Five-loop renormalization of QCD and for a general gauge group

Peter Marquard

DESY

T. Luthe, A. Maier, PM, Y. Schröder
JHEP 1701 (2017) 081, JHEP 1703 (2017) 020

Matter to the Deepest, Podlesice, September 2017

Outline

Outline

Introduction

- The renormalization constants are fundamental quantities of QCD.
- The β function governs the running of the coupling constant.
- The quark mass anomalous dimension together with the β function governs the running of the quark mass in the $\overline{\mathrm{MS}}$ scheme.
- Goal: Extend results in the literature to a general gauge group and perform an independent check.

Existing Approaches

The five-loop renormalization constants are considered using different approaches.

- global R^* operation, all renormalization constants in Feynman gauge, $N_c = 3$,
  $\Longrightarrow$ 4-loop massless propagators [Baikov,Chetyrkin,Kühn]
- local R^* operation, background field gauge, β function, general gauge group,
  $\Longrightarrow$ 4-loop massless propagators [Herzog,Ruijl,Ueda,Vermaseren]
- IRR, all renormalization constants in Feynman gauge, general gauge group,
  $\Longrightarrow$ 5-loop massive tadpoles [Luthe,Maier,PM,Schröder]

Method

- Need to calculate the relevant propagators and vertices at five-loop order.
- Renormalization constants (or anomalous dimensions) can be read off from the single pole.
- Only interested in the poles and since we work in the $\overline{\mathrm{MS}}$ scheme we can use a mass as infrared regulator and calculate fully massive five-loop tadpoles instead.
- Prize to pay: No multiplicative renormalization and a gluon mass counterterm.

Regularization and Renormalization

$$Z_2\mathcal{D}^f(p) = \frac{i}{\not{p} - Z_{m_f} m_f}\,, \qquad Z_{3c}\mathcal{D}^c(p) = \frac{i}{p^2}\,,$$

$$Z_3\mathcal{D}^g_{\mu\nu}(p) = -\frac{i}{p^2}\left[g_{\mu\nu} - (1 - Z_3\xi_L)\frac{p_\mu p_\nu}{p^2}\right].$$

Regularization and Renormalization

$$Z_2\mathcal{D}^f(p) = \frac{i}{\not{p} - Z_{m_f} m_f}\,, \qquad Z_{3c}\mathcal{D}^c(p) = \frac{i}{p^2}\,,$$

$$Z_3\mathcal{D}^g_{\mu\nu}(p) = -\frac{i}{p^2}\left[g_{\mu\nu} - (1 - Z_3\xi_L)\frac{p_\mu p_\nu}{p^2}\right].$$

Introducing an auxiliary mass m

$$\tilde{\mathcal{D}}^f(p) = i\frac{\not{p} + m_f}{p^2 - m^2} + \mathcal{O}(m_f^2)\,,$$

$$\tilde{\mathcal{D}}^c(p) = \frac{i}{p^2 - m^2}\,,$$

$$\tilde{\mathcal{D}}^g_{\mu\nu}(p) = -\frac{i}{p^2 - m^2}\left(g_{\mu\rho} - (1 - \xi_L)\frac{p_\mu p_\rho}{p^2 - m^2}\right)$$

Regularization and Renormalization

$$Z_2\mathcal{D}^f(p) = \frac{i}{\not{p} - Z_{m_f} m_f}\,, \qquad Z_{3c}\mathcal{D}^c(p) = \frac{i}{p^2}\,,$$

$$Z_3\mathcal{D}^g_{\mu\nu}(p) = -\frac{i}{p^2}\left[g_{\mu\nu} - (1 - Z_3\xi_L)\frac{p_\mu p_\nu}{p^2}\right].$$

Introducing an auxiliary mass m and the corresponding gauge boson mass counterterm δZ_{m^2} we obtain

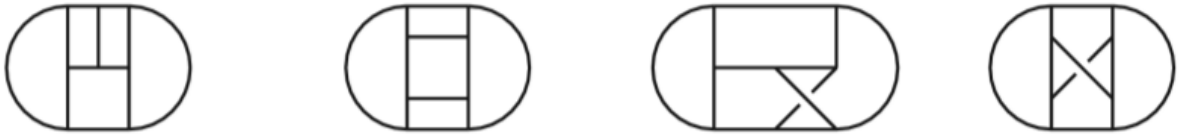
$$\tilde{\mathcal{D}}^f(p) = i\frac{\not{p} + m_f}{p^2 - m^2}\left(1 + i\left[\delta Z_2(\not{p} - m_f) - Z_2\delta Z_{m_f} m_f\right]\tilde{\mathcal{D}}^f(p)\right) + \mathcal{O}(m_f^2)\,,$$

$$\tilde{\mathcal{D}}^c(p) = \frac{i}{p^2 - m^2}\left[1 + i\delta Z_{3c}p^2\tilde{\mathcal{D}}^c(p)\right],$$

$$\tilde{\mathcal{D}}^g_{\mu\nu}(p) = -\frac{i}{p^2 - m^2}\left(g_{\mu\rho} - (1 - \xi_L)\frac{p_\mu p_\rho}{p^2 - m^2}\right)$$
$$\times\left(g_{\rho\nu} - i\left[\delta Z_{m^2} m^2 g_{\rho\sigma} + \delta Z_3(g_{\rho\sigma}p^2 - p_\rho p_\sigma)\right]\tilde{\mathcal{D}}^g_{\sigma\nu}(p)\right)$$

Some details and statistics

- qq - 83,637, cc - 83,637, gg - 509,777, gcc - 1,444,756 five-loop diagrams generated with `QGRAF` [Nogueira '91]
- Mapped onto 4 12-line topologies



  Note: one *master* topology with 15 indices
- Reduction to master integrals using IBPs in a Laporta-like implementation in `Crusher` [Marquard, Seidel] and `TIDE` [Luthe] with `Fermat` [Lewis] as backend $\Rightarrow$ 134 5-loop masters
- Calculation done using `FORM` [Vermaseren]
- Colour algebra done with `color` [van Ritbergen,Schellekens,Vermaseren '99]

Master integrals from Factorial Series

- The idea of the method goes back to Laporta who suggested to calculate Feynman integrals in form of a factorial series. [Laporta '01]
- Take an integral and raise the power of one propagator to the power x e.g. $I(1,1,1) \to I(x) = I(x,1,1)$
- Using IBP relations one can obtain a difference equation for the integral

$$\sum_{k+0}^{R} p_k(x) I(x+k) = \sum_i \sum_{k=0}^{R_i} p_{ik}(x) J_i(x+k)$$

  where J_i are integrals of simpler sectors
- Make an ansatz for $I(x)$ in terms of a factorial series (N.B. not the most general one)

$$I(x) = \sum_{s=0}^{\infty} \frac{\Gamma(x+1)}{\Gamma(x+d/2+s+1)} a_s$$

Master integrals cont'd

- Inserting the ansatz into the difference equation results in a recurrence relation for a_s

$$\sum_{k=0}^{R'} g_k(s) a_{s+k} = \sum_i \sum_{k=0}^{R'_i} g_{ik}(s) a_{i,s+k}$$

- given the initial values $a_0, a_1, \ldots$ are known, an arbitrary number of values for a_n can be calculated.
- using the obtained values for a_n $I(x)$ can be calculated

$$I(x) = \sum_{s=0}^{\infty} \frac{\Gamma(x+1)}{\Gamma(x+d/2+s+1)} a_s$$
$$= \frac{\Gamma(x+1)}{\Gamma(x+d/2+1)} \Big(a_0 + \frac{a_1}{(x+d/2+1)} + \frac{a_2}{(x+d/2+1)(x+d/2+2)} + \cdots \Big)$$

Master integrals cont'd

- In this way we can obtain a numerical result for the master integrals with a very high precision
- For individual master integrals it is not always possible to reconstruct the analytic form since we do not know the alphabet
- For the final results for the anomalous dimensions the expressions are precise enough to reconstruct the analytic form in terms of ζ values using PSLQ

Colour factors

Besides the usual colour factors

$$c_f = \frac{C_{\rm F}}{C_{\rm A}} \quad , \quad n_f = \frac{N_{\rm f}\, T_{\rm F}}{C_{\rm A}} \quad , \quad a \equiv \frac{C_{\rm A}\, g^2(\mu)}{16\pi^2}$$

we also need

$$\begin{aligned} d_1 &= \frac{[{\rm sTr}(T^a T^b T^c T^d)]^2}{N_{\rm A}\, T_{\rm F}^2\, C_{\rm A}^2} \,, \\ d_2 &= \frac{{\rm sTr}(T^a T^b T^c T^d)\, {\rm sTr}(F^a F^b F^c F^d)}{N_{\rm A}\, T_{\rm F}\, C_{\rm A}^3} \,, \\ d_3 &= \frac{[{\rm sTr}(F^a F^b F^c F^d)]^2}{N_{\rm A}\, C_{\rm A}^4} \end{aligned}$$

Color structures with a trace over 3 or 5 generators do not contribute.

Outline

Quark anomalous dimensions

The anomalous dimensions are defined as usual

$$\gamma_2 = -\partial_{\ln \mu^2} \ln Z_2 =$$
$$- c_f\, a \left\{ (1-\xi) + \gamma_{21}\, a + \gamma_{22}\, a^2 + \gamma_{23}\, a^3 + \gamma_{24}\, a^4 + \dots \right\},$$
$$\gamma_m = \partial_{\ln \mu^2} \ln m_{\mathrm{q}}(\mu) \; =$$
$$- c_f\, a \left\{ 3 + \gamma_{m1}\, a + \gamma_{m2}\, a^2 + \gamma_{m3}\, a^3 + \gamma_{m4}\, a^4 + \dots \right\},$$
$$\text{with } a \equiv \frac{C_{\mathrm{A}}\, g^2(\mu)}{16\pi^2}$$

Known up to four loops from

[Tarrach '81; Tarasov; Larin '93; Larin, van Ritbergen, Vermaseren '97; Chetyrkin '97]

γ_m, 2 – 4 loop

The quark mass anomalous dimension is gauge independent

$$3^1\,\gamma_{m1} = n_f\Big[-10\Big] + \Big[(9c_f+97)/2\Big]\,,$$

$$3^3\,\gamma_{m2} = n_f^2\Big[-140\Big] + n_f\Big[54(24\zeta_3-23)c_f - 4(139+324\zeta_3)\Big]$$
$$+\Big[(6966c_f^2-3483c_f+11413)/4\Big],$$

$$3^4\,\gamma_{m3} = n_f^3\Big[-8(83-144\zeta_3)\Big] + n_f^2\Big[48(19-270\zeta_3+162\zeta_4)c_f$$
$$+2(671+6480\zeta_3-3888\zeta_4)\Big] + n_f\Big[-216(35-207\zeta_3+180\zeta_5)c_f^2$$
$$-3(8819-9936\zeta_3+7128\zeta_4-2160\zeta_5)c_f$$
$$-(65459/2+72468\zeta_3-21384\zeta_4-32400\zeta_5)+2592(2-15\zeta_3)d_1\Big]$$
$$+\tfrac{9}{8}\Big[-9(1261+2688\zeta_3)c_f^3+6(15349+3792\zeta_3)c_f^2$$
$$-2(34045+5472\zeta_3-15840\zeta_5)c_f+(70055+11344\zeta_3-31680\zeta_5)$$
$$-1152(2-15\zeta_3)d_2\Big]$$

γ_m, 5 loop

At five loops we get

$$6^5\,\gamma_{m4} = \gamma_{m44}\left[4n_f\right]^4 + \gamma_{m43}\left[4n_f\right]^3 + \gamma_{m42}\left[4n_f\right]^2 + \gamma_{m41}\left[4n_f\right] + \gamma_{m40}\ ,$$

with the coefficients

$$\gamma_{m44} = -6(65 + 80\zeta_3 - 144\zeta_4)\ ,$$

$$\gamma_{m43} = 3(4483 + 4752\zeta_3 - 12960\zeta_4 + 6912\zeta_5)c_f + (18667/2 + 32208\zeta_3 + 29376\zeta_4 - 55296\zeta_5)\ .$$

$$\gamma_{m42} = \left\{c_f^2, c_f, d_1, 1\right\}.\Big\{9(45253 - 230496\zeta_3 + 48384\zeta_3^2 + 70416\zeta_4 + 144000\zeta_5 - 86400\zeta_6),$$
$$375373 + 323784\zeta_3 - 1130112\zeta_3^2 + 905904\zeta_4 - 672192\zeta_5 + 129600\zeta_6,$$
$$-864(431 - 1371\zeta_3 + 432\zeta_4 + 420\zeta_5),$$
$$4(13709 + 394749\zeta_3 + 173664\zeta_3^2 - 379242\zeta_4 - 119232\zeta_5 + 162000\zeta_6)\Big\}\ ,$$

γ_m, 5 loop cont'd

$$\gamma_{m41} = \left\{c_f^3, c_f^2, c_f d_1, c_f, d_1, d_2, 1\right\}.\Big\{-54(48797 - 247968\zeta_3 + 24192\zeta_4 + 444000\zeta_5 - 241920\zeta_7),$$
$$-18(406861 + 216156\zeta_3 - 190080\zeta_3^2 + 254880\zeta_4 - 606960\zeta_5 - 475200\zeta_6 + 362880\zeta_7),$$
$$-62208(11 + 154\zeta_3 - 370\zeta_5),$$
$$753557 + 15593904\zeta_3 - 3535488\zeta_3^2 - 6271344\zeta_4 - 17596224\zeta_5 + 1425600\zeta_6 + 1088640\zeta_7,$$
$$1728(3173 - 6270\zeta_3 + 1584\zeta_3^2 + 2970\zeta_4 - 13380\zeta_5),$$
$$1728(380 - 5595\zeta_3 - 1584\zeta_3^2 - 162\zeta_4 + 1320\zeta_5),$$
$$-2(4994047 + 11517108\zeta_3 - 57024\zeta_3^2 - 5931900\zeta_4 - 15037272\zeta_5 + 4989600\zeta_6 + 3810240\zeta_7)\Big\},$$

$$\gamma_{m40} = \left\{c_f^4, c_f^3, c_f^2, c_f d_2, c_f, d_2, d_3, 1\right\}.\Big\{972(50995 + 6784\zeta_3 + 16640\zeta_5),$$
$$-54(2565029 + 1880640\zeta_3 - 266112\zeta_4 - 1420800\zeta_5),$$
$$108(2625197 + 1740528\zeta_3 - 125136\zeta_4 - 2379360\zeta_5 - 665280\zeta_7),$$
$$373248(141 + 80\zeta_3 - 530\zeta_5),$$
$$-8(25256617 + 16408008\zeta_3 + 627264\zeta_3^2 - 812592\zeta_4 - 40411440\zeta_5 + 3920400\zeta_6 - 5987520\zeta_7),$$
$$-6912(9598 + 453\zeta_3 + 4356\zeta_3^2 + 1485\zeta_4 - 26100\zeta_5 - 1386\zeta_7),$$
$$5184(537 + 2494\zeta_3 + 5808\zeta_3^2 + 396\zeta_4 - 7820\zeta_5 - 1848\zeta_7),$$
$$4(22663417 + 10054464\zeta_3 + 1254528\zeta_3^2 - 1695276\zeta_4 - 41734440\zeta_5 + 7840800\zeta_6 + 5987520\zeta_7)\Big\}.$$

in full agreement with [Baikov, Chetyrkin, Kühn '14, Baikov, Chetyrkin, Kühn '17]

γ_2: 2, 3 loop

The field anomalous dimension γ_2 is gauge dependent.

$$2^2\,\gamma_{21} = n_f\Big[-8\Big] + \Big[-6c_f + 34 - 10\xi + \xi^2\Big]\,,$$

$$2^5 3^2\,\gamma_{22} = n_f^2\Big[640\Big] + n_f\Big[8(108c_f - 1301 + 153\xi)\Big]$$

$$+\Big[432c_f^2 - 72(143 - 48\zeta_3)c_f + 2(10559 - 1080\zeta_3)$$

$$-9\xi(371 + 48\zeta_3) + 27\xi^2(23 + 4\zeta_3) - 90\xi^3\Big]\,,$$

γ_2: 4 loop

$$
\begin{aligned}
2^4 3^5\,\gamma_{23} = {} & n_f^3\Big[13440\Big] + n_f^2\Big[6912(19-18\zeta_3)c_f \\
& + 16(6835+9072\zeta_3) + 64\xi(269-324\zeta_3)\Big] + n_f\Big[5184(19-48\zeta_3)c_f^2 \\
& + \big(-108(2407-1584\zeta_3-1296\zeta_4-5760\zeta_5) \\
& + 324\xi(767-528\zeta_3-144\zeta_4)\big)c_f + 497664 d_1 \\
& - (1365691+154224\zeta_3+97200\zeta_4+311040\zeta_5) \\
& + \xi(48865+152928\zeta_3+29160\zeta_4) - 54\xi^2(109+84\zeta_3-18\zeta_4)\Big] \\
& + \Big[-486(1027+3200\zeta_3-5120\zeta_5)c_f^3 + 324(5131+10176\zeta_3-17280\zeta_5)c_f^2 \\
& + \big(-108(23777+7704\zeta_3+2376\zeta_4-28440\zeta_5) - 1944\xi(6-7\zeta_3+10\zeta_5)\big)c_f
\end{aligned}
$$

γ_2: 4 loop cont'd

$$
\begin{aligned}
&+ 486\big(16(-33 + 95\zeta_3 - 85\zeta_5) - 8\xi(1 + 48\zeta_3 - 70\zeta_5) \\
&- 8\xi^2(7\zeta_3 + 5\zeta_5) + 20\xi^3(2\zeta_3 - \zeta_5) \\
&- \xi^4(7\zeta_3 - 5\zeta_5)\big)d_2 + (10059589/4 - 241218\zeta_3 + 168156\zeta_4 - 604260\zeta_5) \\
&- \xi(2127929/8 + 164106\zeta_3 - 21141\zeta_4 - 107730\zeta_5) \\
&+ 27\xi^2(13883 + 9108\zeta_3 - 1548\zeta_4 - 1920\zeta_5)/8 \\
&- 81\xi^3(263 + 65\zeta_3 - 9\zeta_4 + 20\zeta_5)/2 + 81\xi^4(57 + \zeta_3 + 10\zeta_5)/4\Big] \,.
\end{aligned}
$$

We obtained the result including the full ξ dependence.

γ_2: 5 loop

At five loop we only have the result in the Feynman gauge, $\xi = 0$

$$24^3\,\gamma_{24} = \frac{83-144\zeta_3}{72}\left[16n_f\right]^4 + \gamma_{243}\left[16n_f\right]^3 + \gamma_{242}\left[16n_f\right]^2 + \gamma_{241}\left[16n_f\right] + \gamma_{240} + \mathcal{O}(\xi)$$

For the coefficients γ_{24i} we obtain

$$\gamma_{243} = \left\{c_f, 1\right\}.\left\{-659/18 + 312\zeta_3 - 216\zeta_4, -3443/48 - 255\zeta_3 + 252\zeta_4\right\},$$

$$\gamma_{242} = \left\{c_f^2, c_f, d_1, 1\right\}.\left\{-2(2497 - 1200\zeta_3 + 3456\zeta_4 - 8640\zeta_5),\right.$$
$$477433/12 - 45636\zeta_3 + 4608\zeta_3^2 + 11448\zeta_4 - 65088\zeta_5 + 28800\zeta_6,$$
$$-384(115 - 33\zeta_3 - 90\zeta_5), 3015955/72 + 69509\zeta_3 - 2304\zeta_3^2$$
$$\left.-12861\zeta_4 + 16662\zeta_5 - 14400\zeta_6 - 11907\zeta_7\right\}$$

γ_2: 5 loop cont'd

$$\gamma_{241} = \left\{c_f^3, c_f^2, c_f d_1, c_f, d_1, d_2, 1\right\}.\Big\{24(29209 + 89984\zeta_3 + 12288\zeta_3^2$$
$$-28800\zeta_4 - 187520\zeta_5 + 76800\zeta_6),$$
$$-4(296177 + 517020\zeta_3 + 26784\zeta_3^2 - 469908\zeta_4 - 4104720\zeta_5$$
$$+1069200\zeta_6 + 3011904\zeta_7),$$
$$-2304(748 + 4536\zeta_3 - 1368\zeta_3^2 - 6780\zeta_5 + 3255\zeta_7),$$
$$8(115334 - 37764\zeta_3 - 123012\zeta_3^2 - 49923\zeta_4 - 1124556\zeta_5$$
$$+133650\zeta_6 + 1519308\zeta_7),$$
$$192(16732 + 39912\zeta_3 - 10944\zeta_3^2 - 72960\zeta_5 + 36771\zeta_7),$$
$$96(6158 - 13952\zeta_3 - 372\zeta_3^2 + 2880\zeta_4 - 39475\zeta_5 - 3900\zeta_6 + 45696\zeta_7),$$
$$-34919359/9 - 753797\zeta_3 + 548148\zeta_3^2 - 135063\zeta_4 + 1759474\zeta_5$$
$$+265350\zeta_6 - 2647806\zeta_7\Big\},$$

γ_2: 5 loop cont'd

$$\gamma_{240} = \left\{c_f^4, c_f^3, c_f^2, c_f d_2, c_f, d_2, d_3, 1\right\}.\Big\{1728(4977 + 128000\zeta_3 + 19968\zeta_3^2 + 180800\zeta_5 - 381024\zeta_7),$$
$$-96(835739 + 8494144\zeta_3 + 1182336\zeta_3^2 - 316800\zeta_4 + 3983360\zeta_5 + 844800\zeta_6 - 17852688\zeta_7),$$
$$192(825361 + 5472068\zeta_3 + 651816\zeta_3^2 - 335808\zeta_4 - 1140420\zeta_5 + 950400\zeta_6 - 8056377\zeta_7),$$
$$4608(10 + 53226\zeta_3 - 15264\zeta_3^2 + 2145\zeta_5 - 45885\zeta_7),\ -16(84040774/9$$
$$+33396648\zeta_3 + 2804616\zeta_3^2 - 838782\zeta_4 - 18160944\zeta_5 + 6252300\zeta_6 - 41015331\zeta_7),$$
$$-384(43066 + 628802\zeta_3 - 160998\zeta_3^2 + 36540\zeta_4 - 201125\zeta_5 - 53475\zeta_6 - 403263\zeta_7),$$
$$-72(20566 - 218812\zeta_3 - 79080\zeta_3^2 - 13212\zeta_4 + 760220\zeta_5 + 20100\zeta_6 - 660667\zeta_7),\ 804023630/9$$
$$+101490400\zeta_3 + 3143352\zeta_3^2 + 7356024\zeta_4 - 86186276\zeta_5 + 18372900\zeta_6 - 115799439\zeta_7\Big\}.$$

in full agreement with [Baikov, Chetyrkin, Kühn '14] for $N_c = 3$

Outline

β function

We introduce the renormalization constants as

$$\begin{aligned} \psi_b &= \sqrt{Z_2}\psi_r \ , \quad A_b = \sqrt{Z_3}A_r \ , \quad c_b = \sqrt{Z_3^c}c_r \ , \\ m_b &= Z_m m_r \ , \quad g_b = \mu^\varepsilon Z_g g_r \ , \quad \xi_{L,b} = Z_\xi \xi_{L,r} \ , \end{aligned}$$

or alternatively for the vertices

$$Z_1^j \text{ where } j \in \{\text{3g,4g,ccg,}\psi\psi\text{g}\}$$

The anomalous dimensions are related through Ward identities

$$\begin{aligned} \gamma_3 &= 2(\gamma_1^{ccg} - \gamma_3^c) - \beta \ , \quad \gamma_1^{3g} = 3(\gamma_1^{ccg} - \gamma_3^c) - \beta \ , \\ \gamma_1^{4g} &= 4(\gamma_1^{ccg} - \gamma_3^c) - \beta \ , \quad \gamma_1^{\psi\psi g} = \gamma_1^{ccg} - \gamma_3^c + \gamma_2 \ , \end{aligned}$$

and thus we choose to evaluate

$$Z_1^{ccg} = \sqrt{Z_3}\, Z_3^c\, Z_g$$

Ghost propagator: γ_3^c: full gauge dependence

$$
\begin{aligned}
\gamma_3^c &= -a\Big[-\tfrac{1}{4}(2+\xi)+\gamma_{31}^c a+\gamma_{32}^c a^2+\gamma_{33}^c a^3+\gamma_{34}^c a^4+\ldots\Big] \\
{}^5\,3^1\,\gamma_{31}^c &= 5[16n_f]-2(98-3\xi)\ , \\
2^8\,3^3\,\gamma_{32}^c &= 35[16n_f]^2+\big(324(15-16\zeta_3)c_f+2(5+189\xi+1944\zeta_3)\big)[16n_f] \\
&\quad -4(14656+1485\xi-405\xi^2+81\xi^3)-648(4-\xi)(2-\xi)\zeta_3\ .
\end{aligned}
$$

γ_3^c: 4-loop, full gauge dependence

$$
\begin{aligned}
2^{11}3^4\gamma_{33}^c \quad = \quad & (83-144\zeta_3)[16n_f]^3+\Big\{c_f,1\Big\}.\Big\{24(1080\zeta_3-648\zeta_4-115),\\
& 2(779\xi-8315)/3-432(43+2\xi)\zeta_3+11664\zeta_4\Big\}[16n_f]^2\\
& +\Big\{c_f^2,d_2,c_f,1\Big\}.\Big\{-864(271+888\zeta_3-1440\zeta_5),124416(4\zeta_3-5\zeta_5),\\
& 24\big(22517+3825\xi-864(43+\xi)\zeta_3+1296(23-\xi)\zeta_4-25920\zeta_5\big),\\
& 432(2983+42\xi-6\xi^2)\zeta_3-648(846-46\xi+\xi^2)\zeta_4-570240\zeta_5\\
& +14(128354-722\xi-837\xi^2)/3\Big\}[16n_f]\\
& +\Big\{d_3,1\Big\}.\Big\{1296(12(28-6\xi+\xi^2)-4(2392+108\xi-63\xi^2-17\xi^3+16\xi^4)\zeta_3\\
& +5(1696+544\xi-252\xi^2+42\xi^3+7\xi^4)\zeta_5),\\
& -4(8202784+512546\xi-111402\xi^2+28107\xi^3-3888\xi^4)/3\\
& -36(159040-19104\xi-162\xi^2+1092\xi^3-123\xi^4)\zeta_3\\
& +1296(492-376\xi+91\xi^2-9\xi^3)\zeta_4\\
& +270(28832+320\xi-732\xi^2+186\xi^3-7\xi^4)\zeta_5\Big\}\ .
\end{aligned}
$$

5-loop γ_3^c, Feynman gauge $\xi = 0$

$$
\begin{aligned}
2^{14}\,3^5\,\gamma_{34}^c &= \gamma_{344}^c\,[16n_f]^4 + \gamma_{343}^c\,[16n_f]^3 + \gamma_{342}^c\,[16n_f]^2 + \gamma_{341}^c\,[16n_f] + \gamma_{340}^c\,, \quad \gamma_{344}^c = 3(65 + 80\zeta_3 - 144\zeta_4)\,, \\
\gamma_{343}^c &= \Big\{c_f, 1\Big\}.\Big\{-2(14765 + 12528\zeta_3 - 38880\zeta_4 + 20736\zeta_5), -3(8325 + 15664\zeta_3 + 12240\zeta_4 - 33408\zeta_5)\Big\}\,, \\
\gamma_{342}^c &= \Big\{c_f^2, c_f, d_1, d_2, 1\Big\}.\Big\{-72(53927 - 182112\zeta_3 + 48384\zeta_3^2 + 42768\zeta_4 + 144000\zeta_5 - 86400\zeta_6), \\
&\quad -4(364361 + 484488\zeta_3 - 1804032\zeta_3^2 + 1868184\zeta_4 - 2239488\zeta_5 + 777600\zeta_6), \\
&\quad 20736(107 - 109\zeta_3 - 96\zeta_3^2 - 36\zeta_4 + 180\zeta_5), -41472(52\zeta_3 + 18\zeta_3^2 - 36\zeta_4 - 125\zeta_5 + 75\zeta_6), \\
&\quad 2(239495 - 3082212\zeta_3 - 1721088\zeta_3^2 + 3863376\zeta_4 - 156384\zeta_5 - 1425600\zeta_6)\Big\}\,, \\
\gamma_{341}^c &= \Big\{c_f^3, c_f^2, c_f d_2, c_f, d_2, d_3, 1\Big\}.\Big\{746496(7 + 26\zeta_3 + 490\zeta_5 - 560\zeta_7), 576(24617 - 301866\zeta_3 \\
&\quad -196560\zeta_3^2 + 177066\zeta_4 + 274680\zeta_5 - 491400\zeta_6 + 725760\zeta_7), 165888(4 + 66\zeta_3 + 216\zeta_3^2 \\
&\quad -705\zeta_5 + 357\zeta_7), 16(4796303 - 9571932\zeta_3 + 6399648\zeta_3^2 + 11100240\zeta_4 - 16127424\zeta_5 + 8845200\zeta_6 \\
&\quad -10809288\zeta_7), -5184(4192 - 87152\zeta_3 + 21432\zeta_3^2 + 5616\zeta_4 + 89300\zeta_5 - 27300\zeta_6 - 20139\zeta_7), \\
&\quad -864(2805 - 86018\zeta_3 - 15960\zeta_3^2 + 43542\zeta_4 - 70360\zeta_5 - 68700\zeta_6 + 192906\zeta_7), \\
&\quad 2(52725013 + 136974540\zeta_3 + 1505088\zeta_3^2 - 118046052\zeta_4 - 226012536\zeta_5 \\
&\quad +84380400\zeta_6 + 143718624\zeta_7)\Big\}\,, \\
\gamma_{340}^c &= \Big\{d_3, 1\Big\}.\Big\{-6912(5326 + 771746\zeta_3 - 17934\zeta_3^2 - 209916\zeta_4 - 1172870\zeta_5 + 377625\zeta_6 \\
&\quad +396669\zeta_7), -8(192342607 + 174080040\zeta_3 + 36201384\zeta_3^2 - 103216464\zeta_4 - 855002232\zeta_5 \\
&\quad +222650100\zeta_6 + 492202872\zeta_7)\Big\}\,,
\end{aligned}
$$

Ghost-gluon vertex: γ_1^{ccg}: full gauge dependence

$$
\begin{aligned}
\gamma_1^{ccg} &= -a(1-\xi)\left[\tfrac{1}{2} + \tfrac{6-\xi}{8}\,a + \gamma_{12}^{ccg}a^2 + \gamma_{13}^{ccg}a^3 + \gamma_{14}^{ccg}a^4 + \dots\right] , \\
2^7\,\gamma_{12}^{ccg} &= -15[16n_f] + 2(250 - 59\xi + 10\xi^2) \; . \\
2^7\,3^5\,\gamma_{13}^{ccg} &= (-251 + 324\zeta_3)[16n_f]^2 \\
&\quad +\big(324(96\zeta_3 + 36\zeta_4 - 161)c_f - 6166 + 4077\xi/2 \\
&\quad -162(164 - 5\xi)\zeta_3 - 8748\zeta_4\big)[16n_f] \\
&\quad +1944\big((272 - 60\xi + 3\xi^2 + 7\xi^3)\zeta_3 \\
&\quad -5(56 - 12\xi + 3\xi^2 + \xi^3)\zeta_5\big)d_3 \\
&\quad +751120 - 27\xi(5434 - 1332\xi + 171\xi^2) \\
&\quad +81(2528 - 548\xi + 99\xi^2 - \xi^3)\zeta_3 \\
&\quad +1458(4-\xi)(2-\xi)\zeta_4 - 405(496 - 72\xi + 9\xi^2 + 2\xi^3)\zeta_5 \; .
\end{aligned}
$$

5-loop γ_1^{ccg}: Feynman gauge

$$
\begin{aligned}
2^{14}\,3^5\,\gamma_{14}^{ccg} &= \gamma_{143}^{ccg}\,[16n_f]^3 + \gamma_{142}^{ccg}\,[16n_f]^2 + \gamma_{141}^{ccg}\,[16n_f] + \gamma_{140}^{ccg}\,, \\
\gamma_{143}^{ccg} &= -2989 - 1440\zeta_3 + 5184\zeta_4\,, \\
\gamma_{142}^{ccg} &= \Big\{c_f, 1\Big\}.\Big\{1296(557 - 736\zeta_3 + 108\zeta_4 + 192\zeta_5), \\
&\quad 251891 + 1591056\zeta_3 - 335016\zeta_4 - 717984\zeta_5\Big\}\,, \\
\gamma_{141}^{ccg} &= \Big\{c_f^2, c_f, d_2, d_3, 1\Big\}.\Big\{5184(3731 + 9588\,zeta_3 - 1440\zeta_3^2 + 1332\zeta_4 - 10800\zeta_5 - 3600\zeta_6), \\
&\quad -1296(45129 - 14192\zeta_3 - 4032\zeta_3^2 + 5616\zeta_4 - 19296\zeta_5 - 7200\zeta_6), \\
&\quad -31104(1360\zeta_3 + 168\zeta_3^2 + 144\zeta_4 - 1260\zeta_5 - 300\zeta_6 - 441\zeta_7), \\
&\quad -10368(1126\zeta_3 + 150\zeta_3^2 - 567\zeta_4 - 1200\zeta_5 + 975\zeta_6 - 441\zeta_7), -42165410 \\
&\quad -432(145015\zeta_3 + 3564\zeta_3^2 - 9168\zeta_4 - 114001\zeta_5 - 10950\zeta_6 + 17640\zeta_7)\Big\}\,, \\
\gamma_{140}^{ccg} &= \Big\{d_3, 1\Big\}.\Big\{20736(70330\zeta_3 + 11076\zeta_3^2 - 8856\zeta_4 - 81380\zeta_5 + 16500\zeta_6 - 12607\zeta_7 - 2451), \\
&\quad 8(114251711 + 54643392\zeta_3 + 7060608\zeta_3^2 - 7531704\zeta_4 - 143288568\zeta_5 + 9023400\zeta_6 \\
&\quad +52599078\zeta_7)\Big\}\,.
\end{aligned}
$$

gluon propagator: γ_3: full gauge dependence

$$
\begin{aligned}
\gamma_3 &= a\Big[\tfrac{1}{6}(10+3\xi-8n_f)+\gamma_{31}a+\gamma_{32}a^2+\gamma_{33}a^3+\gamma_{34}a^4+\ldots\Big] \\
2^3\gamma_{31} &= \Big\{c_f,1\Big\}.\Big\{-32n_f,-40n_f-2\xi^2+15\xi+46\Big\} \\
2^5 3^2\gamma_{32} &= \Big\{c_f^2,c_f,1\Big\}.\Big\{576n_f,1408n_f^2-16\,(432\zeta_3+5)\,n_f, \\
&\quad -54\,(\zeta_3+9)\,\xi^2+n_f\,(16\,(324\zeta_3-875)-576\xi) \\
&\quad +18\,(18\zeta_3+127)\,\xi+2432n_f^2-2\,(216\zeta_3-4051)+63\xi^3\Big\}
\end{aligned}
$$

γ_3: 4-loop, full gauge dependence

$$
\begin{aligned}
2^9 3^5 \gamma_{33} \quad = \quad & \left\{d_1, d_2, d_3, c_f^3, c_f^2, c_f, 1\right\}.\Big\{884736\left(24\zeta_3 - 11\right)n_f^2, -110592\left(516\zeta_3 + 135\zeta_5 - 64\right)n_f, \\
& -1944\left(8\zeta_3 + 5\zeta_5\right)\xi^4 + 3888\left(18\zeta_3 + 65\zeta_5\right)\xi^3 - 23328\left(21\zeta_3 + 55\zeta_5 - 1\right)\xi^2 \\
& +15552\left(278\zeta_3 - 9\right)\xi + 3456\left(1842\zeta_3 + 6030\zeta_5 - 131\right), 5723136 n_f, -36864\left(264\zeta_3 - 169\right)n_f^2 \\
& -2304\left(5880\zeta_3 - 12960\zeta_5 + 10847\right)n_f, \left(5184\left(672\zeta_3 + 144\zeta_4 - 863\right)\xi\right. \\
& \left.-64\left(538272\zeta_3 - 244944\zeta_4 + 233280\zeta_5 - 363565\right)\right)n_f \\
& +1024\left(15768\zeta_3 - 5832\zeta_4 + 7541\right)n_f^2 + 630784 n_f^3, \\
& 81\left(74\zeta_3 - 115\zeta_5 - 360\right)\xi^4 + 162\left(36\zeta_3 - 36\zeta_4 + 325\zeta_5 + 1657\right)\xi^3 \\
& -324\left(1267\zeta_3 - 294\zeta_4 + 105\zeta_5 + 3823\right)\xi^2 + n_f\left(1296\left(32\zeta_3\right.\right. \\
& \left.-12\zeta_4 + 129\right)\xi^2 - 32\left(95904\zeta_3 + 12636\zeta_4 + 35345\right)\xi \\
& +32\left(1249020\zeta_3 - 376164\zeta_4 - 427680\zeta_5 - 1404961\right)\Big) \\
& +n_f^2\left(256\left(1296\zeta_3 - 1229\right)\xi - 256\left(18360\zeta_3 - 17496\zeta_4 - 41273\right)\right) \\
& -2048\left(432\zeta_3 - 355\right)n_f^3 + 4\left(756216\zeta_3 - 141912\zeta_4 - 427680\zeta_5 + 1539403\right)\xi \\
& -32\left(338580\zeta_3 - 26973\zeta_4 - 415125\zeta_5 - 504770\right)\Big\}
\end{aligned}
$$

5-loop γ_3: Feynman gauge

$$
\begin{aligned}
2^9\, 3^5\, \gamma_{34} &= \gamma_{344}\,[16 n_f]^4 + \gamma_{343}\,[16 n_f]^3 + \gamma_{342}\,[16 n_f]^2 + \gamma_{341}\,[16 n_f] + \gamma_{340} \\
\gamma_{344} &= \Big\{ c_f\,,1 \Big\}.\Big\{ -2\,(144\zeta_3 + 107)\,, 619 - 864\zeta_4 \Big\} \\
\gamma_{343} &= \Big\{ c_f^2\,, c_f\,, 1\,, d_1 \Big\}.\Big\{ 24\,(11424\zeta_3 - 4752\zeta_4 - 4961)\,, -4\,(63648\zeta_3 - 57024\zeta_4 + 20736\zeta_5 + 16973)\,, \\
&\quad -8\,(25992\zeta_3 + 5940\zeta_4 - 29376\zeta_5 + 14843)\,, -6912\,(123\zeta_3 - 36\zeta_4 - 60\zeta_5 - 55) \Big\} \\
\gamma_{342} &= \Big\{ c_f^3\,, c_f^2\,, c_f\,, 1\,, c_f d_1\,, d_1\,, d_2 \Big\}.\Big\{ -3456\,(3216\zeta_3 - 6960\zeta_5 + 2509)\,, \\
&\quad -144\left(48384\zeta_3^2 - 67600\zeta_3 + 1584\zeta_4 + 302400\zeta_5 - 86400\zeta_6 - 135571\right), \\
&\quad 8\left(1804032\zeta_3^2 + 1658520\zeta_3 - 2514888\zeta_4 + 3214080\zeta_5 - 777600\zeta_6 - 476417\right), \\
&\quad -18\left(382464\zeta_3^2 + 271448\zeta_3 - 865800\zeta_4 + 588576\zeta_5 + 316800\zeta_6 - 1524019\right), \\
&\quad 1658880\,(16\zeta_3 - 40\zeta_5 + 13)\,, 13824\left(-288\zeta_3^2 + 4715\zeta_3 - 900\zeta_4 + 820\zeta_5 - 2373\right), \\
&\quad -27648\left(54\zeta_3^2 - 2354\zeta_3 + 360\zeta_4 - 295\zeta_5 + 225\zeta_6 + 230\right) \Big\}
\end{aligned}
$$

5-loop γ_3: Feynman gauge

$$
\begin{aligned}
\gamma_{341} \quad = \quad & \left\{c_f^4, c_f^3, c_f^2, c_f, 1, c_f d_2, d_3, d_2\right\}.\Big\{ -41472\,(768\zeta_3 + 4157)\,, 82944\,(6164\zeta_3 + 1340\zeta_5 - 10080\zeta_7 + 11277)\,, \\
& -768\left(275400\zeta_3^2 + 1189209\zeta_3 - 195345\zeta_4 - 1942380\zeta_5 + 688500\zeta_6 - 1088640\zeta_7 + 2208371\right), \\
& 1152\left(168696\zeta_3^2 - 266283\zeta_3 + 385920\zeta_4 - 1119240\zeta_5 + 229500\zeta_6 - 300258\zeta_7 + 1139437\right), \\
& -8\left(-1137456\zeta_3^2 - 88182810\zeta_3 + 66057714\zeta_4 + 87060456\zeta_5 - 41007600\zeta_6 - 73764432\zeta_7 + 124662829\right), \\
& 331776\left(216\zeta_3^2 + 386\zeta_3 + 895\zeta_5 + 357\zeta_7 - 236\right), \\
& 1728\left(17760\zeta_3^2 - 232502\zeta_3 + 342\zeta_4 + 119960\zeta_5 + 80400\zeta_6 - 198198\zeta_7 + 11659\right), \\
& 3456\left(-61272\zeta_3^2 - 735952\zeta_3 + 150480\zeta_4 + 249580\zeta_5 + 76500\zeta_6 + 52479\zeta_7 + 77920\right)\Big\} \\
\gamma_{340} \quad = \quad & \left\{1, c_f d_2, d_2, d_3\right\}.\Big\{ -32\big(21630996\zeta_3^2 + 116865396\zeta_3 - 57883140\zeta_4 - 484699320\zeta_5 + 115836750\zeta_6 \\
& +272400975\zeta_7 - 112182361\big), 331776\big(216\zeta_3^2 + 386\zeta_3 + 895\zeta_5 + 357\zeta_7 - 236\big), 3456\big(-61272\zeta_3^2 \\
& -735952\zeta_3 + 150480\zeta_4 + 249580\zeta_5 + 76500\zeta_6 + 52479\zeta_7 + 77920\big), \\
& 1728\left(17760\zeta_3^2 - 232502\zeta_3 + 342\zeta_4 + 119960\zeta_5 + 80400\zeta_6 - 198198\zeta_7 + 11659\right)\Big\}
\end{aligned}
$$

β function

$$\partial_{\ln \mu^2} a = -a\Big[\varepsilon - \beta\Big] = -a\Big[\varepsilon + b_0\, a + b_1\, a^2 + b_2\, a^3 + b_3\, a^4 + b_4\, a^5 + \ldots\Big]$$

$$\begin{aligned}
3^1\, b_0 &= \big[-4\big] n_f + 11\ , \\
3^2\, b_1 &= \big[-36 c_f - 60\big] n_f + 102\ , \\
3^3\, b_2 &= \big[132 c_f + 158\big] n_f^2 + \big[54 c_f^2 - 615 c_f - 1415\big] n_f + 2857/2\ , \\
3^5\, b_3 &= \big[1232 c_f + 424\big] n_f^3 + 432(132\zeta_3 - 5) d_3 + (150653/2 - 1188\zeta_3) + \\
&\quad \big[72(169 - 264\zeta_3) c_f^2 + 64(268 + 189\zeta_3) c_f + 1728(24\zeta_3 - 11) d_1 \\
&\quad + 6(3965 + 1008\zeta_3)\big] n_f^2 + \big[11178 c_f^3 + 36(264\zeta_3 - 1051) c_f^2 \\
&\quad + (7073 - 17712\zeta_3) c_f + 3456(4 - 39\zeta_3) d_2 + 3(3672\zeta_3 - 39143)\big] n_f,
\end{aligned}$$

5-loop β function

$$
\begin{aligned}
3^5\, b_4 &= b_{44}\, n_f^4 + b_{43}\, n_f^3 + b_{42}\, n_f^2 + b_{41}\, n_f + b_{40}\ , \\
b_{44} &= \Big\{c_f, 1\Big\}.\Big\{ -8(107 + 144\zeta_3), 4(229 - 480\zeta_3)\Big\}\ , \\
b_{43} &= \Big\{c_f^2, c_f, d_1, 1\Big\}.\Big\{ -6(4961 - 11424\zeta_3 + 4752\zeta_4), -48(46 + 1065\zeta_3 - 378\zeta_4), \\
&\quad 1728(55 - 123\zeta_3 + 36\zeta_4 + 60\zeta_5), -3(6231 + 9736\zeta_3 - 3024\zeta_4 - 2880\zeta_5)\Big\}\ , \\
b_{42} &= \Big\{c_f^3, c_f^2, c_f d_1, c_f, d_2, d_1, 1\Big\}.\Big\{ -54(2509 + 3216\zeta_3 - 6960\zeta_5), \\
&\quad 9(94749/2 - 28628\zeta_3 + 10296\zeta_4 - 39600\zeta_5), 25920(13 + 16\zeta_3 - 40\zeta_5), \\
&\quad 3(5701/2 + 79356\zeta_3 - 25488\zeta_4 + 43200\zeta_5), -864(115 - 1255\zeta_3 + 234\zeta_4 + 40\zeta_5), \\
&\quad -432(1347 - 2521\zeta_3 + 396\zeta_4 - 140\zeta_5), 843067/2 + 166014\zeta_3 - 8424\zeta_4 - 178200\zeta_5\Big\}\ , \\
b_{41} &= \Big\{c_f^4, c_f^3, c_f^2, c_f d_2, c_f, d_3, d_2, 1\Big\}.\Big\{ -81(4157/2 + 384\zeta_3), 81(11151 + 5696\zeta_3 - 7480\zeta_5), \\
&\quad -3(548732 + 151743\zeta_3 + 13068\zeta_4 - 346140\zeta_5), -25920(3 - 4\zeta_3 - 20\zeta_5), \\
&\quad 8141995/8 + 35478\zeta_3 + 73062\zeta_4 - 706320\zeta_5, 216(113 - 2594\zeta_3 + 396\zeta_4 + 500\zeta_5), \\
&\quad 216(1414 - 15967\zeta_3 + 2574\zeta_4 + 8440\zeta_5), -5048959/4 + 31515\zeta_3 - 47223\zeta_4 + 298890\zeta_5\Big\}\ , \\
b_{40} &= \Big\{d_3, 1\Big\}.\Big\{ -162(257 - 9358\zeta_3 + 1452\zeta_4 + 7700\zeta_5), \\
&\quad 8296235/16 - 4890\zeta_3 + 9801\zeta_4/2 - 28215\zeta_5\Big\}\ .
\end{aligned}
$$

all results in full agreement with the literature

[v. Ritbergen, Vermaseren, Larin; Czakon] [Baikov,Chetyrkin,Kühn; Herzog,Ruijl,Ueda,Vermaseren]

β function: numerical results

$$\beta = +(0.166667 n_f - 2.75)\left(\frac{\alpha_s}{\pi}\right) + (0.791667 n_f - 6.375)\left(\frac{\alpha_s}{\pi}\right)^2$$
$$+ \left(-0.0940394 n_f^2 + 4.36892 n_f - 22.3203\right)\left(\frac{\alpha_s}{\pi}\right)^3$$
$$+ \left(-0.0058567 n_f^3 - 1.58238 n_f^2 + 27.1339 n_f - 114.23\right)\left(\frac{\alpha_s}{\pi}\right)^4$$
$$+ \left(0.00179929 n_f^4 + 0.225857 n_f^3 - 17.156 n_f^2 + 181.799 n_f - 524.558\right)\left(\frac{\alpha_s}{\pi}\right)^5$$

	1	2	3	4	5
$n_f = 4$	-2.08333	-3.20833	-6.34925	-31.3874	-56.9439
$n_f = 5$	-1.91667	-2.41667	-2.82668	-18.8522	-15.108
$n_f = 6$	-1.75	-1.625	0.507812	-9.65736	-0.265067

Conclusions

- Presented full gauge dependence up to four loops for QCD renormalization constants
- Presented the five-loop results for QCD renormalization constants (in Feynman gauge) including the β function
- All results are available for a general gauge group
- Full agreement with available results in the literature
- Completely independent calculation