QCD corrections to evolution equations for operator matrix elements

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Based on work done in collaboration with:

- *Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:1707.08315`
- *Three-loop evolution equation for flavor-nonsinglet operators in off-forward kinematics*
  V.M. Braun, A.N. Manashov, S. M. and M. Strohmaier `arXiv:1703.09532`
- *Two-loop conformal generators for leading-twist operators in QCD*
  V.M. Braun, A.N. Manashov, S. M. and M. Strohmaier `arXiv:1601.05937`
- Many more papers of MVV and friends . . .
  `2001 - ...`

Motivation

Operator matrix elements

- QCD applications to hard processes use nonlocal operators of partons at light-like separation

$$\mathcal{O}_\mu(x; z_1, z_2) = \bar{\psi}(x + z_1)\gamma_\mu[z_1, z_2]\psi(x + z_2)$$

  - quark and anti-quark fields joined by Wilson line along '$+$'-direction

$$[z_1, z_2] = \text{Pexp}\left[ig\int_{z_2}^{z_1} dt A_+(t)\right]$$

- Expansion of $\mathcal{O}_\mu(x; z_1, z_2)$ at short distances leads to local operators
  - (anti-)quark fields with covariant derivatives $D_\mu = \partial_+ - igA_+$

$$\bar{\psi}(x)(\overleftarrow{D}_+)^m\gamma_\mu(\overrightarrow{D}_+)^k\psi(x)$$

Applications

- (Generalized) parton distributions: PDFs and GPDs
- Hard exclusive reactions with identified hadrons $N(p)$ and $N(p')$ in initial and final state: $\gamma^* N(p) \to \gamma N(p')$ (DVCS)
- Meson-photon transition form factors $\gamma^* \to \gamma\pi$

Light-ray operators

- Short-distance expansion yields light-ray operators $\mathcal{O}_\mu(x; z_1, z_2)$ with light-like direction n

$$[\mathcal{O}](x; z_1, z_2) \quad \equiv \quad \sum_{m,k} \frac{z_1^m z_2^k}{m!k!} \left[\bar{\psi}(x)(\overleftarrow{D}\cdot n)^m \not{n}(n\cdot \overrightarrow{D})^k \psi(x)\right]$$

- multiplicative renormalization $[\mathcal{O}] = Z\mathcal{O}$
- Light-ray operators satisfy renormalization group equation Balitsky, Braun '87

$$\Big(\mu\partial_\mu + \beta(a)\partial_a + \mathbb{H}(a)\Big)[\mathcal{O}](x; z_1, z_2) = 0$$

- Integral operator $\mathbb{H}(a)$ acts on light-cone coordinates of fields $z_{12}^\alpha = z_1(1-\alpha) + z_2\alpha$
  - evolution kernel $h(\alpha, \beta)$

$$\mathbb{H}(a)[\mathcal{O}](z_1, z_2) = \int_0^1 d\alpha \int_0^1 d\beta\, h(\alpha, \beta)\, [\mathcal{O}](z_{12}^\alpha, z_{21}^\beta)$$

- Powers $[\mathcal{O}](z_1, z_2) \mapsto (z_1 - z_2)^N$ are eigenfunctions of $\mathbb{H}$
- Eigenvalues $\gamma(N) = \int d\alpha d\beta\, h(\alpha, \beta)(1-\alpha-\beta)^N$ are anomalous dimensions of leading-twist local operators with $N = m + k$ derivatives

Evolution equations

- Leading-order result for evolution kernel

$$\mathbb{H}^{(1)} f(z_1, z_2) = 4C_F \left\{ \int_0^1 d\alpha \frac{\bar{\alpha}}{\alpha} \left[2f(z_1, z_2) - f(z_{12}^{\alpha}, z_2) - f(z_1, z_{21}^{\alpha}) \right] - \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \, f(z_{12}^{\alpha}, z_{21}^{\beta}) + \frac{1}{2} f(z_1, z_2) \right\}$$

- Expression comprises all classical leading-order QCD evolution equations
  - PDFs Altarelli, Parisi; $\gamma^{(0)}(N)$
  - meson light-cone distribution amplitudes Efremov, Radyushkin, Brodsky, Lepage
  - general evolution equation for GPDs Belitsky, Müller; $h^{(1)}(\alpha, \beta)$

Task

- Push accuracy of evolution equations to NNLO and beyond
- Computation of $h(\alpha, \beta)$ to three loops and $\gamma(N)$ to four loops

Off-forward kinematics

Conformal symmetry

- Full conformal algebra in 4 dimensions includes fifteen generators
  Mack, Salam '69; Treiman, Jackiw, Gross '72

$\mathbf{P}_\mu$ (4 translations)	$\mathbf{M}_{\mu\nu}$ (6 Lorentz rotations)
$\mathbf{D}$ (dilatation)	$\mathbf{K}_\mu$ (4 special conformal transformations)

Collinear subgroup $SL(2,\mathbb{R})$

- Leading order evolution operator $\mathbb{H}^{(1)}$ commutes with (canonical) generators of collinear conformal transformations
  - Special conformal transformation $x_- \to x' = \dfrac{x_-}{1+2ax_-}$
  - Translations $x_- \to x' = x_- + c$ and dilatations $x_- \to x' = \lambda x_-$
- Evolution kernel $h^{(1)}(\alpha,\beta) = \bar{h}(\tau)$ effectively function of one variable
  $\tau = \dfrac{\alpha\beta}{\bar{\alpha}\bar{\beta}}$ (conformal ratio) Braun, Derkachov, Korchemsky, Manashov '99

$$h^{(1)}(\alpha,\beta) = -4C_F\left[\delta_+(\tau) + \theta(1-\tau) - \frac{1}{2}\delta(\alpha)\delta(\beta)\right],$$

Higher orders

- Conformal symmetry is broken in any realistic four-dimensional QFT
  - $\beta(a) \neq 0$
- Leading order generators of $SL(2,\mathbb{R})$ commute with $\mathbb{H}^{(1)}$ satisfy the usual $SL(2,\mathbb{R})$ algebra $[S_0, S_\pm] = \pm S_\pm$, $[S_+, S_-] = 2S_0$

$$
\begin{aligned}
S_-^{(0)} &= -\partial_{z_1} - \partial_{z_2}\,, \\
S_0^{(0)} &= z_1\partial_{z_1} + z_2\partial_{z_2} + 2, \\
S_+^{(0)} &= z_1^2\partial_{z_1} + z_2^2\partial_{z_2} + 2(z_1 + z_2)
\end{aligned}
$$

Idea

- Instead of considering consequences of broken conformal symmetry in QCD make use of exact conformal symmetry of modified theory
  - $\beta(a) = 2a(-\epsilon - \beta_0 a - \beta_1 a^2 - \dots)$ with $a = \dfrac{\alpha_s}{4\pi}$
  - large-n_f QCD in $4 - 2\epsilon$ dimensions at critical coupling a_* with $\beta(a_*) = 0$ Banks, Zaks '82
- Maintain exact conformal symmetry, but the generators of $SL(2,\mathbb{R})$ are modified by quantum corrections

Conformal anomaly (I)

- Generators in interacting theory (at the critical point a_*) are nontrivial

$$
\begin{aligned}
S_- &= S_-^{(0)} \\
S_0 &= S_0^{(0)} + \Delta S_0 = S_0^{(0)} - \epsilon + \frac{1}{2}\mathbb{H}(a_*) \\
S_+ &= S_+^{(0)} + \Delta S_+ = S_+^{(0)} + (z_1 + z_2)\Big(- \epsilon + \frac{1}{2}\mathbb{H}(a_*)\Big) + (z_1 - z_2)\,\Delta_+(a_*)
\end{aligned}
$$

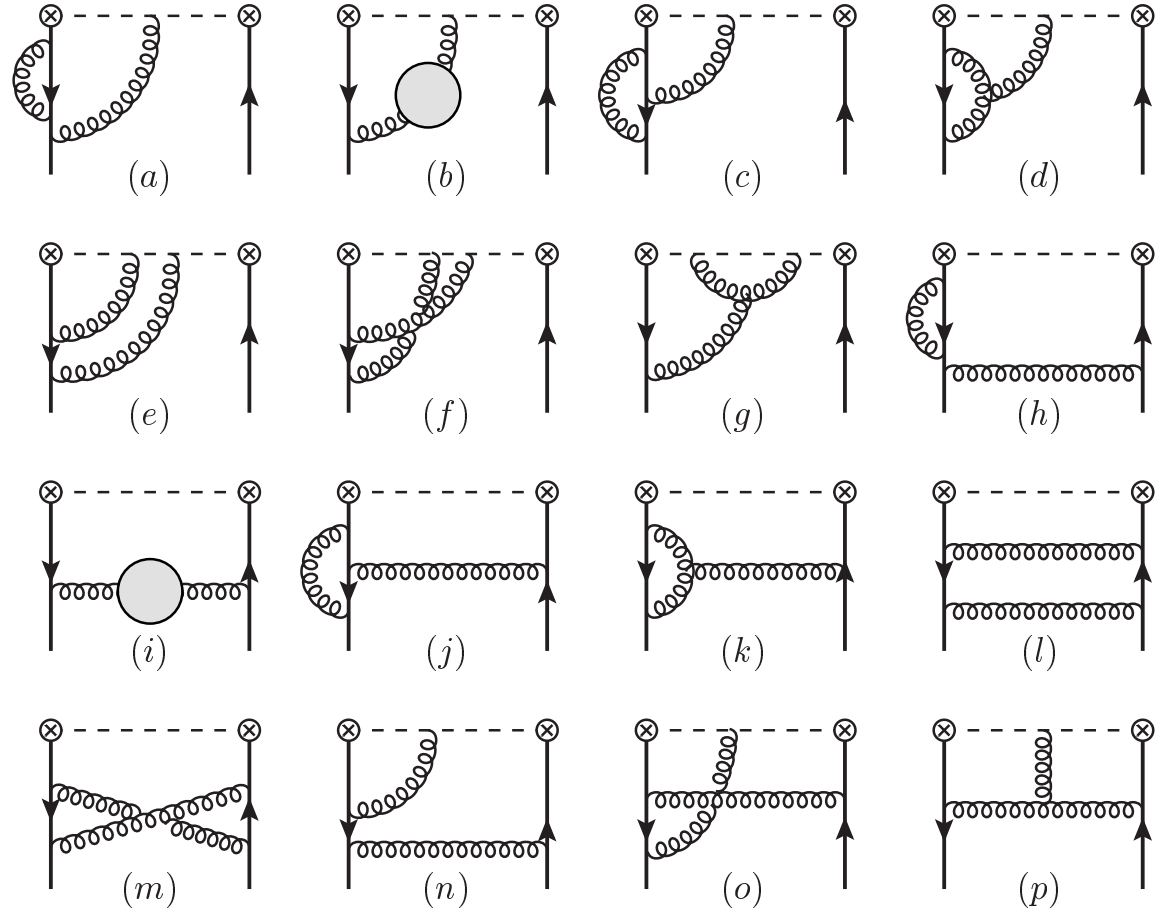
  - generator of special conformal transformations ΔS_+
  - conformal anomaly $\Delta_+(a_*)$ restores conformal Ward identity

- Integral operator $\mathbb{H}(a_*)$ commutes with all generators
  - eigenvalues at order k satisfy $\mathbb{H}^{(k)}(z_1 - z_2)^N = \gamma^{(k)}(N)\,(z_1 - z_2)^N$ where $SL(2,\mathbb{R})$-invariant is function of conformal spin $N(N+1)$
  - full anomalous dimension constrained by 'self-tuning' with universal evolution kernel $\gamma_{\rm u}$ reciprocity-respecting (invariant for $N \to -N-1$)

$$
\gamma(N) = \gamma_{\rm u}\,(N + \gamma(N) - \beta(a))
$$

Conformal anomaly (II)

- Feynman diagrams for deformation of generator of special conformal transformations ΔS_+



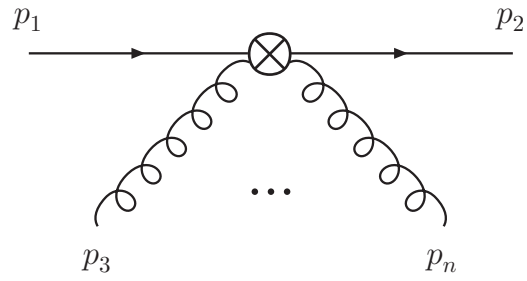
Upshot

- Three-loop evolution equation for flavor-nonsinglet operators in off-forward kinematics computed with two-loop conformal anomaly and algebraic solution of constraint for conformal Ward identity

Forward kinematics

Anomalous dimensions

- Anomalous dimensions $\gamma(N)$ of leading twist non-singlet local operators
  - ultraviolet divergence of loop corrections to operator in (anti-)quark two-point function



  - expressible in harmonic sums up to weight 7

$$S_{\pm m_1,\ldots,m_k}(N) = \sum_{i=1}^{N} \frac{(\pm 1)^i}{i^{m_1}} S_{m_2,\ldots,m_k}(i)$$

  - $2 \cdot 3^{w-1}$ sums at weight w

- Reciprocity relation $\gamma(N) = \gamma_{\rm u}\left(N + \gamma(N) - \beta(a)\right)$ reduces number of 2^{w-1} sums at weight w for $\gamma_{\rm u}$
  - additional denominators with powers $1/(N+1)$ give $2^{w+1} - 1$ objects (255 at weight 7)

- Constraints at large-x/small-x ($N \to \infty$/$N \to 0$) give additional 46 conditions

Calculation

- Non-singlet operator of spin-N and twist two
$$O^{\rm ns}_{\{\mu_1,\ldots,\mu_N\}} \;=\; \overline{\psi}\,\lambda^\alpha\,\gamma_{\{\mu_1}D_{\mu_2}\ldots D_{\mu_N\}}\,\psi\;, \qquad \alpha\,=\,3,8,\ldots,(n_f^2-1)$$
- Feynman diagrams for operator matrix elements generated up to four loops with **Qgraf** Nogueira '91
- Parametric reduction of four-loop massless propagator diagrams with **Forcer** Ruijl, Ueda, Vermaseren '17
- Symbolic manipulations with **Form** Vermaseren '00; Kuipers, Ueda, Vermaseren, Vollinga '12 and multi-threaded version **TForm** Tentyukov, Vermaseren '07
- Diagrams of same topology and color factor combined to meta diagrams
  - 1 one-, 7 two-, 53 three- and 650 four-loop meta diagrams for $\gamma_{\rm ns}^{\pm}$
  - 1 three- and 29 four-loop meta diagrams for $\gamma_{\rm ns}^{\rm s}$

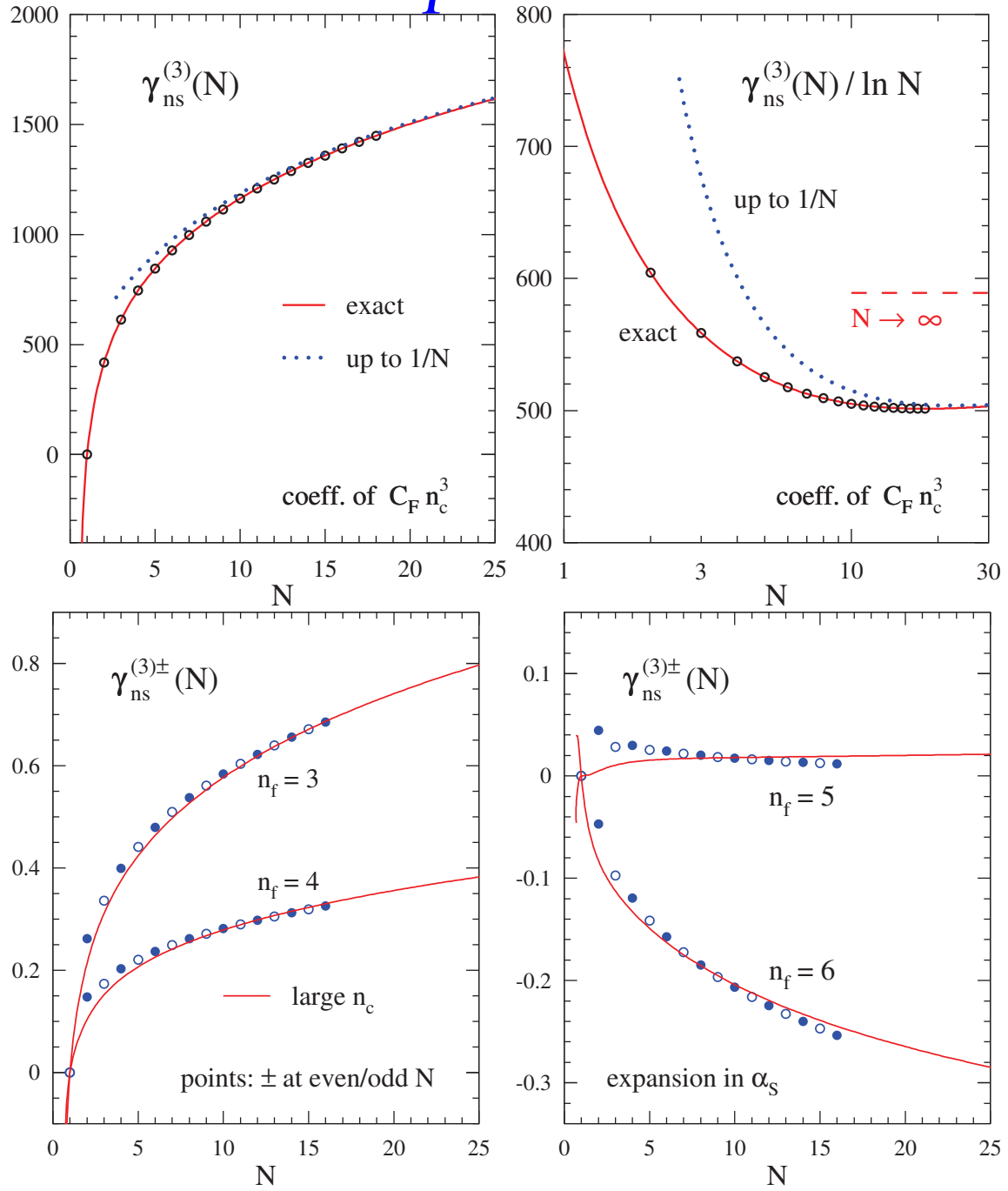
Upshot

- Computation of Mellin moments up to $N = 18$ for anomalous dimensions feasible
- Reconstruction of analytic all-N expressions in large-n_c limit from solution of Diophantine equations

Mellin moments at four loops

- Top:
  n_f^0 part of anomalous dimensions $\gamma_{\rm ns}^{(3)\pm}(N)$ in large-n_c limit and large-N expansion

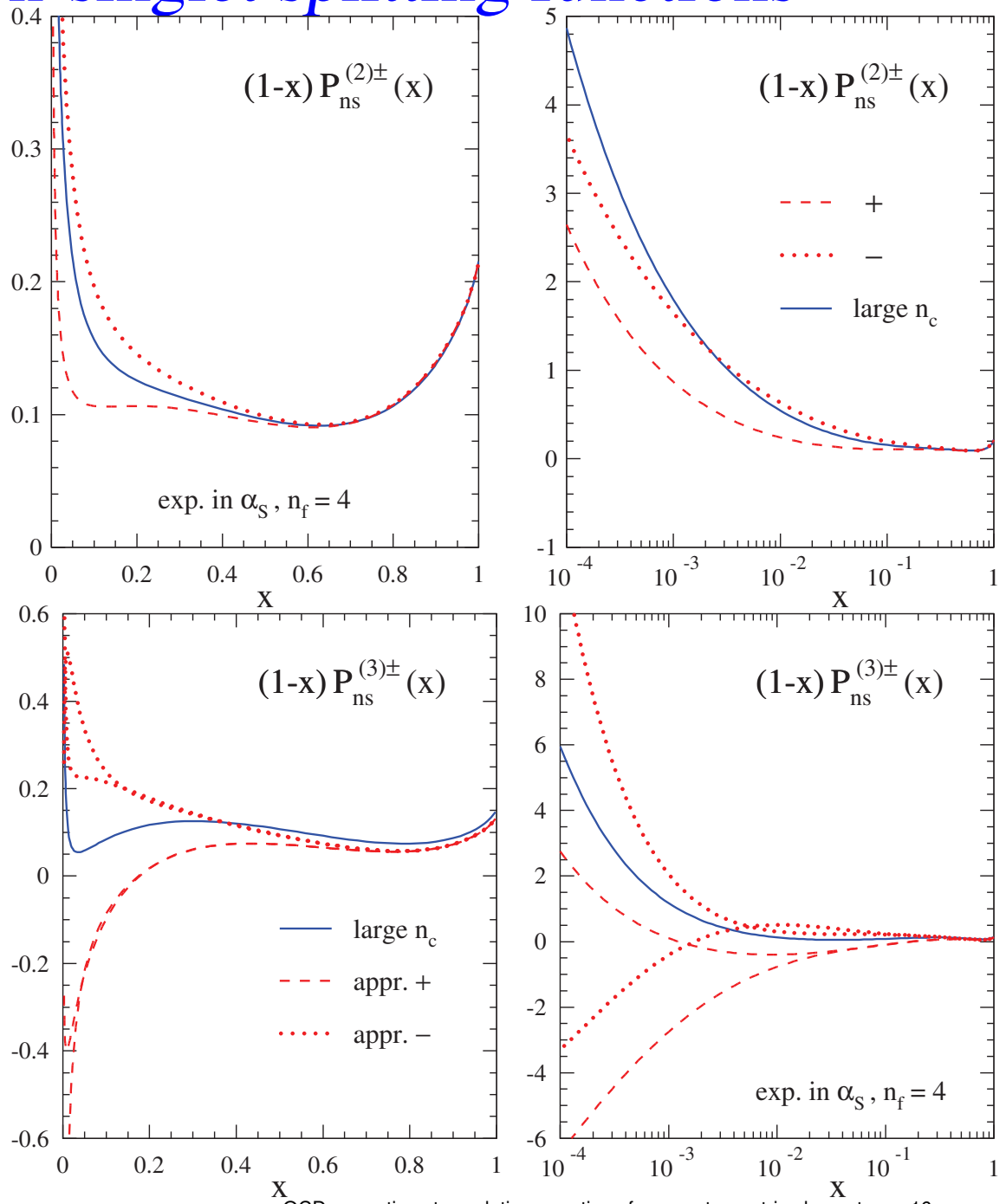
- Bottom: results for even-N ($\gamma_{\rm ns}^{(3)+}(N)$) and odd-N ($\gamma_{\rm ns}^{(3)-}(N)$) in large-n_c limit for $n_f = 3, \ldots 6$



Four-loop non-singlet splitting functions

- Top:
  three-loop $P_{\rm ns}^{(2)\pm}(x)$
  and large-n_c limit
  with $n_f = 4$

- Bottom:
  four-loop $P_{\rm ns}^{(3)\pm}(x)$
  and uncertainty bands
  beyond large-n_c limit
  with $n_f = 4$



Numerical implications

- Large-x limit

$$P_{\rm ns}^{(n-1)}(x) \;=\; \frac{A_n}{(1-x)_+} \;+\; B_n\,\delta(1-x) \;+\; C_n\,\ln(1-x) \;+\; D_n$$

Cusp anomalous dimension

- Large-n_c limit (agrees with Henn, Lee, Smirnov, Smirnov, Steinhauser '16)

$$\begin{aligned} A_{L,4} \;=\; & C_F n_c^3 \left(\frac{84278}{81} - \frac{88832}{81}\,\zeta_2 + \frac{20992}{27}\,\zeta_3 + 1804\zeta_4 - \frac{352}{3}\,\zeta_2\zeta_3 - 352\zeta_5 \right. \\ & \left. -32\zeta_3^2 - 876\,\zeta_6 \right) \\ & - C_F n_c^2\, n_f \left(\frac{39883}{81} - \frac{26692}{81}\,\zeta_2 + \frac{16252}{27}\,\zeta_3 + \frac{440}{3}\,\zeta_4 - \frac{256}{3}\,\zeta_2\zeta_3 - 224\,\zeta_5 \right) \\ & + C_F n_c\, n_f^2 \left(\frac{2119}{81} - \frac{608}{81}\,\zeta_2 + \frac{1280}{27}\,\zeta_3 - \frac{64}{3}\,\zeta_4 \right) - C_F n_f^3 \left(\frac{32}{81} - \frac{64}{27}\,\zeta_3 \right) \end{aligned}$$

- Full QCD result

$$A_4 \;=\; 20702(2) \;-\; 5171.9(2)\,n_f \;+\; 195.5772\,n_f^2 \;+\; 3.272344\,n_f^3$$

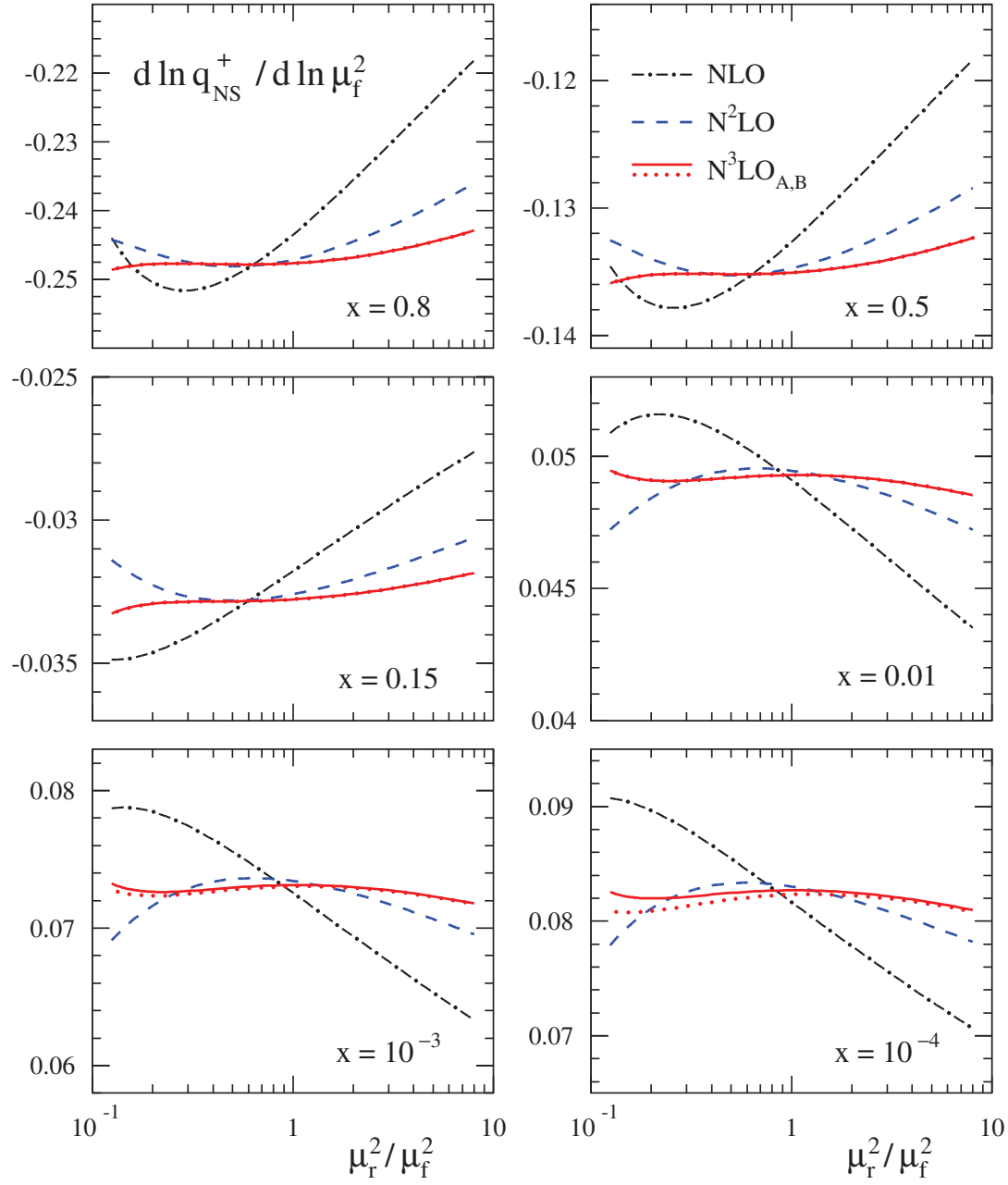
  - includes non-vanishing coefficients of quartic Casimir contributions $\dfrac{d_F^{abcd} d_A^{abcd}}{N_R}$ and $\dfrac{d_F^{abcd} d_F^{abcd}}{N_R}$

Scale stability of evolution

- Renormalization scale dependence of evolution kernel $d\ln q_{\rm ns}^{+}/d\ln\mu_f^2$
  - non-singlet shape

$$xq_{\rm ns}^{+}(x,\mu_0^2) \;=\; x^{0.5}(1-x)^3$$

- NLO, NNLO and N^{3}LO predictions
  - remaining uncertainty of four-loop splitting function $P_{\rm ns}^{(3)+}$ almost invisible



Summary

- QCD radiative corrections to evolution equations for operator matrix elements
  - necessary for precision of non-perturbative quantities: PDFs, GPDs, DAs
- Light-ray operators
  - QCD evolution equations possess a "hidden" conformal symmetry
  - evolution kernel $h(\alpha, \beta)$ at NNLO for off-forward kinematics
- Matrix elements of non-singlet local operators of twist two
  - anomalous dimension $\gamma(N)$ (fixed Mellin moments and exact results for large-n_c) at N^{3}LO
- Non-singlet splitting functions at four loops