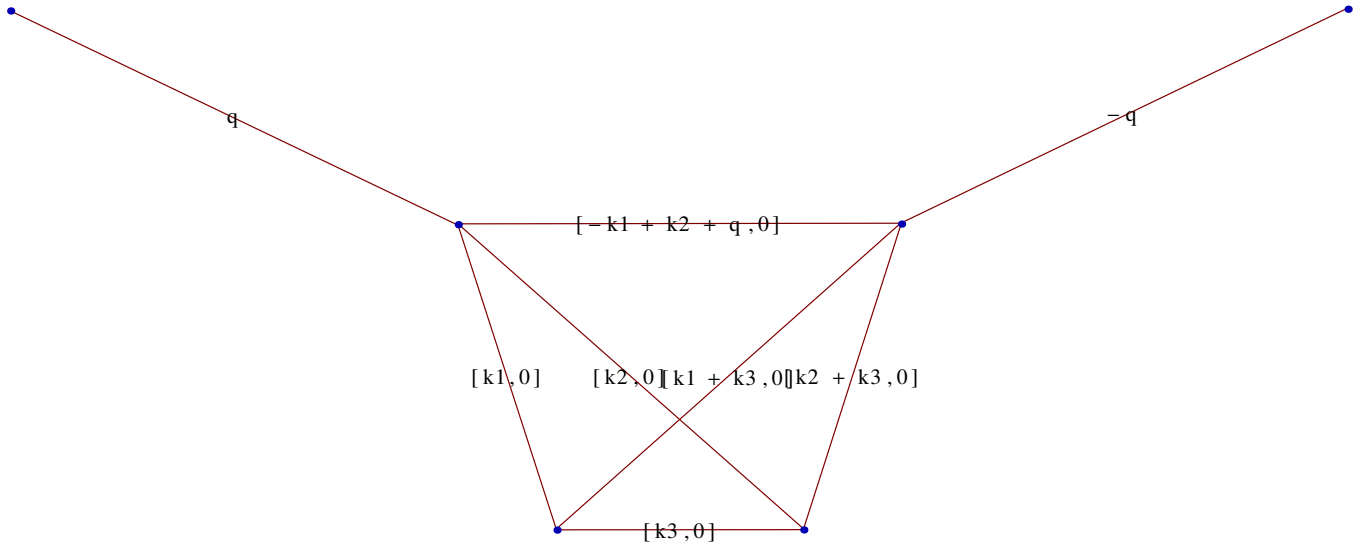
Limitations of GA approach

U polynomial for non-planar 3-loop box (64 terms)

```
x[1] x[2] x[4] + x[1] x[3] x[4] + x[2] x[3] x[4] + x[1] x[2] x[5] +
x[1] x[3] x[5] + x[2] x[3] x[5] + x[1] x[4] x[5] + x[2] x[4] x[5] +
x[2] x[4] x[6] + x[3] x[4] x[6] + x[2] x[5] x[6] + x[3] x[5] x[6] +
x[4] x[5] x[6] + x[2] x[4] x[7] + x[3] x[4] x[7] + x[2] x[5] x[7] +
x[3] x[5] x[7] + x[4] x[5] x[7] + x[1] x[2] x[8] + x[1] x[3] x[8] +
x[2] x[3] x[8] + x[1] x[4] x[8] + x[2] x[4] x[8] + x[2] x[6] x[8] +
x[3] x[6] x[8] + x[4] x[6] x[8] + x[2] x[7] x[8] + x[3] x[7] x[8] +
x[4] x[7] x[8] + x[1] x[2] x[9] + x[1] x[3] x[9] + x[2] x[3] x[9] +
x[2] x[4] x[9] + x[3] x[4] x[9] + x[1] x[5] x[9] + x[3] x[5] x[9] +
x[4] x[5] x[9] + x[2] x[6] x[9] + x[3] x[6] x[9] + x[5] x[6] x[9] +
x[2] x[7] x[9] + x[3] x[7] x[9] + x[5] x[7] x[9] + x[1] x[8] x[9] +
x[3] x[8] x[9] + x[4] x[8] x[9] + x[6] x[8] x[9] + x[7] x[8] x[9] +
x[1] x[2] x[10] + x[1] x[3] x[10] + x[2] x[3] x[10] +
x[1] x[4] x[10] + x[2] x[4] x[10] + x[2] x[6] x[10] +
x[3] x[6] x[10] + x[4] x[6] x[10] + x[2] x[7] x[10] +
x[3] x[7] x[10] + x[4] x[7] x[10] + x[1] x[9] x[10] +
x[3] x[9] x[10] + x[4] x[9] x[10] + x[6] x[9] x[10] + x[7] x[9] x[10]
```

AMBRE HOW TO I

- LA is basicaly for planar diagrams and GA – for non-planar.

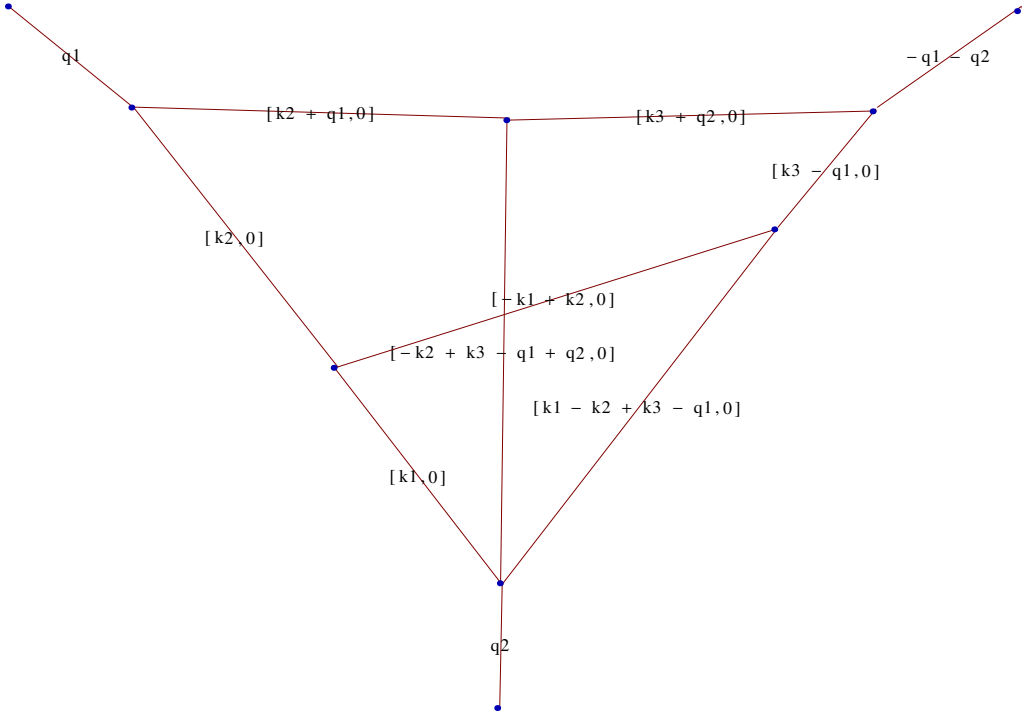


- For LA momentum flow really matters.
- 1st and 2nd Barnes lemmas are key ingredients.

$$-sx_1x_2 - sx_1x_4 + \ldots = -sx_1(x_2 + x_4) + \ldots \leftrightarrow \text{1st Barnes lemma}$$

AMBRE HOW TO II

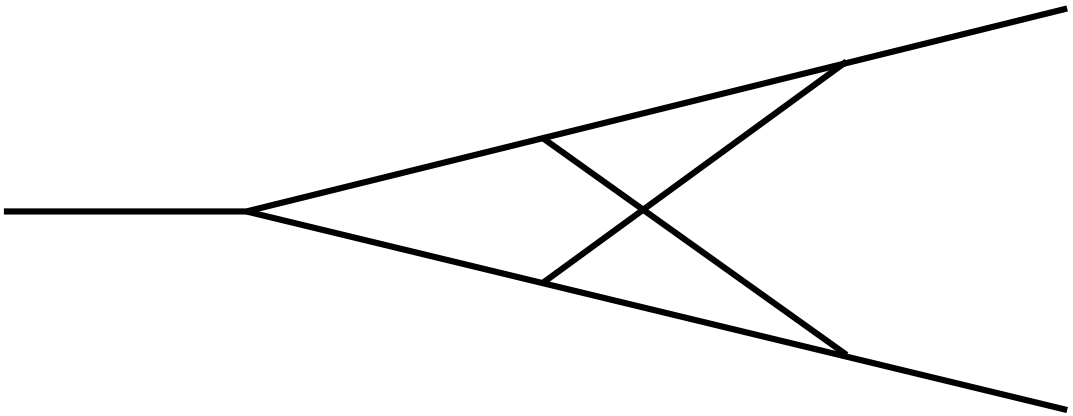
- LA works also for non-planar diagrams:



  - pros: AMBREv2.1.1 - 6-dim vs AMBREv3.1.1 - 13-dim
  - cons: always "minkowskian" - representation contains $(-s)^z s^z$

- GA works for both planar and non-planar diagrams

GA 2-loop example (non-planar vertex)



$$\iint d^d k_1 d^d k_2 \frac{1}{[k_1^2]^{n_1}[(p_1-k_1)^2]^{n_2}[(p_1-k_1-k_2)^2]^{n_3}} \frac{1}{[(p_2+k_1+k_2)^2]^{n_4}[(p_2+k_2)^2]^{n_5}[k_2^2]^{n_6}}$$

$$U = x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + x_3x_4 + x_1x_5 + x_2x_5 + x_4x_5 + x_2x_6 + x_3x_6 + x_4x_6 + x_5x_6$$

$$F = U\sum_{i=1}^{6} m_i^2 x_i - sx_1x_4x_5 - sx_1x_2x_6 - sx_1x_3x_6 - sx_2x_3x_6 - sx_1x_4x_6 - sx_1x_5x_6$$

Variables transformation

$$\{\vec{x}\}_i : x_k \to v_i \xi_{ik}$$

i denotes a subset of feynman parameters associated to propagators with different combinations of loop momenta

$$\begin{aligned} m^2 = \sum x_i D_i \quad &= \quad x_1(p_1 - k_1 - k_2)^2 && x_1 \to v_1 \xi_{11} \\ &+ \quad x_2(p_2 + k_1 + k_2)^2 && x_2 \to v_1 \xi_{12} \\ &+ \quad x_3(k_1)^2 && x_3 \to v_2 \xi_{21} \\ &+ \quad x_4(p_1 - k_1)^2 && x_4 \to v_2 \xi_{22} \\ &+ \quad x_5(p_2 + k_2)^2 && x_5 \to v_3 \xi_{31} \\ &+ \quad x_6(k_2)^2 && x_6 \to v_3 \xi_{32} \end{aligned}$$

$$\delta\left(1 - \sum_{i=1}^{6} x_i\right) \Rightarrow \delta(1 - v_1 - v_2 - v_3)\delta(1 - \xi_{11} - \xi_{12})\delta(1 - \xi_{21} - \xi_{22})\delta(1 - \xi_{31} - \xi_{32})$$

Jacobian:

$$J = v_1^{N_{\xi_1} - 1} v_2^{N_{\xi_2} - 1} v_3^{N_{\xi_3} - 1} = v_1 v_2 v_3$$

- Using $\prod_i \delta\left(1-\sum_k \xi_{ik}\right)$ we can simplify U and F

$$U = v_1 v_2 + v_1 v_3 + v_2 v_3 \quad F = -s\xi_{11}\xi_{22}\xi_{31}v_1 v_2 v_3 - s\xi_{12}\xi_{21}\xi_{32}v_1 v_2 v_3 \\ -s\xi_{31}\xi_{32}v_1 v_3^2 - s\xi_{31}\xi_{32}v_2 v_3^2$$

Chang–Wu theorem:
delta function in the feyman parameters representation can be replaced by

$$\delta\left(\sum_{i\in\Omega} x_i - 1\right)$$

where Ω is an arbitrary subset of the lines $1,\ldots,L$, when the integration over the rest of the variables, i.e. for $i \notin \Omega$, is extended to the integration from zero to infinity.

- Choose now v_3 as Chang-Wu variable $\int_0^\infty dv_3 \int_0^1 dv_1 dv_2 \delta(1-v_1-v_2)$

$$U = v_3 + v_1 v_2 \quad F = -s\xi_{11}\xi_{22}\xi_{31}v_1 v_2 v_3 - s\xi_{12}\xi_{21}\xi_{32}v_1 v_2 v_3 \\ -s\xi_{31}\xi_{32}v_1 v_3^2$$

- Apply MB relation for F

$$\frac{1}{(A_1+\ldots+A_n)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{(2\pi i)^{n-1}}\int_{-i\infty}^{i\infty} dz_1\ldots dz_{n-1}$$
$$\times\prod_{i=1}^{n-1} A_i^{z_i}\ A_n^{-\lambda-z_1-\ldots-z_{n-1}}\prod_{i=1}^{n-1}\Gamma(-z_i)\ \Gamma(\lambda+z_1+\ldots+z_{n-1})$$

- lintegrate over $v3$ using

$$\int_0^\infty dx\ x^{z_1}(x+y)^{z_2} = \frac{y^{1+z_1+z_2}\Gamma(1+z_1)\Gamma(-1-z_1-z_2)}{\Gamma(-z_2)}$$

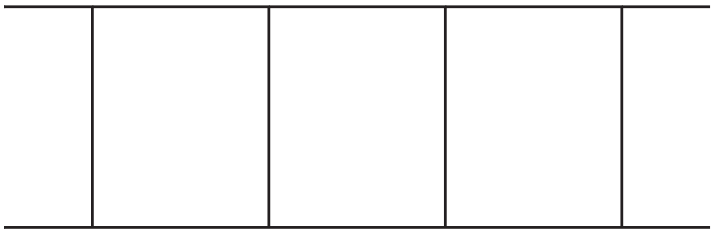
- lintegrate over each subset of variables $\{v,\xi_i\}$ separately using

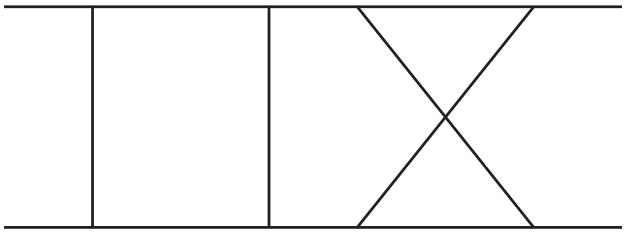
$$\int_0^1\prod_{i=1}^{N} dx_i\ x_i^{n_i-1}\ \delta(1-x_1-\ldots-x_N) = \frac{\Gamma(n_1)\ldots\Gamma(n_N)}{\Gamma(n_1+\ldots+n_N)}$$

U polynomial gives no additional MB integration and final dimentionality depends only on length of F $\longrightarrow$ similar to one loop integrals and/or LA approach

3-loop GA

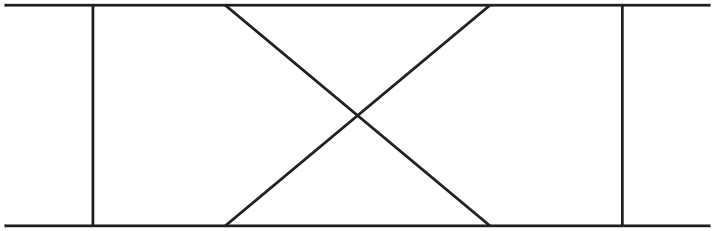
Case I:





$$U = v_1v_2v_3 + v_1v_2v_4 + v_2v_3v_4 + v_1v_2v_5 + v_1v_3v_5 + v_2v_3v_5 + v_1v_4v_5 + v_3v_4v_5$$

Case II:



$$\begin{aligned} U = {} & v_1v_2v_3 + v_1v_2v_4 + v_1v_3v_4 + v_1v_2v_5 + v_1v_3v_5 + v_2v_3v_5 + v_2v_4v_5 + v_3v_4v_5 \\ & + v_1v_2v_6 + v_2v_3v_6 + v_1v_4v_6 + v_2v_4v_6 + v_3v_4v_6 + v_1v_5v_6 + v_3v_5v_6 + v_4v_5v_6 \end{aligned}$$

Now in the Chang-Wu theorem we choose 3 variables

$$\int_0^\infty dv_2 dv_3 dv_4 \int_0^1 dv_1 dv_5 dv_6 \delta(1 - v_1 - v_5 - v_6)$$

$$U_{CW} = v_2v_3 + v_2v_4 + v_3v_4 + v_1v_2v_5 + v_1v_3v_5 + v_1v_2v_6 + v_1v_4v_6 + v_1v_5v_6 + v_3v_5v_6 + v_4v_5v_6$$

Factorization trick:

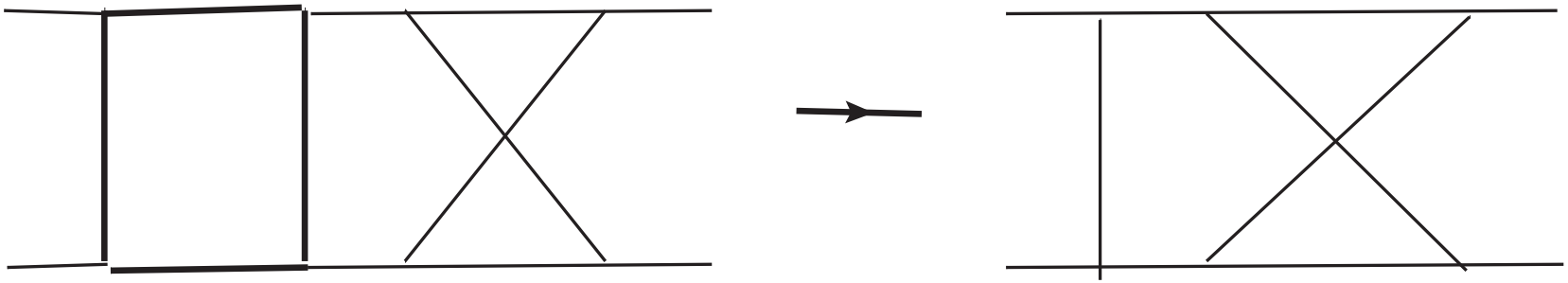
$$U_{CW} = v_2(v_3 + v_4 + v_1v_5) + v_3(v_4 + v_1v_5) + v_1v_6(v_2 + v_5) + v_4v_6(v_1 + v_5) + v_3v_5v_6$$

U polynomial gives 4 additional MB integration!

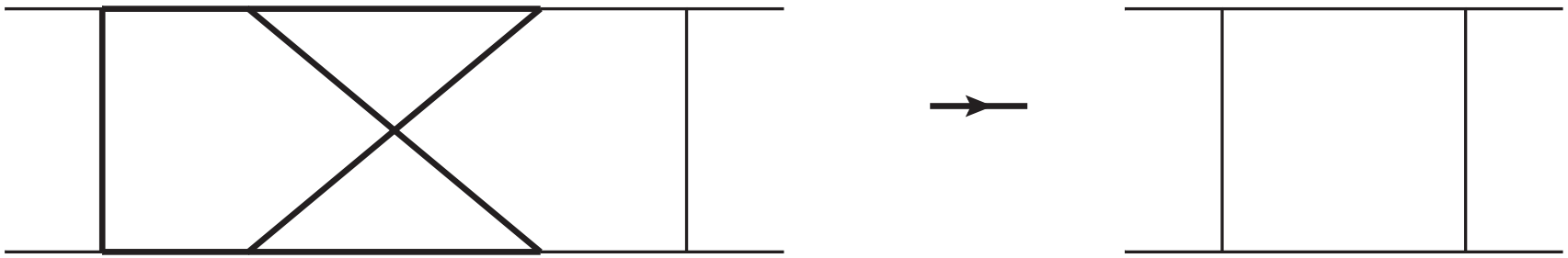
GA usually gives optimal representation if from the beginning Length(U) $\gtrsim$ Length(F)

3-loop mixed approach

Mixed approach starting with planar subloop:



Mixed approach starting with non-planar subloop:

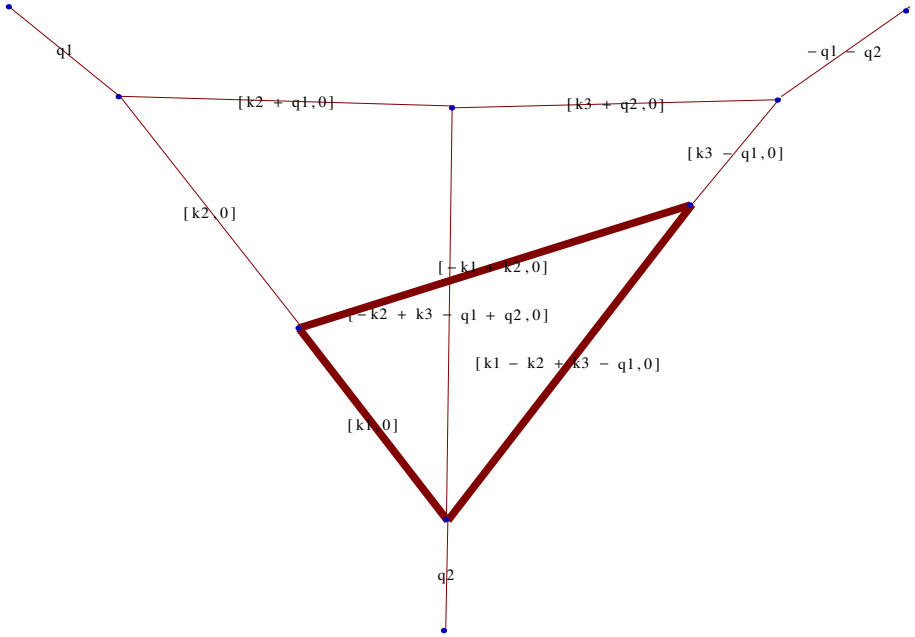


3-loop mixed approach: example

```
PR[k1, 0, n1] PR[-k1 + k2, 0, n2] PR[k2, 0, n3] PR[k2 + q1, 0, n4] PR[k3 + q2, 0, n5]
PR[k3 - q1, 0, n6] PR[-k2 + k3 - q1 + q2, 0, n7] PR[k1 - k2 + k3 - q1, 0, n8]

--iteration nr: 1 with momentum: k1

   F polynomial during this iteration
-PR[k2,0] X[1] X[2]-PR[k2-k3+q1,0] X[1] X[3]-PR[k3-q1,0] X[2] X[3]
```




```
PR[k2, 0, nz3] PR[k3 - q1, 0, nz6] PR[k2 + q1, 0, n4] PR[k3 + q2, 0,n5] PR[-k2 + k3 - q1 + q2, 0, n7] PR[k2 - k3 + q1, 0, z]
```

Conclusions and Outlook

- MB approach to Feynman integrals reached Minkowskian region
- Dimensionality of MB representations depends on topology, number of legs, internal and external masses.
- `AMBRE` software is based on two different approaches:
  LA – general planar and some non-planar diagrams
  GA – 2-loop planar and non-planar, 3-loop non-planar diagrams with massless external legs
- new `AMBREv4` which combines all advantages of methods above is underway
- On larger time scale - 3 loops prospect is a beautiful place for theoretical work
- and still much must be done on tools and methods to get it beyond the present status