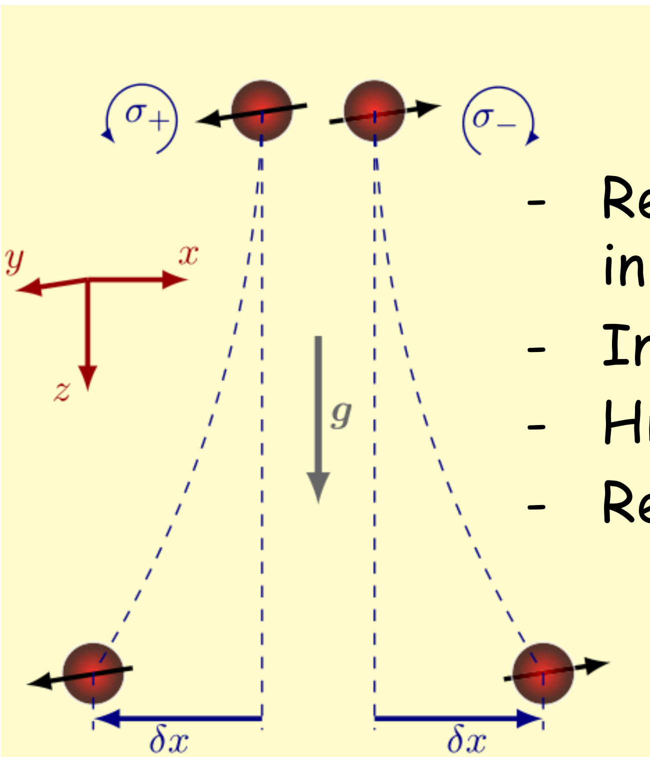
Energy-momentum tensor and the gravitational spin-Hall effect

based on work with Ting Gao



Outline:

- Recent claim of the gravitational spin Hall effect in a uniform field (like near the Earth)
- Introduction: spin Hall effects
- Hidden momentum and a model of a spinning mass
- Resolution of the claim

Matter to the Deepest 2025, Katowice

Andrzej Czarnecki, University of Alberta

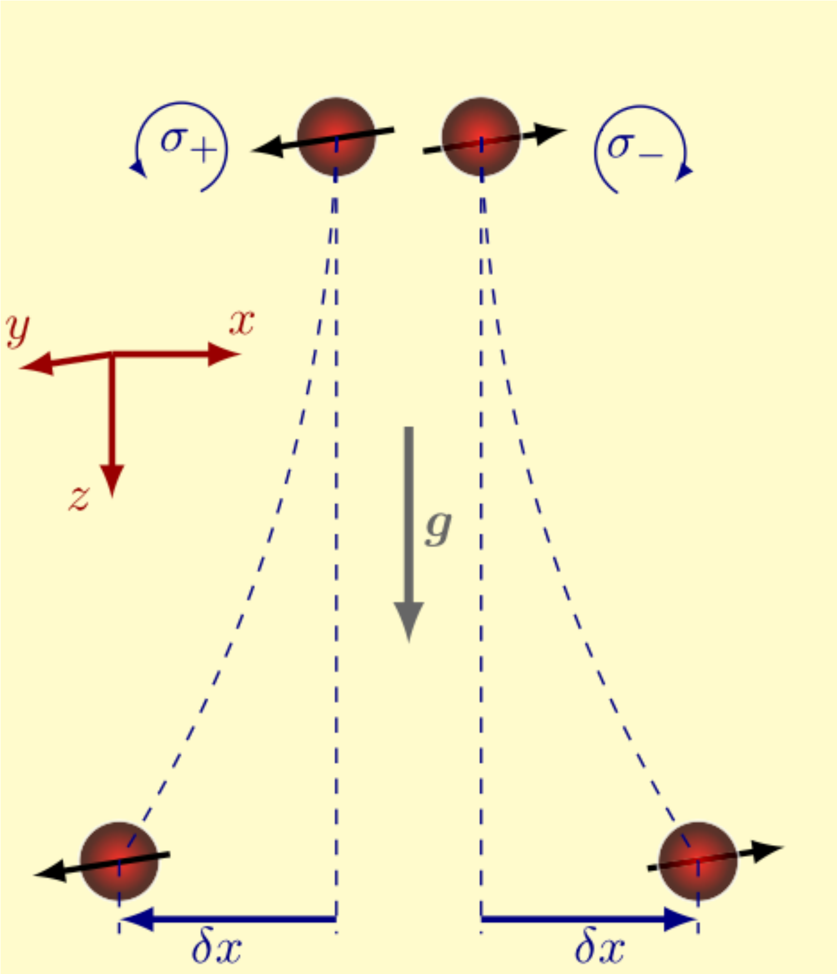
September 19, 2025

Gravitational spin Hall effect

PHYSICAL REVIEW D **109**, 044060 (2024)

Gravitational spin Hall effect of Dirac particle and the weak equivalence principle

Zhen-Lai Wang*



normalized Gaussian wave packet

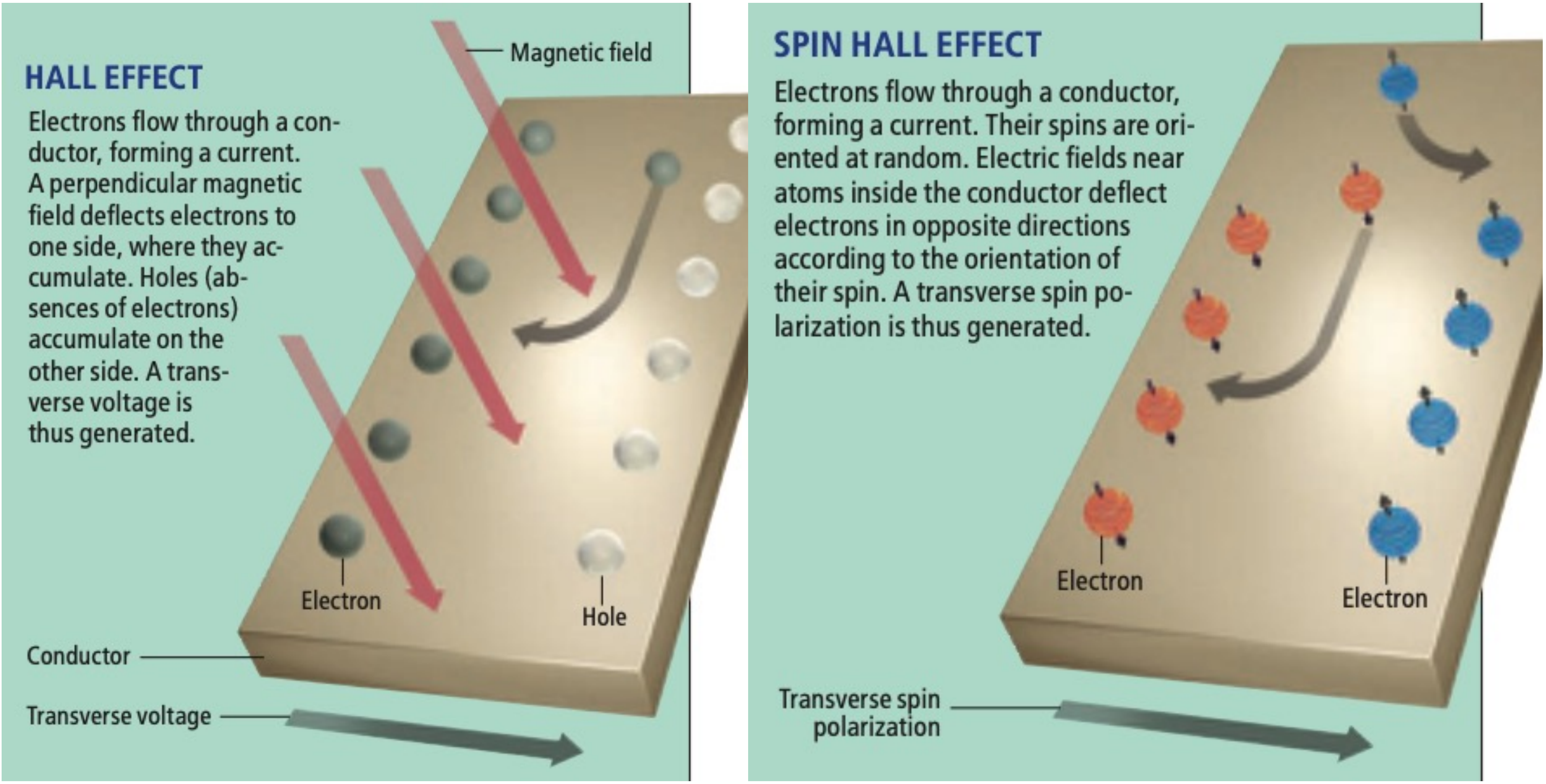
$$\phi(\boldsymbol{x}, 0) = (a^2\pi)^{-3/4}\mathrm{Exp}\left[-\frac{\boldsymbol{x}^2}{2a^2}\right]$$

$$\langle x(t)\rangle_+ = -\frac{gt}{4m}, \qquad \langle x(t)\rangle_- = \frac{gt}{4m}$$

$$\langle y(t)\rangle_+ = \langle y(t)\rangle_- = 0$$

$$\langle z(t)\rangle_+ = \langle z(t)\rangle_- = \frac{1}{2}gt^2$$

Spin Hall effect



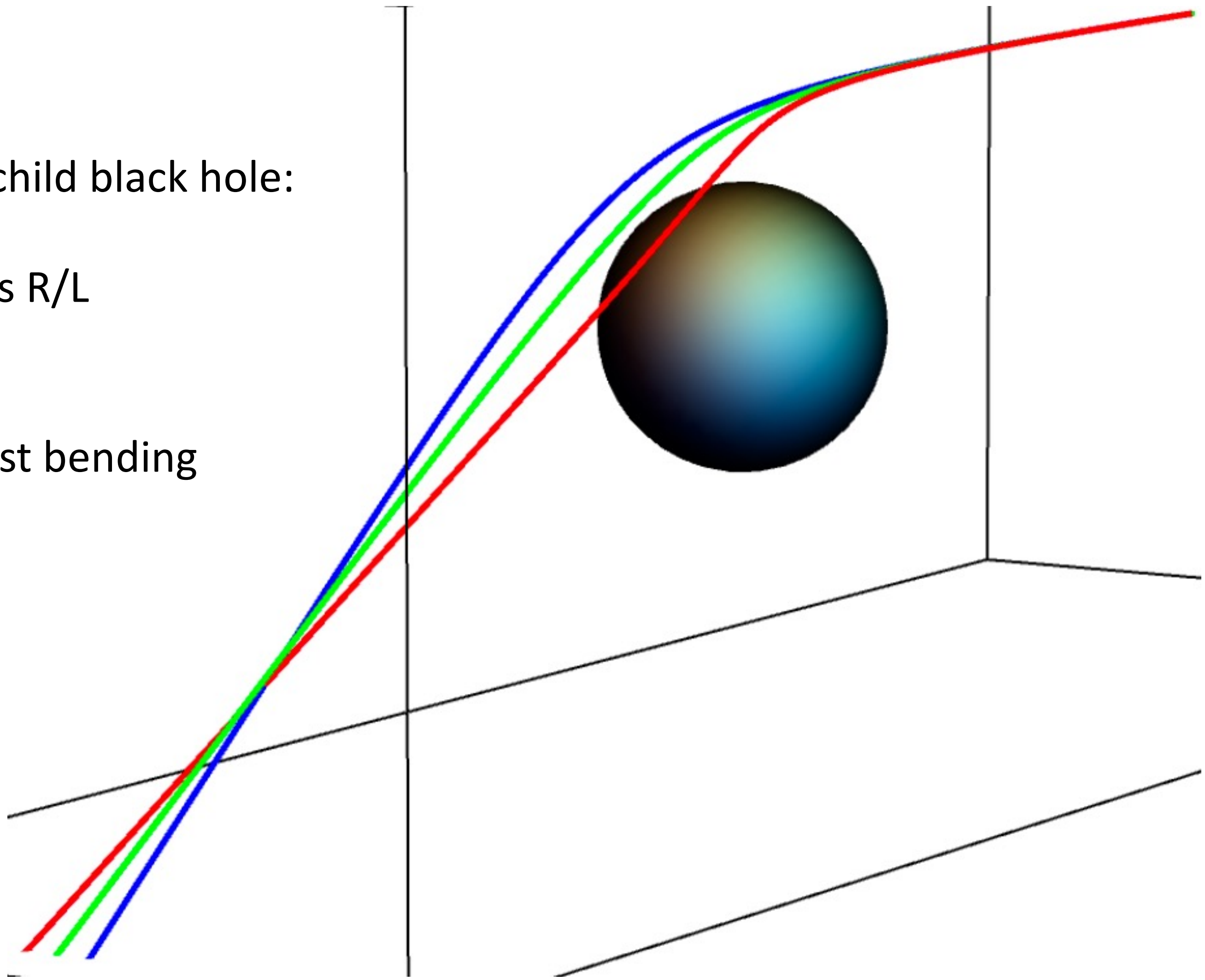
Awschalom, Epstein, Hanson; Kato, Sc. Am. Oct 2007

Gravitational spin Hall effect

Photon near a Schwarzschild black hole:

- blue/red lines: helicities R/L
  (non-planar orbits!)

- green: null geodesic, just bending



Oancea, Paganini, Joudioux, Andersson: 1904.09963

Gravitational spin Hall effect

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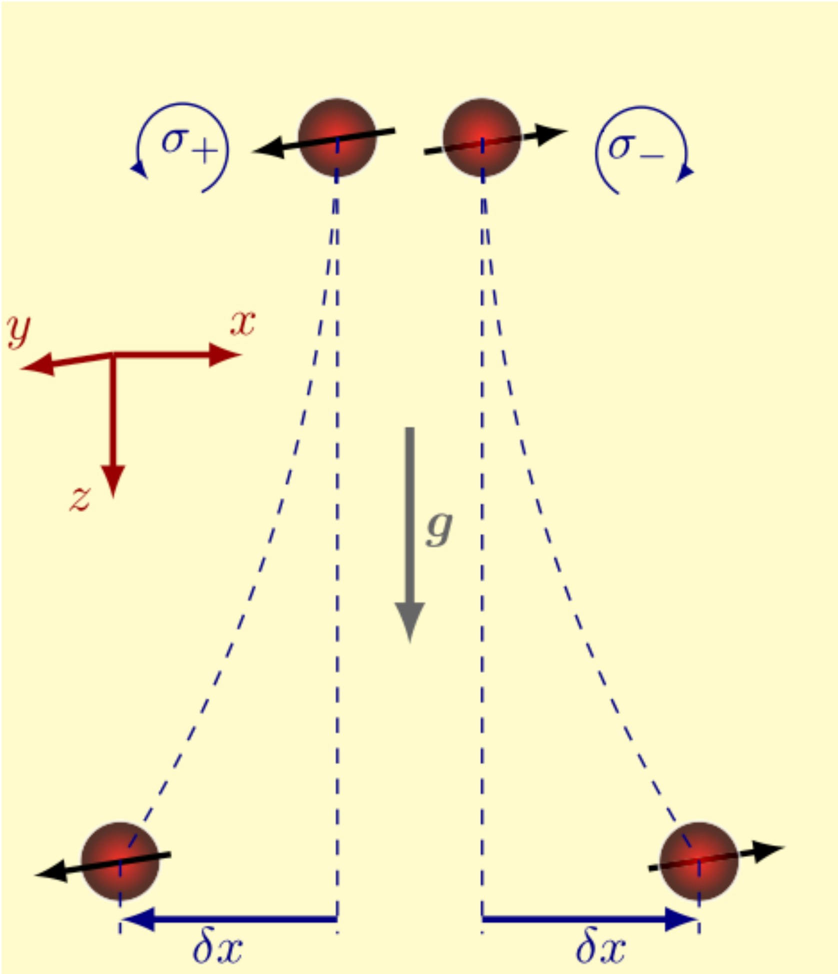

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Contradiction: figure vs formula

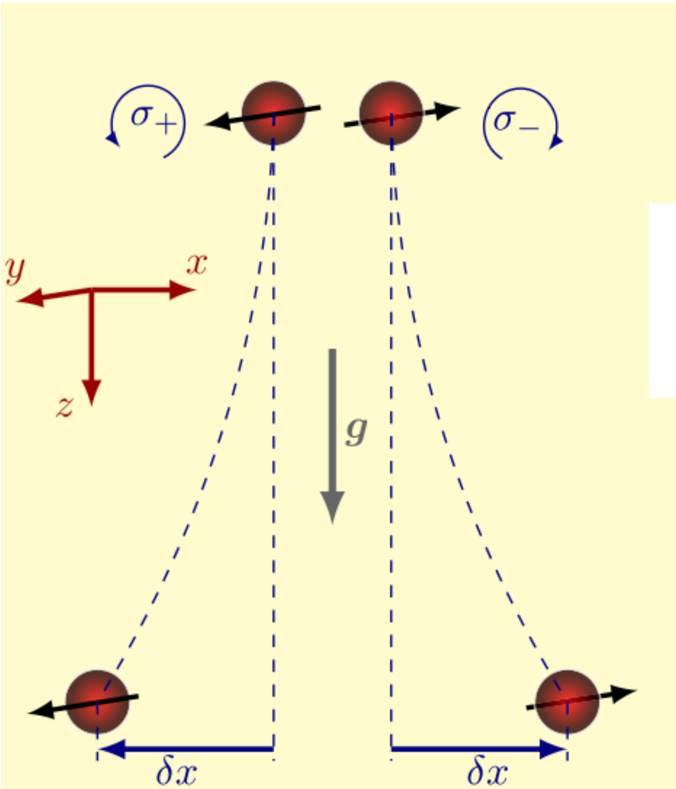
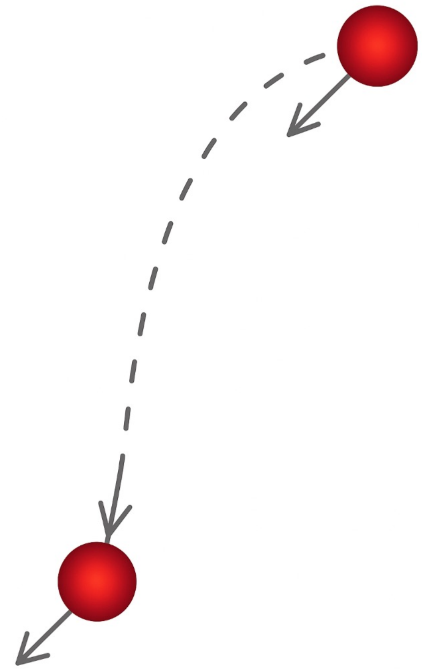


Figure suggests zero initial x-velocity;
but the formula

$$\langle x(t)\rangle_+ = -\frac{gt}{4m},$$

constant speed g/4m:
projectile motion,



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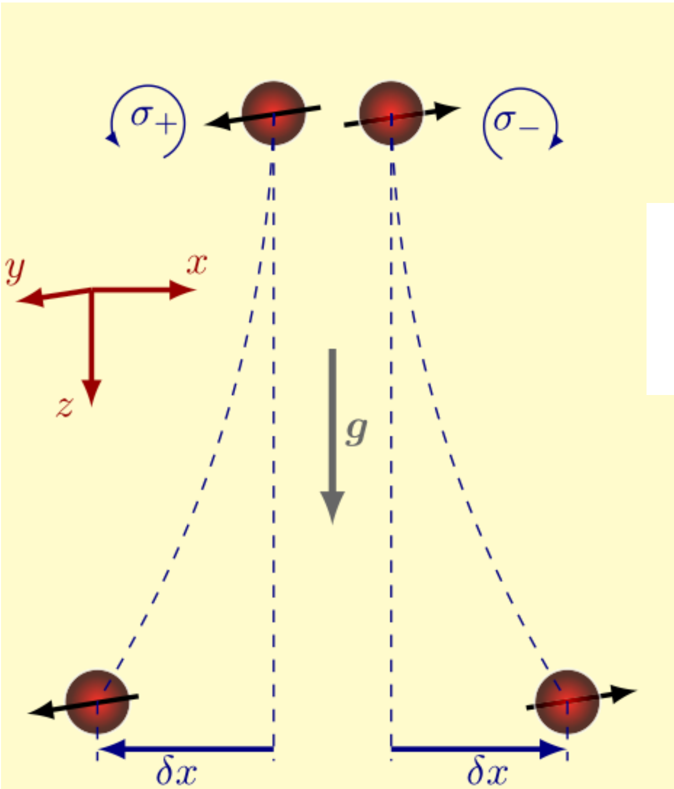

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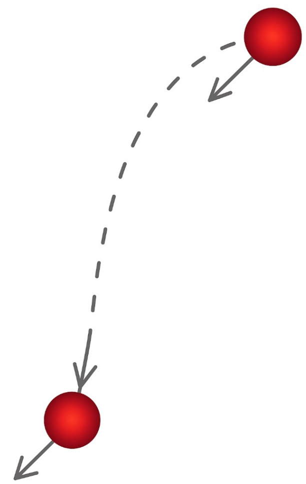
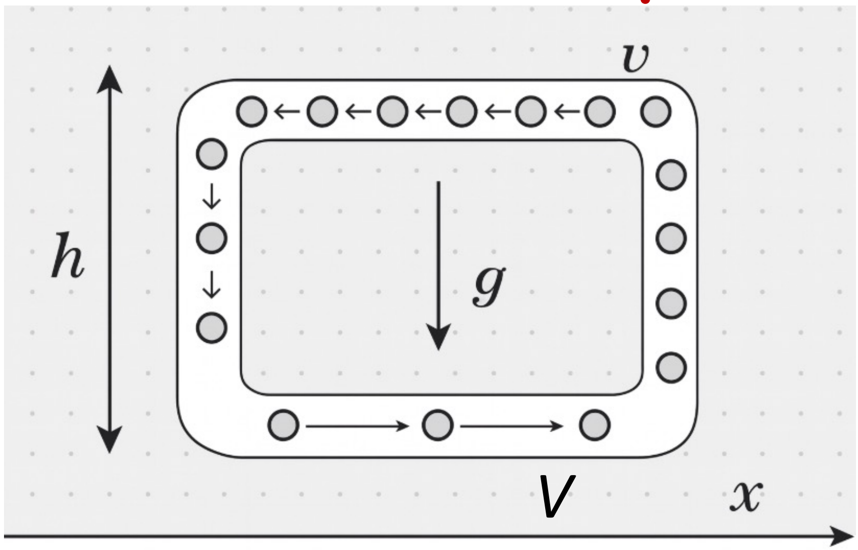


But how to reconcile non-zero speed at t=0 with the symmetric wave packet?

$$\phi(\boldsymbol{x},0) = (a^2\pi)^{-3/4}\mathrm{Exp}\left[-\frac{\boldsymbol{x}^2}{2a^2}\right]$$

Hidden momentum: a simple model



What is the x-component of momentum if the frame is at rest?

GENERAL ARGUMENTS IN A FULLY RELATIVISTIC THEORY

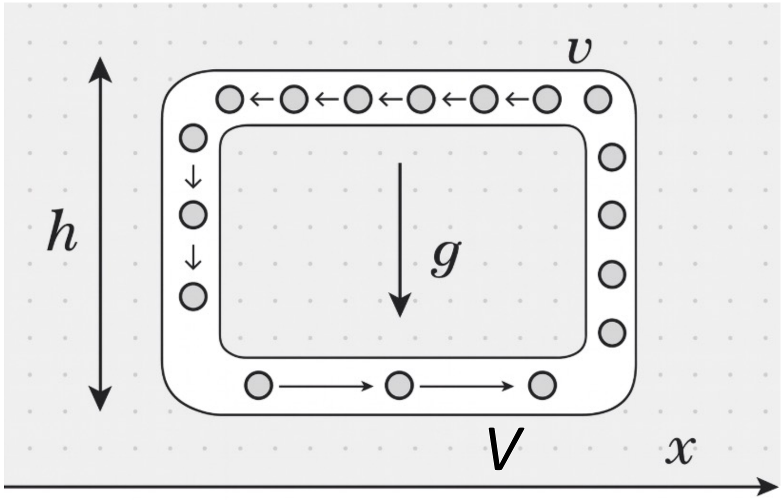
Coleman & Van Vleck, 1968

$$P^i = c^{-1}\int T^{0i} d^3x\,, \qquad d\mathbf{P}/dt = 0\,,$$

$$X^i = E^{-1}\int T^{00} x^i d^3x\,, \qquad d\mathbf{X}/dt = c^2\mathbf{P}/E\,.$$

If the center of energy *X* is at rest, total momentum must vanish;
but momentum can be hidden in circulating motion,
even if the whole system is at rest.

Non-relativistic vs relativistic momentum



n_v particles with speed v

n_V particles with speed V

Current conservation:

$n_v\, v = n_V\, V$

Momentum (NR)

$m\, n_V\, V - m\, n_v\, v = 0.$

$$V^2 - v^2 = 2gh,$$

Relativistic correction to momentum!

$$\begin{aligned}\Delta_{\rm sr} p_x &= n_V \gamma_V m V - n_v \gamma_v m v \\ &= n_V \left(\gamma_V m V - \frac{V}{v} \gamma_v m v \right) \\ &\simeq n_V m V \frac{V^2 - v^2}{2c^2}.\end{aligned}$$

$$\boxed{\Delta_{\rm sr} p_x = n_V m V \frac{gh}{c^2}}$$

Non-zero momentum?!

Gravitational red-shift effect

"gamma" from line element:

$$c^2\,d\tau^2 = c^2\,dt^2 - d\mathbf{x}^2 \quad \Rightarrow \quad \gamma_{\rm SR} \equiv \frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}.$$

GR generalization:

$$c^2\,d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

Near-Earth, weak-field metric:

$$c^2 d\tau^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2\Phi}{c^2}\right) d\boldsymbol{x}^2$$

$$\frac{d\tau}{dt} = \underbrace{\sqrt{1+2\Phi/c^2}}_{\text{grav. redshift}} \times \underbrace{\sqrt{1-v_{\rm phys}^2/c^2}}_{\text{SR with local speed}}.$$

$$\Phi \simeq mgz$$

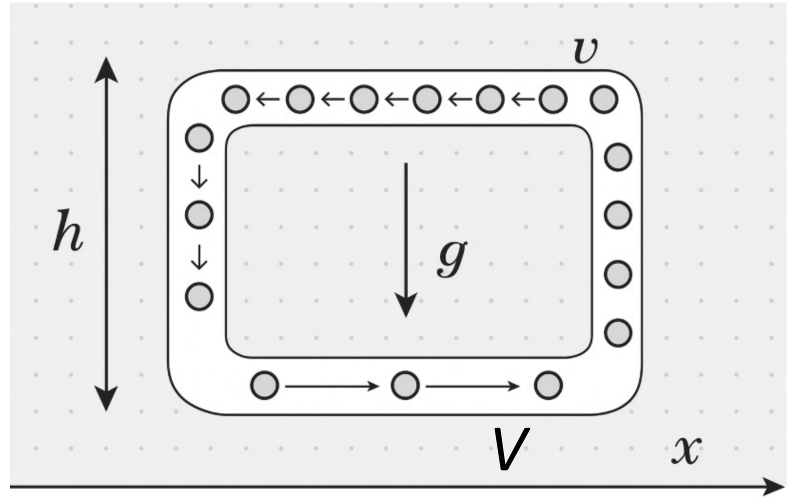
Summary of the red-shift effect on momentum:

$$\boxed{\Delta_{\rm rs} p_x = n_V \left(\frac{gh}{2c^2} mV + \frac{V}{v}\frac{gh}{2c^2} mv\right) = n_V \frac{gh}{c^2} mV,}$$

Doubles the SR effect!

Relation to angular momentum

With respect to the center of the frame, circulating masses have angular momentum,



$$L_v = 2 \cdot \frac{b}{2} \cdot mu \frac{n_V V h}{ub} = m n_V V h = L_h.$$

Compact expression of the mechanical momentum in terms of the angular momentum,

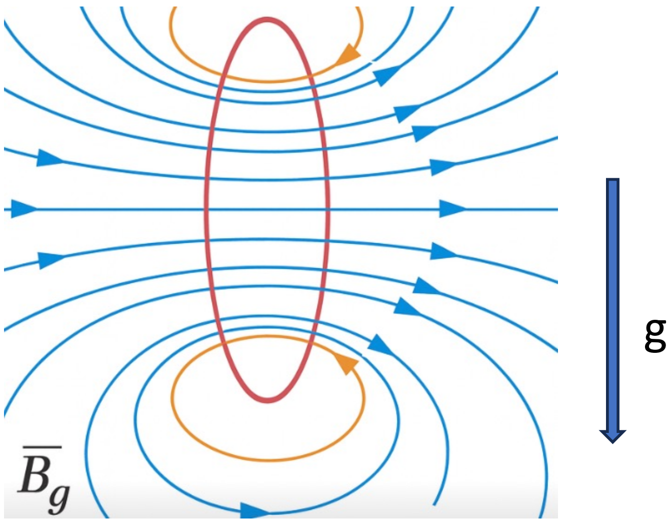
$$\boldsymbol{p}_{\text{mech}} = \frac{\boldsymbol{L} \times \boldsymbol{g}}{c^2}, \quad |\boldsymbol{L}| = L_v + L_h.$$

Momentum of the gravitational field

Review: Bahram Mashhoon gr-qc/0311030

Circulating masses generate gravito-magnetic field

Analog of the Poynting vector: $$\mathcal{S} = -\frac{c}{2\pi G}\mathbf{E} \times \mathbf{B}.$$



$$\boldsymbol{p}_{\text{field}} = -\frac{\boldsymbol{L} \times \boldsymbol{g}}{c^2}.$$

$$\boldsymbol{p}_{\text{mech}} + \boldsymbol{p}_{\text{field}} = 0,$$

in accordance with the center-of-energy theorem

So what went wrong in Wang's analysis?

It's true that the expectation value of the CANONICAL momentum in the symmetric wave packet vanishes,

$$\left\langle \phi(\boldsymbol{x},0) \left| \frac{\hbar}{i}\boldsymbol{\nabla} \right| \phi(\boldsymbol{x},0) \right\rangle = 0 \qquad \phi(\boldsymbol{x},0) = (a^2\pi)^{-3/4}\mathrm{Exp}\left[-\frac{\boldsymbol{x}^2}{2a^2}\right]$$

but the MECHANICAL momentum does not vanish,

$$\mathcal{H}_{FW} = \beta V(m + \frac{\boldsymbol{p}^2}{2m}) + \frac{i\beta g}{2m}p_3 - \frac{\beta\boldsymbol{\Sigma}\cdot(\boldsymbol{g}\times\boldsymbol{p})}{4m}, \qquad m\frac{dx}{dt} = m\frac{\partial\mathcal{H}}{\partial p_x}$$

$$= Vp_x - \frac{S_y g}{2}$$

$$c = \hbar = 1$$

Summary

- Tantalizing recent claim of the gravitational spin-Hall effect near Earth's surface
- Model of spinning matter: interesting, complex picture of linear momentum
- Hidden momentum, when treated carefully, clarifies the apparent acceleration of the wave packet.