Advances in amplitudes with off-shell partons

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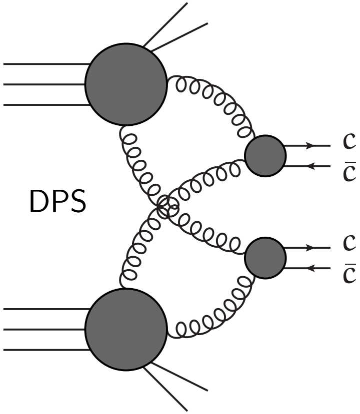
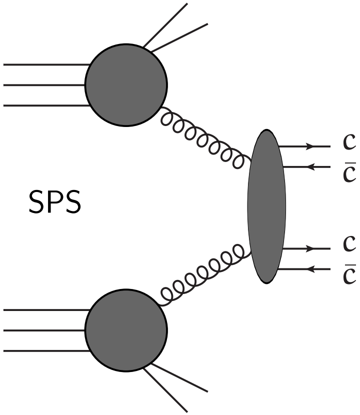
Outline

- DPS vs SPS for $\mathrm{pp} \to \mathrm{c\bar{c}\,c\bar{c}}$
- $\mathrm{k_T}$-factorization vs collinear factorization for $\mathrm{pp} \to \mathrm{c\bar{c}\,c\bar{c}}$
- $\mathrm{pp} \to 4\mathrm{j}$ with $\mathrm{k_T}$-factorization
- Off-shell amplitudes
- BCFW recursion for amplitudes with off-shell partons
- Towards NLO
- Conclusions

DPS vs SPS for $pp \to c\bar{c}\,c\bar{c}$

- production of $c\bar{c}\,c\bar{c}$ is a good place to study DPS effects Łuszczak, Maciuła, Szczurek 2012
- DPS $c\bar{c}\,c\bar{c}$ cross section approaches $c\bar{c}$ cross section for large energies
- DPS $c\bar{c}\,c\bar{c}$ cross section is orders of magnitude larger than LO SPS $c\bar{c}\,c\bar{c}$ cross section Schäfer, Szczurek 2012, Maciuła, Szczurek, AvH 2014
- LHCb measured a surprisingly large cross section for the production of D-meson pairs JHEP 06 141 (2012)





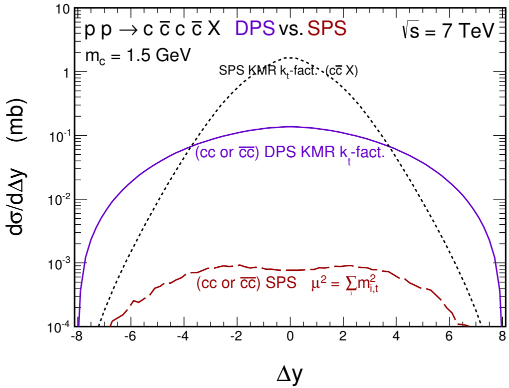
DPS vs SPS for $pp \longrightarrow c\bar{c}\, c\bar{c}$

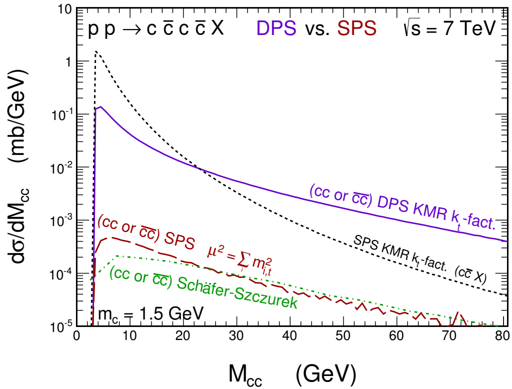
Maciuła, Szczurek, AvH 2014

Simple factorized model

$$d\sigma^{DPS}(pp \to c\bar{c}\, c\bar{c}X) = \frac{1}{2\sigma_{eff}}\, d\sigma^{SPS}(pp \to c\bar{c}X_1)\, d\sigma^{SPS}(pp \to c\bar{c}X_2)$$

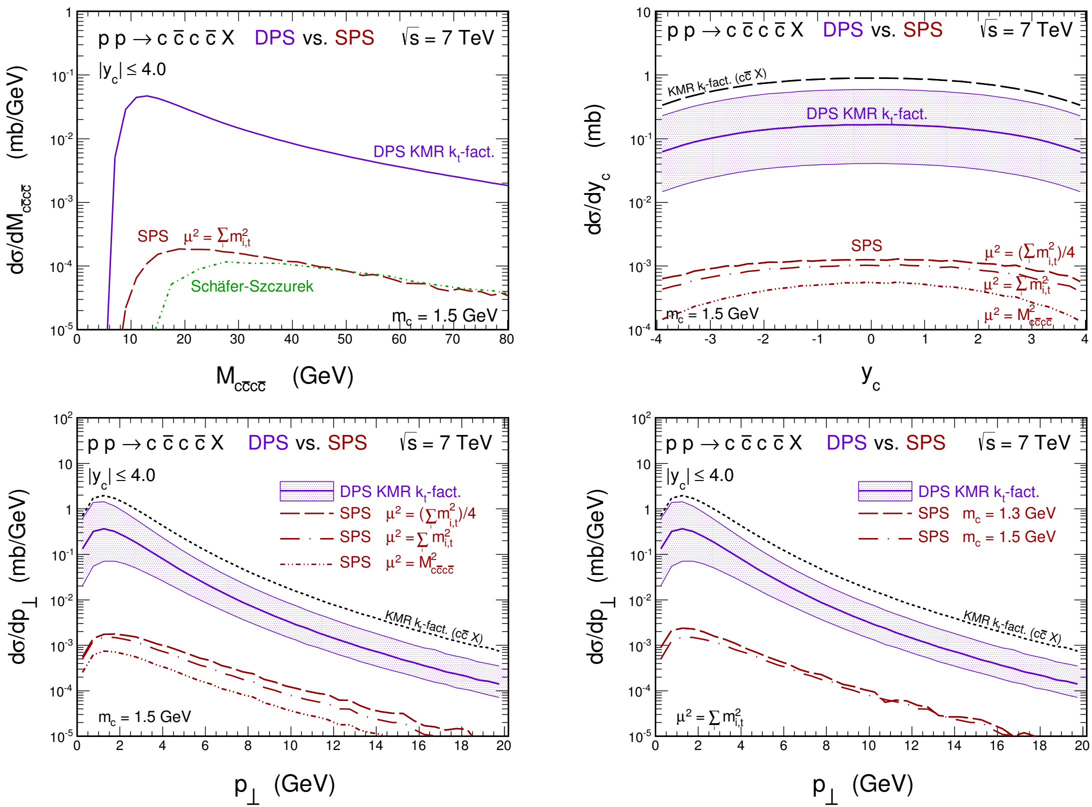
with $\sigma_{eff} = 15\text{mb}$.





DPS vs SPS for $pp \longrightarrow c\bar{c}\,c\bar{c}$

Maciuła, Szczurek, AvH 2014



High Energy Factorization

a.k.a. k_T-factorization

Catani, Ciafaloni, Hautmann 1991 | Collins, Ellis 1991

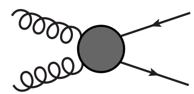
$$\sigma_{h_1,h_2\to QQ} = \int d^2k_{1\perp}\,\frac{dx_1}{x_1}\,\mathcal{F}(x_1,k_{1\perp})\,d^2k_{2\perp}\,\frac{dx_2}{x_2}\,\mathcal{F}(x_2,k_{1\perp})\,\hat{\sigma}_{gg}\left(\frac{m^2}{x_1x_2s},\frac{k_{1\perp}}{m},\frac{k_{2\perp}}{m}\right)$$

- reduces to collinear factorization for $s \gg m^2 \gg k_\perp^2$, but holds al so for $s \gg m^2 \sim k_\perp^2$
- typically associated with small-x physics, forward physics, saturation, heavy-ions . . .
- allows for higher-order kinematical effects at leading order

- requires matrix elements with *off-shell* initial-state partons with $k_i^2 = k_{i\perp}^2 < 0$

$k_1 = x_1p_A + k_{1\perp}$

$k_2 = x_2p_B + k_{2\perp}$



- $k_\perp$-dependent $\mathcal{F}$ may satisfy BFKL-eqn, CCFM-eqn, BK-eqn, KGBJS-eqn, . . .

High Energy Factorization a.k.a. k_T-factorization

Catani, Ciafaloni, Hautmann 1991 Collins, Ellis 1991

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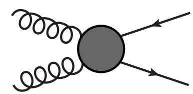


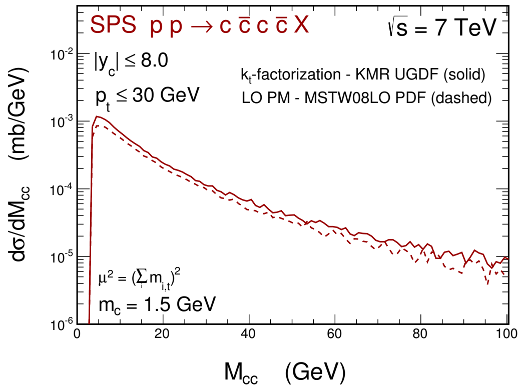
- $k_\perp$-dependent $\mathcal{F}$ may satisfy BFKL-eqn, CCFM-eqn, BK-eqn, KGBJS-eqn, ...
- in particular KMR-type unintegrated pdfs (Kimber, Martin, Ryskin 2000) contain essential hard scale dependence via Sudakov resummation

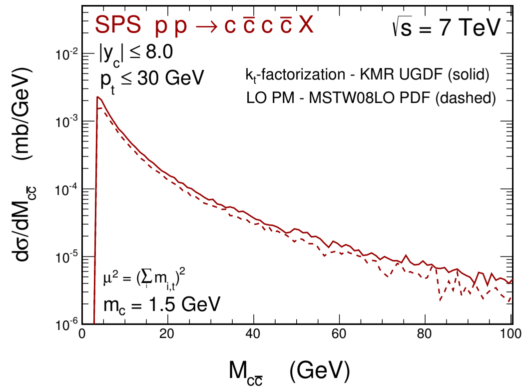
$$T_a(k^2,\mu^2) = \exp\left(-\int_{k^2}^{\mu^2}\frac{dp^2}{p^2}\frac{\alpha_S(p^2)}{2\pi}\sum_b\int_0^{k/(\mu+k)} dz\,P_{ba}(z)\right)$$

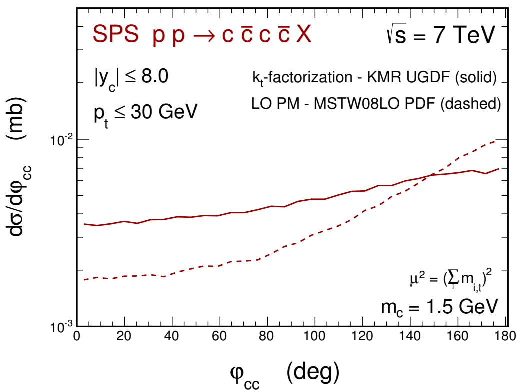
$$\mathcal{F}_a(x,k^2,\mu^2) = \partial_\lambda\big[T_a(\lambda,\mu^2)\,xg_a(x,\lambda)\big]_{\lambda=k^2}$$

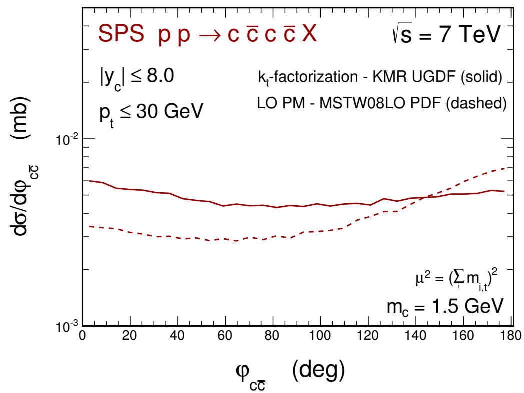
k_T vs collinear for $pp \to c\bar{c}\,c\bar{c}$

Maciuła, Szczurek, AvH 2015



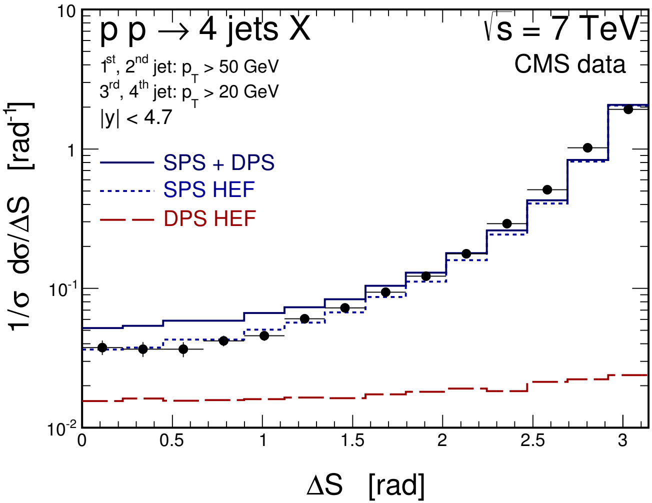


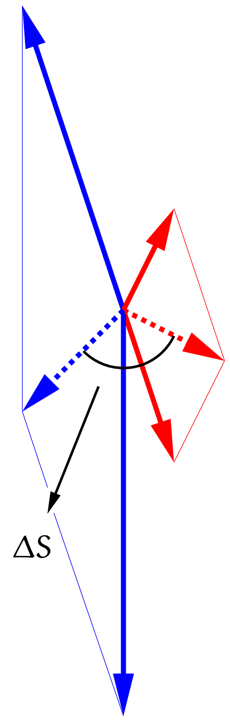




Four jets with k_T-factorization

Maciuła, Szczurek, Kutak, Serino, AvH 2016



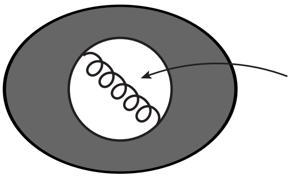


- ΔS is the azimutal angle between the sum of the two hardest jets and the sum of the two softest jets.
- This variable has no distribution at LO in collinear factorization: pairs would have to be back-to-back.
- Our (KMR-type) updfs DLC2016 describe data remarkably well.

Gauge invariance

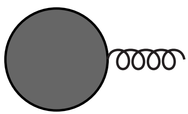
In order to be physically relevant, any scattering amplitude following the constructive definition given before must satisfy the following

Freedom in choice of gluon propagator:

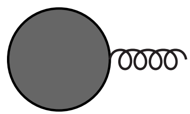


$$\left\{ \begin{array}{l} \dfrac{-i}{k^2}\left[g^{\mu\nu} - (1-\xi)\dfrac{k^\mu k^\nu}{k^2}\right] \\[2ex] \dfrac{-i}{k^2}\left[g^{\mu\nu} - \dfrac{k^\mu n^\nu + n^\mu k^\nu}{k\cdot n} + (n^2 + \xi k^2)\dfrac{k^\mu k^\nu}{(k\cdot n)^2}\right] \end{array} \right.$$

Ward identity:



$${}_\mu\varepsilon^\mu(k) \quad \to$$



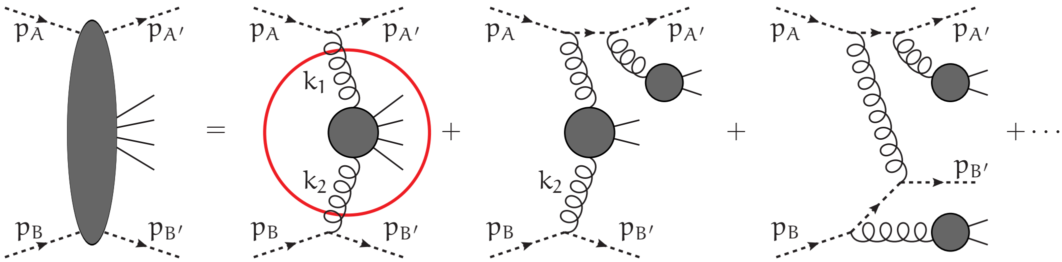
$${}_\mu k^\mu \; = \; 0$$

- Only holds if all external particles are on-shell.
- k_T-factorization requires off-shell initial-state momenta $k^\mu = p^\mu + k_T^\mu$.
- How to define amplitudes with off-shell intial-state momenta?

Amplitudes with off-shell partons

AvH, Kutak, Kotko 2013:

Embed the process in an on-shell process with auxiliary partons



Hadron momenta p_1, p_2:

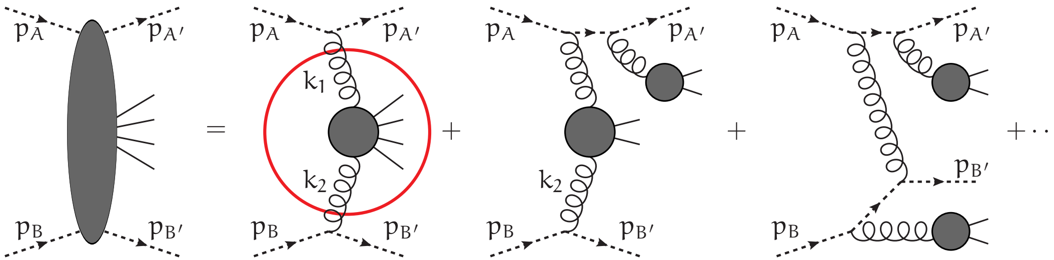
$p_1 \cdot p_A = p_1 \cdot p_{A'} = p_1 \cdot k_1 = 0$

$p_2 \cdot p_B = p_2 \cdot p_{B'} = p_2 \cdot k_2 = 0$

Amplitudes with off-shell partons

AvH, Kutak, Kotko 2013:

Embed the process in an on-shell process with auxiliary partons and eikonal Feynman rules.



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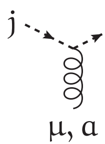
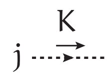


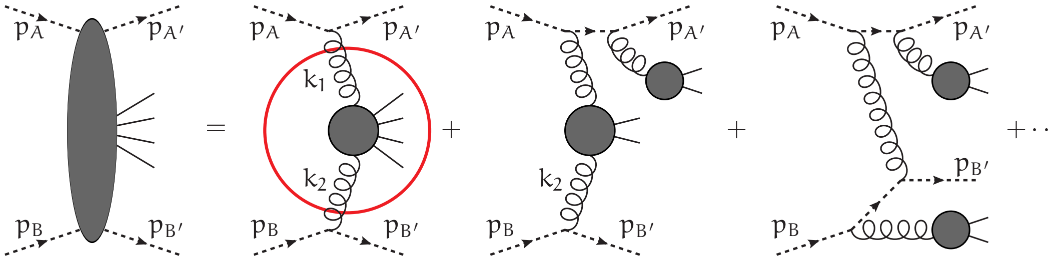
$= -i\, T^a_{i,j}\, p_1^\mu$

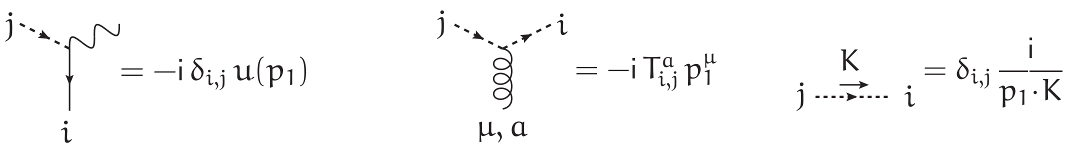


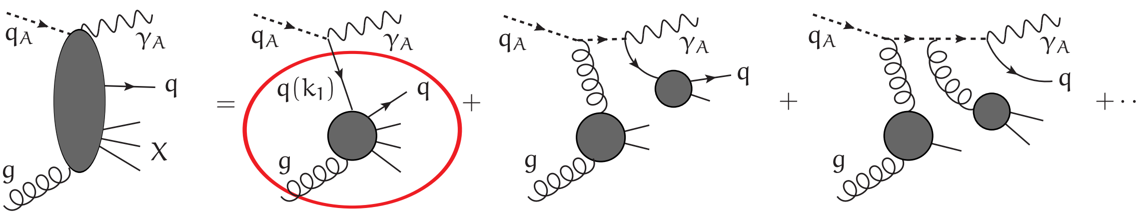
$= \delta_{i,j} \dfrac{i}{p_1 \cdot K}$

Amplitudes with off-shell partons

AvH, Kutak, Kotko 2013, AvH, Kutak, Salwa 2013:
Embed the process in an on-shell process with auxiliary partons and eikonal Feynman rules.







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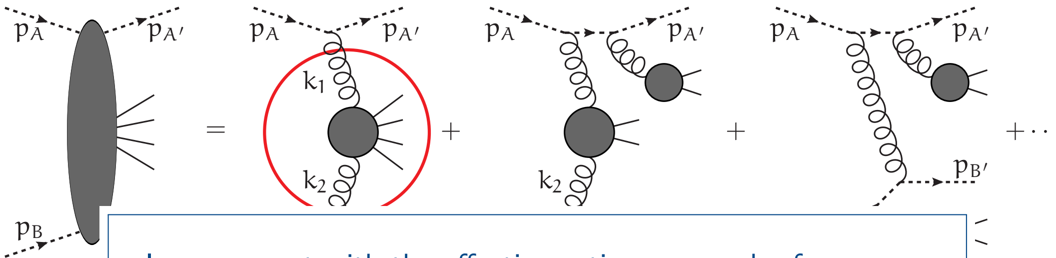
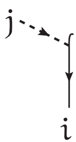
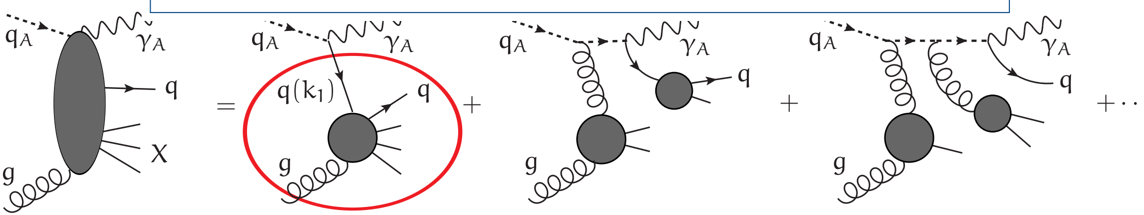


In agreement with the effective action approach of
Lipatov 1995, Antonov, Lipatov, Kuraev, Cherednikov 2005
Lipatov, Vyazovsky 2000, Nefedov, Saleev, Shipilova 2013
and the Wilson-line approach of
Kotko 2014



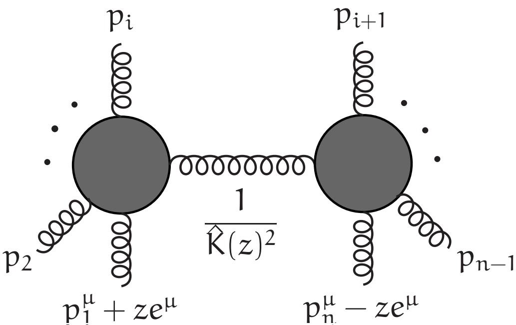
$\delta_{i,j}\,\frac{i}{p_1\cdot K}$



BCFW recursion for on-shell amplitudes

Britto, Cachazo, Feng, Witten 2005

Gives compact expression through recursion of *on-shell* amplitudes.



$$\hat{K}^\mu(z) = p_1^\mu + \cdots + p_i^\mu + ze^\mu$$
$$= -p_{i+1}^\mu - \cdots - p_n^\mu + ze^\mu$$

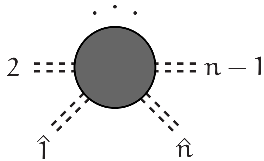
$$e^\mu = \tfrac{1}{2}\langle p_1|\gamma^\mu|p_n]$$

$$\hat{K}(z)^2 = 0 \quad \Leftrightarrow \quad z = -\frac{(p_1 + \cdots + p_i)^2}{2(p_2 + \cdots + p_i)\cdot e}$$

$$\mathcal{A}(1^+, 2, \ldots, n{-}1, n^-) = \sum_{i=2}^{n-1} \sum_{h=+,-} \mathcal{A}(\hat{1}^+, 2, \ldots, i, -\hat{K}_{1,i}^h)\, \frac{1}{K_{1,i}^2}\, \mathcal{A}(\hat{K}_{1,i}^{-h}, i{+}1, \ldots, n{-}1, \hat{n}^-)$$

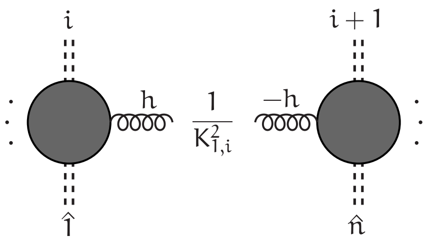
$$\mathcal{A}(1^+, 2^-, 3^-) = \frac{\langle 23\rangle^3}{\langle 31\rangle\langle 12\rangle} \quad , \quad \mathcal{A}(1^-, 2^+, 3^+) = \frac{[32]^3}{[21][13]}$$

The BCFW recursion formula becomes

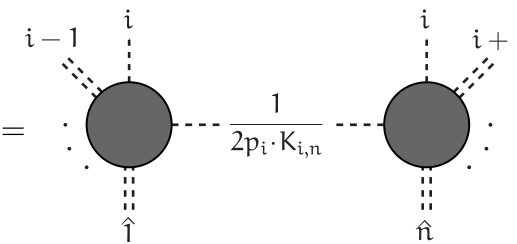


$$= \sum_{i=2}^{n-2}\sum_{h=+,-} A_{i,h} + \sum_{i=2}^{n-1} B_i + C + D\,,$$

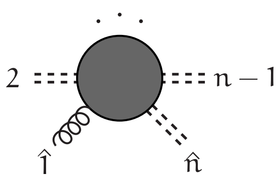
$A_{i,h} =$



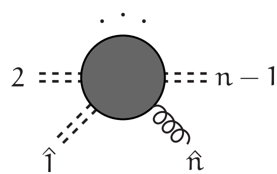
$B_i =$



$C = \frac{1}{\kappa_1}$



$D = \frac{1}{\kappa_1^*}$

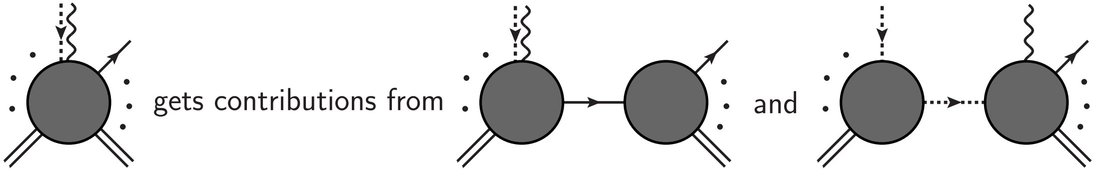


The hatted numbers label the shifted external gluons.

BCFW recursion with (off-shell) quarks

AvH, Serino 2015

- on-shell case treated in Luo, Wen 2005
- any off-shell parton can be shifted: propagators of "external" off-shell partons give the correct power of z in order to vanish at infinity
- different kinds of contributions in the recursion



- many of the MHV amplitudes come out as expected
- some more-than-MHV amplitudes do not vanish, but are sub-leading in k_T

$$\mathcal{A}(1^+, 2^+, \dots, n^+, \bar{q}^*, q^-) = \frac{-\langle \bar{q}q \rangle^3}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n\bar{q} \rangle \langle \bar{q}q \rangle \langle q1 \rangle}$$

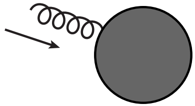
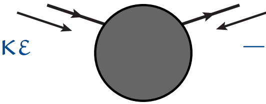
- off-shell quarks have helicity

$$\mathcal{A}(1, 2, \dots, n, \bar{q}^{*(+)}, q^{*(-)}) \neq \mathcal{A}(1, 2, \dots, n, \bar{q}^{*(-)}, q^{*(+)})$$

Towards NLO

with O. Gituliar

Go back to derivation of eikonal Feynman rules for off-shell gluons:

$k = p_1 + k_\perp$  $\Longrightarrow$ $(\Lambda+1)p_1 - \kappa\varepsilon$  $-\Lambda p_1 - \kappa^*\varepsilon^*$

$$\varepsilon\cdot\varepsilon = \varepsilon\cdot p_1 = 0 \quad , \quad k_\perp = -\kappa\varepsilon - \kappa^*\varepsilon^*$$

Towards NLO

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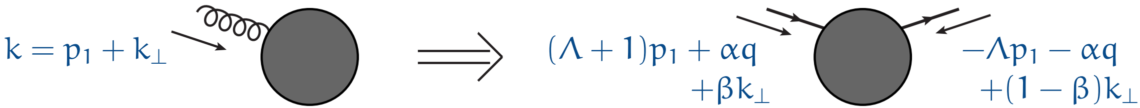

$$q \cdot k_\perp = 0 \quad \alpha = \frac{-\beta^2 k_\perp^2}{(\Lambda+1)(p_1+q)^2} \quad \beta = \frac{\sqrt{\Lambda+1}}{\sqrt{\Lambda+1}+\sqrt{\Lambda}}$$

Towards NLO

with O. Gituliar

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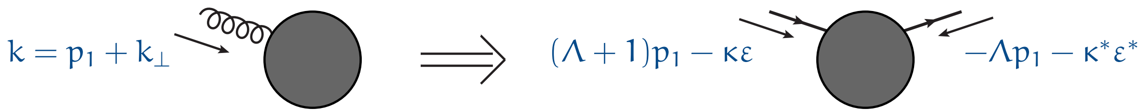

Divide amplitude by Λ (each auxiliary quark spinor gives factor $\sqrt{\Lambda}$) and take $\Lambda \to \infty$.

Towards NLO

with O. Gituliar

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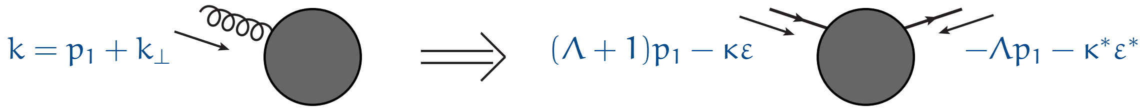

Divide amplitude by Λ (each auxiliary quark spinor gives factor $\sqrt{\Lambda}$) and take $\Lambda \to \infty$.

This can also be done with complete(ly summed) matrix elements at tree-level:

$$\frac{1}{\Lambda^2} M(q, g_2, g_3, g_4, \bar{q}) \left[\begin{array}{c} s_{q,\bar{q}} \to k_\perp^2 \\ s_{q,i} \to 2\Lambda p_1 \cdot p_i \ , \ s_{\bar{q},i} \to -2\Lambda p_1 \cdot p_i \end{array} \right] \Longrightarrow M(g_1^*, g_2, g_3, g_4)$$

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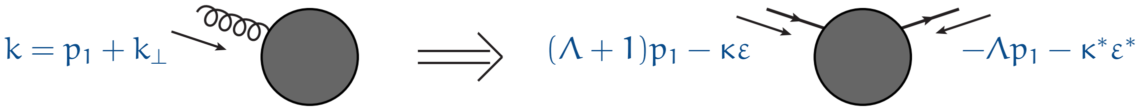

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Trying the same with one-loop expressions (eg. from Ellis, Sexton 1986) leads to terms with $\log\Lambda$, which can be traced back to integrals with linear denominators

$$\int \frac{d^{4-2\epsilon} l}{[p_1 \cdot l]\,[l^2]\,[(l + p_2)^2] \cdots}$$

Towards NLO

with O. Gituliar

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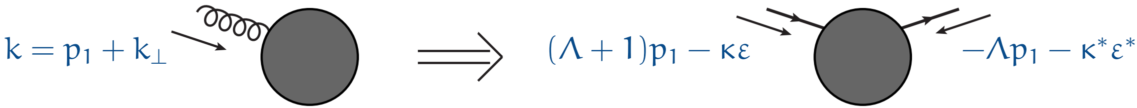

Divide amplitude by Λ (each auxiliary quark spinor gives factor $\sqrt{\Lambda}$) and take $\Lambda \to \infty$.

This can also be done with complete(ly summed) matrix elements at tree-level:

$$\frac{1}{\Lambda^2} M(q, g_2, g_3, g_4, \bar{q}) \left[\begin{array}{c} s_{q,\bar{q}} \to k_\perp^2 \\ s_{q,i} \to 2\Lambda p_1 \cdot p_i \ , \ s_{\bar{q},i} \to -2\Lambda p_1 \cdot p_i \end{array} \right] \Longrightarrow M(g_1^*, g_2, g_3, g_4)$$

Trying the same with one-loop expressions (eg. from Ellis, Sexton 1986) leads to terms with $\log\Lambda$, which can be traced back to integrals with linear denominators

$$\int \frac{d^{4-2\epsilon} l}{[p_1 \cdot l] \, [l^2] \, [(l + p_2)^2] \cdots}$$

Tree-level matrix elements have denominators $\propto p_1 \cdot p_i$, *i.e.* there are singularities despite off-shellness. Corresponding splitting function is given by $(1 - x)^3 / x$.

Conclusions

- Double-parton scattering gives an important contribution to the cross section for the process $\mathrm{pp} \to \mathrm{c\bar{c}\,c\bar{c}}$.
- This is confirmed by comparing with single-parton scattering at tree-level both in collinear factorization and $\mathrm{k_T}$-factorization.
- $\mathrm{k_T}$-factorization allows for the description of kinematical situations inaccessible with LO collinear factorization with parton shower, eg. ΔS for four jets.
- Factorization prescriptions with explicit $\mathrm{k_T}$ dependence in the pdfs ask for hard matrix elements with off-shell initial-state partons.
- The necessary amplitudes can be defined in a manifestly gauge invariang manner that allows for Dyson-Schwinger recursion and BCFW recursion, both for off-shell gluons and off-shell quarks.
- Upgrade to NLO in progress.

Public programs http://bitbucket.org/hameren/

AVHLIB (A Very Handy LIBrary)

- complete Monte Carlo program for tree-level calculations
- any process within the Standard Model
- any initial-state partons on-shell or off-shell
- employs numerical Dyson-Schwinger recursion to calculate helicity amplitudes
- automatic phase space optimization
- flexibility at the cost of user-friendliness

AMP4HEF (AvH, M.Bury, K.Bilko, H.Milczarek, M.Serino)

- only provides tree-level matrix elements (or color-ordered helicity amplitudes)
- available processes (plus those with fewer on-shell gluons and fewer off-shell partons):

$$\emptyset \to g^* \, g^* + 5g \qquad \emptyset \to \bar{q} \, q^* + 3g \qquad \emptyset \to \bar{q}^* \, q^* + 2g$$

$$\emptyset \to \bar{q}^* \, q + 3g \qquad \emptyset \to g^* \, \bar{q}^* + q \, g$$

$$\emptyset \to g^* + \bar{q} \, q + 2g \qquad \emptyset \to q^* \, g^* + g \, \bar{q}$$

- employs BCFW recursion to calculate color-ordered helicity amplitudes
- easy to use, both in Fortran and C++

Backup

Amplitudes with off-shell gluons

n-parton amplitude is a function of n momenta $k_1, k_2, \ldots, k_n$
and n *directions* $p_1, p_2, \ldots, p_n$, satisfying the conditions

$$k_1^\mu + k_2^\mu + \cdots + k_n^\mu = 0 \qquad \text{momentum conservation}$$
$$p_1^2 = p_2^2 = \cdots = p_n^2 = 0 \qquad \text{light-likeness}$$
$$p_1 \cdot k_1 = p_2 \cdot k_2 = \cdots = p_n \cdot k_n = 0 \qquad \text{eikonal condition}$$

With the help of an auxiliary four-vector q^μ with $q^2 = 0$, we define

$$k_T^\mu(q) = k^\mu - x(q)p^\mu \quad \text{with} \quad x(q) \equiv \frac{q\cdot k}{q\cdot p}$$

Construct k_T^μ explicitly in terms of p^μ and q^μ:

$$k_T^\mu(q) = -\frac{\kappa}{2}\frac{\langle p|\gamma^\mu|q]}{[pq]} - \frac{\kappa^*}{2}\frac{\langle q|\gamma^\mu|p]}{\langle qp\rangle} \quad \text{with} \quad \kappa = \frac{\langle q|\not{k}|p]}{\langle qp\rangle} \;,\; \kappa^* = \frac{\langle p|\not{k}|q]}{[pq]}$$

$k^2 = -\kappa\kappa^*$ is independent of q^μ, but also individually κ and κ^* are independent of q^μ.

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$$\frac{\langle q|\not{k}|p]}{\langle qp\rangle} = \frac{\langle q|\not{k}|p]\langle pr\rangle}{\langle qp\rangle\langle pr\rangle} = \frac{\langle q|\not{k}\not{p}|r\rangle}{\langle qp\rangle\langle pr\rangle} = \frac{\langle q|2k\cdot p - \not{p}\not{k}|r\rangle}{\langle qp\rangle\langle pr\rangle} = -\frac{\langle qp\rangle[p|\not{k}|r\rangle}{\langle qp\rangle\langle pr\rangle} = \frac{\langle r|\not{k}|p]}{\langle rp\rangle}$$

Amplitudes with off-shell gluons

n-parton amplitude is a function of n momenta $k_1, k_2, \ldots, k_n$
and n *directions* $p_1, p_2, \ldots, p_n$, satisfying the conditions

$$
\begin{aligned}
k_1^\mu + k_2^\mu + \cdots + k_n^\mu &= 0 && \text{momentum conservation}\\
p_1^2 = p_2^2 = \cdots = p_n^2 &= 0 && \text{light-likeness}\\
p_1\cdot k_1 = p_2\cdot k_2 = \cdots = p_n\cdot k_n &= 0 && \text{eikonal condition}
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$$

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$k^2 = -\kappa\kappa^*$ is independent of q^μ, but also individually κ and κ^* are independent of q^μ.

Besides the spinors of directions and light-like momenta, κ and κ^* will show up in expressions for off-shell amplitudes.

Example of a 4-gluon amplitude

$$\mathcal{A}(1^*,2^-,3^*,4^+) = \frac{\langle 13\rangle^3[13]^3}{\langle 34\rangle\langle 41\rangle\langle 1|\not{k}_3+\not{p}_4|3]\langle 3|\not{k}_1+\not{p}_4|1][32][21]}$$
$$+\frac{1}{\kappa_1^*\kappa_3}\frac{\langle 12\rangle^3[43]^3}{\langle 2|\not{k}_3|4]\langle 1|\not{k}_3+\not{p}_4|3](k_3+p_4)^2}+\frac{1}{\kappa_1\kappa_3^*}\frac{\langle 23\rangle^3[14]^3}{\langle 2|\not{k}_1|4]\langle 3|\not{k}_1+\not{p}_4|1](k_1+p_4)^2}$$

- Eventual matrix element needs factor $k_1^2k_3^2 = |\kappa_1|^2|\kappa_3|^2$.
  This *must not* be included at the amplitude level not to spoil analytic structure.

- Last two terms dominate for $|k_1| \to 0$ and $|k_3| \to 0$, and give the on-shell helicity amplitudes in that limit.

$$\mathcal{A}(1^*,2^-,3^*,4^+) \xrightarrow{|k_1|,|k_3|\to 0} \frac{1}{\kappa_1^*\kappa_3}\mathcal{A}(1^-,2^-,3^+,4^+) + \frac{1}{\kappa_1\kappa_3^*}\mathcal{A}(1^+,2^-,3^-,4^+)$$

- Coherent sum of amplitudes becomes incoherent sum of squared amplitudes via angular integrations for $\vec{k}_{1T}$ and $\vec{k}_{3T}$.