Energy Spectrum of Magnetic Fields in the Universe at the EWPT

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Table of contents

1. Introduction to primordial magnetic fields
2. Magnetic fields from the electroweak phase transition
3. Lattice simulations vs Analytic approach
4. Continuous case and the form of spectra

Introduction & Motivation

Introduction

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  - ▷ Amplification of weak "seed magnetic fields" by the rotating galactic plasma
  - ▷ But what is the origin of this seed field?

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  - ▷ Amplification of weak "seed magnetic fields" by the rotating galactic plasma
  - ▷ But what is the origin of this seed field?
- Evidence for weak galactic fields outside of galaxies (in voids)
  - ▷ Observed in TeV Blazar data [Neronov et al., 2010]
  - ▷ Diffuse image caused by magnetic fields between the blazar and observer
  - ▷ Result: $B_{\text{void}} \sim \mathcal{O}(10^{-16})$ G field with $\lambda_{\text{void}} \sim 1$ Mpc correlation length
  - ▷ Compare: $\lambda_{\text{void}} \sim 10^{21}\,\text{m} \longrightarrow$ Weak field but enormous distances

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  - ▷ Compare: $\lambda_{\text{void}} \sim 10^{21}\,\text{m} \longrightarrow$ Weak field but enormous distances
- Weak and homogeneous fields $\rightarrow$ Possibly early Universe origin
  - ▷ A possible explanation is through topological effects [Vachaspati, 1991]
  - ▷ Despite the absence of stable defects in the SM, unstable structures–called Nambu dumbbells–may exist [Nambu, 1977]

Primordial Magnetic Fields

$$\mathcal{L}_{\rm SM} = |D_\mu \Phi|^2 + V(\Phi) + \dots$$

$$\Phi_{\rm min}: \quad D_\mu \Phi_{\rm min} \equiv \Big(\partial_\mu - \frac{\mathrm{i}g}{2}\tau^a A^a_\mu - \frac{\mathrm{i}g'}{2} A^0_\mu\Big)\Phi_{\rm min} = 0 \quad \text{and} \quad V(\Phi_{\rm min}) = 0\,.$$

$$g' A^0_\mu = -\mathrm{i}\sin^2\theta_{\rm w}\big(\Phi^\dagger \cdot \partial_\mu \Phi - \partial_\mu \Phi^\dagger \cdot \Phi\big)$$

$$g A^a_\mu = -\mathrm{i}\cos^2\theta_{\rm w} n^a \big(\Phi^\dagger \cdot \partial_\mu \Phi - \partial_\mu \Phi^\dagger \cdot \Phi\big) - \epsilon^{abc} n^b \partial_\mu n^c \qquad \big(n^a = \Phi^\dagger \cdot \tau^a \cdot \Phi\big)$$

$$\mathcal{F}_{\mu\nu} = \cos\theta_{\rm w} F^0_{\mu\nu} - \sin\theta_{\rm w} F^a_{\mu\nu} n^a = -\frac{2\mathrm{i}}{g}\sin\theta_{\rm w}\big(\partial_\mu \Phi^\dagger \partial_\nu \Phi - \partial_\nu \Phi^\dagger \partial_\mu \Phi\big)\,.$$

$$\mathbf{B} = -\mathrm{i}\frac{2\sin\theta_{\rm w}}{g}\big(\boldsymbol{\nabla}\Phi^\dagger \times \boldsymbol{\nabla}\Phi\big) \quad \text{and} \quad \mathbf{A} = -\mathrm{i}\frac{2\sin\theta_{\rm w}}{g}\Phi^\dagger \boldsymbol{\nabla}\Phi$$

Correlation functions

- To describe the statistical features of the $\mathbf{B}(\mathbf{r})$ field we use **correlation functions**
  - ▷ Assuming isotropy and $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ (defects have decayed)
  - ▷ Spatial correlation function:

$$\langle B_i(\mathbf{x}+\mathbf{r}) B_j(\mathbf{r}) \rangle = M_{\mathrm{N}}(r) \mathrm{P}_{ij} + M_{\mathrm{L}}(r) \hat{r}_i \hat{r}_j + \epsilon_{ijk} \hat{r}_k M_{\mathrm{H}}(r)$$

  - ▷ Fourier transform convention:

$$\mathbf{b}(\mathbf{k}) = \int \mathrm{d}^3 r \, \mathbf{B}(\mathbf{r}) \exp\left(\mathrm{i}\mathbf{k} \cdot \mathbf{r}\right)$$

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  - ▷ Fourier-space correlation function:

$$\langle b_i(\mathbf{k}) b_j(\mathbf{k}') \rangle = (2\pi)^6 \delta^{(3)}(\mathbf{k}-\mathbf{k}') \Big[\frac{E_{\mathrm{M}}(k)}{4\pi k^2} \mathrm{P}_{ij}^{(k)} + \mathrm{i} \epsilon_{ijl} k_l \frac{H_{\mathrm{M}}(k)}{8\pi k^2} \Big]$$

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  - ▷ Trace of these $\rightarrow$ correlation function and power spectrum

Energy spectrum

- Using the ergodic principle we define the **correlation function**:

$$\xi(\mathbf{r}) \equiv \langle \mathbf{B}(\mathbf{r}) \cdot \mathbf{B}(\mathbf{x}+\mathbf{r}) \rangle = \frac{1}{V} \int_V \mathrm{d}^3 x \, \mathbf{B}(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x}+\mathbf{r})$$

- Using $V = (2\pi)^3 \delta^{(3)}(\mathbf{k} = \mathbf{0})$ we extract the **energy spectrum**:

$$E_{\mathrm{M}}(k) = \frac{k^2 \langle |\mathbf{b}(\mathbf{k})|^2 \rangle}{(2\pi)^2 V}$$

- Relationship between the correlation function and the energy spectrum:

$$\xi(\mathbf{r}) = \frac{1}{2} \int_0^\infty \mathrm{d}k \, E_{\mathrm{M}}(k) j_0(kr) \quad \longleftrightarrow \quad E_{\mathrm{M}}(k) = \frac{k^2}{\pi} \int_0^\infty \mathrm{d}r \, r^2 \xi(\mathbf{r}) j_0(kr)$$

Magnetic fields from EWPT

EWPT and the Scalar Field

- After EWPT the scalar field $\Phi(\mathbf{r}) \equiv v\tilde{\Phi}(\mathbf{r})$ is on its vacuum manifold:

$$|\tilde{\Phi}|^2 = \tilde{\phi}_1^2 + \tilde{\phi}_2^2 + \tilde{\phi}_3^2 + \tilde{\phi}_4^2 \overset{!}{=} 1 \quad \longrightarrow \quad \tilde{\Phi} \sim S_3$$

- *Side note*: topological defects require non-trivial homotopy classes, however

$$\pi_0(S_3) = \pi_1(S_3) = \pi_2(S_3) = 0 \quad \longrightarrow \quad \text{No topological defects!}$$

- Hopf-parametrization of the scalar field on S_3:

$$\tilde{\Phi} = \begin{pmatrix} \cos\alpha \exp(\mathrm{i}\beta) \\ \sin\alpha \exp(\mathrm{i}\gamma) \end{pmatrix}, \quad \text{where} \quad \beta, \gamma \in [0, 2\pi] \quad \text{and} \quad 2u \equiv \cos(2\alpha) \in [-1, 1]$$

- Hopf-parameters u, β, γ distributed uniformly at random

Lattice Setup I.

- Idea based on [Vachaspati et al., 2025]
- Cubic lattice of size N^3 with lattice spacing δx and $L = N\delta x = V^{1/3}$
- Choose random Hopf-angles α, β, γ at each lattice site:

  Random realization of the scalar field: $\tilde{\Phi}(\mathbf{x}_{ijk})$, where $\mathbf{x}_{ijk} = i\hat{\mathbf{x}} + j\hat{\mathbf{y}} + k\hat{\mathbf{z}}$

- Let $2\sin\theta_{\mathrm{w}}/g \to 1$ so that $\mathbf{A} = -\mathrm{i}\tilde{\Phi}^\dagger \boldsymbol{\nabla} \tilde{\Phi}$
- Define the vector potential through the **link variable**:

$$T_a(\mathbf{x}) \equiv \mathbf{A}(\mathbf{x}) \cdot (\delta x\, \hat{\mathbf{a}}) \simeq \mathfrak{Im}\big[\tilde{\Phi}^\dagger(\mathbf{x}) \cdot \tilde{\Phi}(\mathbf{x} + \delta x\, \hat{\mathbf{a}})\big] \equiv \mathbf{x} \longrightarrow \mathbf{x}+\delta x\hat{\mathbf{a}}$$

- Important step: **Periodic boundary condition** when defining link variables!

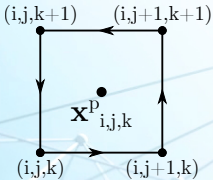
Lattice Setup II.

- Magnetic field defined via the loop integral:

$$\mathbf{B}(\mathbf{x}^p)\delta x^2 = \hat{\mathbf{p}} \oint_{\partial p} \mathrm{d}\mathbf{l} \cdot \mathbf{A}(\mathbf{x})$$

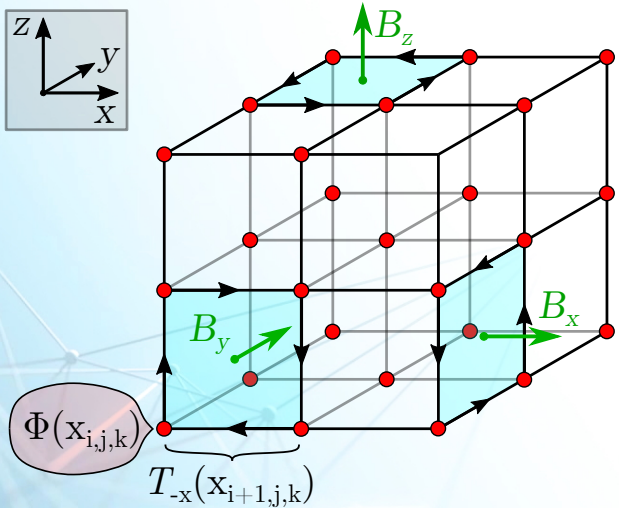
- On the lattice this reduces to a sum of 4 link variables, *e.g.*:

$$B_x(\mathbf{x}^p_{i,j,k})\delta x^2 =$$

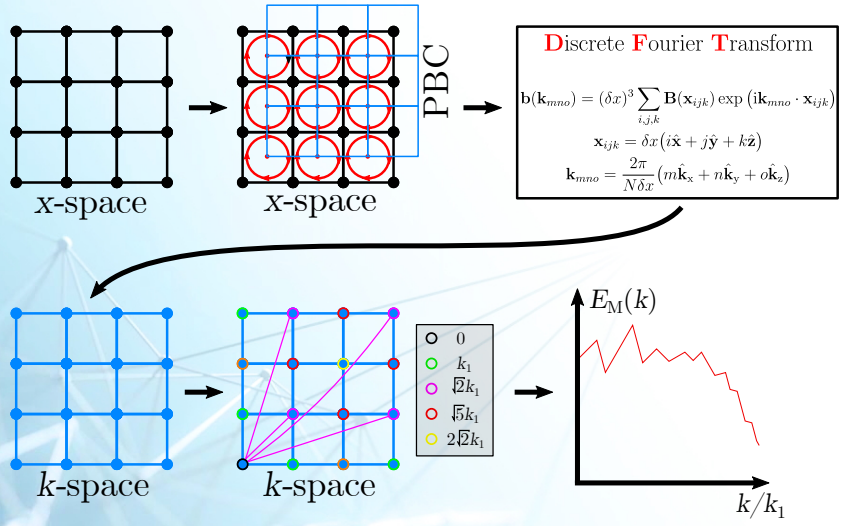


$$= T_y(\mathbf{x}_{i,j,k}) + T_z(\mathbf{x}_{i,j+1,k}) + T_{-y}(\mathbf{x}_{i,j+1,k+1}) + T_{-z}(\mathbf{x}_{i,j,k+1})$$

Visualizing the Lattice



Basic Simulation Approach



An analytic approach for the spectra

"Simulation on Paper"

- Basic simulation approach: Slow for large lattice size N and very noisy for small k (large distances)
  - ▷ Personal problem: cannot run on a laptop due to limited RAM for $N \gtrsim 256$

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  - ▷ We can exploit the randomness of $\Phi(\mathbf{x}_{ijk})$ and the law of large numbers (nr. of lattice points $\ggg 1$)
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- Approach:
  1. Find the spatial correlation function → $\xi(\mathbf{r})$
  2. Average over polar angles to get the isotropic correlation function → $\xi_{\text{iso}}(r)$
  3. Use relationship between power spectrum and correlation function → $E_{\text{M}}(k)$

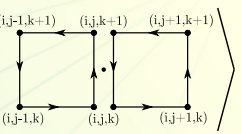
Analytic Approach for the Spectrum

- Approach:
  - ▷ Evaluate the correlation function $\xi(\mathbf{r})$ for the lattice:

$$\xi(\mathbf{r}) = \frac{1}{N^3}\sum_{i=1}^{3}\sum_{\mathbf{x}} B_i(\mathbf{x})B_i(\mathbf{x}+\mathbf{r}) \simeq \sum_{i=1}^{3}\langle B_i(\mathbf{x})B_i(\mathbf{x}+\mathbf{r})\rangle_{\mathbf{x}}$$

  - ▷ Using $\langle T_a(\mathbf{x})T_b(\mathbf{y})\rangle = 0$ unless $T_a(\mathbf{x})$ and $T_b(\mathbf{y})$ are the "same" link:

$$\langle B_x(\mathbf{x}^p_{i,j-1,k})B_x(\mathbf{x}^p_{i,j,k})\rangle =$$



$$= \langle \text{(i,j,k)} \text{ (i,j,k+1)} \rangle$$

$$= \langle T_z(\mathbf{x}_{i,j,k})T_{-z}(\mathbf{x}_{i,j,k+1})\rangle \equiv -\langle [T_z(\mathbf{x}_{i,j,k})]^2\rangle$$

Analytic Approach – The Correlation Function

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  - ▷ Obviously, correlations vanish for $r > \delta x$ as there will be no shared links:

$$\xi(\mathbf{r}) = \frac{1}{\delta x^4} \sum_{i=1}^{3} \left(\delta_{\mathbf{r},\mathbf{0}} - \frac{1}{4} \sum_{j=1}^{3} \delta_{\mathbf{r},\pm\delta x \hat{\mathbf{e}}_j} (1 - \delta_{ij}) \right)$$

Discrete approach I.

- Isotropic limit on the lattice:
  - ▷ Introduce the set of lattice points that are at a given distance:

$$\mathcal{P}(r) = \left\{ \mathbf{r} \,\middle|\, |\mathbf{r}| = r;\, \mathbf{r} \in \text{lattice} \right\} \qquad \rightarrow \qquad |\mathcal{P}(0)| = 1\,,\ |\mathcal{P}(\delta x)| = 6$$

  - ▷ We average over the direction-dependence of $\xi(\mathbf{r})$:

$$\xi_{\text{iso}}(r) = \frac{3}{\delta x^4 |\mathcal{P}(r)|} \left(\delta_{r,0} - \delta_{r,\delta x} \right) = \frac{3}{\delta x^4} \left(\delta_{r,0} - \frac{1}{6} \delta_{r,\delta x} \right)$$

  - ▷ The power spectrum is the DFT of the correlation function:

$$\langle |\mathbf{b}(\mathbf{k})|^2 \rangle = V \delta x^3 \sum_{\mathbf{r}} \xi(\mathbf{r}) \exp(\mathrm{i}\mathbf{k} \cdot \mathbf{r}) = \delta x^2 N^3 \sum_{i=1}^{3} \left[1 - \cos\left(k_i \delta x \right) \right]$$

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  - ▷ We neglect the direction-dependence of the 2nd term in $\xi(\mathbf{r})$:

$$\xi_{\text{iso}}(r) = \frac{3}{\delta x^4 |\mathcal{P}(r)|}\big(\delta_{r,0} - \delta_{r,\delta x}\big) = \frac{3}{\delta x^4}\Big(\delta_{r,0} - \frac{1}{6}\delta_{r,\delta x}\Big)$$

  - ▷ The power spectrum is the DFT of the correlation function:

$$\langle|\mathbf{b}(\mathbf{k})|^2\rangle = \delta x^2 N^3 \sum_{i=1}^{3}\Big[1 - \Big(1 - \frac{k_i^2 L^2}{2N^2} + \mathcal{O}(k_i^4 L^4)\Big)\Big] = \frac{N}{2}\delta x^2 (kL)^2\Big[1 + \mathcal{O}\big((kL)^2\big)\Big]$$

Discrete Approach II.

- The energy spectrum is obtained from the power spectrum:

$$8\pi^2 \frac{E_{\mathrm{M}}(k)}{k^4} = \delta x \left[1 + \mathcal{O}\left((kL)^2\right)\right] \qquad \rightarrow \qquad E_{\mathrm{M}}(k) \propto k^4$$

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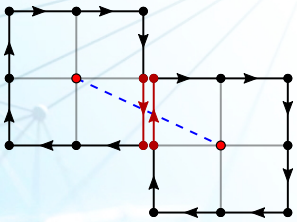
- More detailed spectrum $\rightarrow$ generalized coarse-grain algorithms
  - ▷ Effectively: averaging over $n \times n$ sized plaquettes
  - ▷ Correlations will exist up to larger distances for n^2-plaquette averaged fields, *e.g.*:



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- More detailed spectrum $\rightarrow$ generalized coarse-grain algorithms
  - ▷ Effectively: averaging over $n \times n$ sized plaquettes
  - ▷ Correlations will exist up to larger distances for n^2-plaquette averaged fields
  - ▷ More complicated correlation function and power spectrum...

$$\xi_{\mathrm{iso}}^{(4\mathrm{PA})}(r) = \frac{2 \cdot 3}{(2\delta x)^4 |\mathcal{P}(r)|}\Big[\delta_{r,0} + \delta_{r,\delta x} - \delta_{r,2\delta x} - \delta_{r,\sqrt{5}\delta x}\Big]$$

  - ▷ Scaling of $E_{\mathrm{M}}(k)$ remains the same for small k

Continuous Approach

- Starting point is the same $\xi(\mathbf{r})$ lattice correlation function:

$$\xi_{\text{iso}}(r) = \frac{3}{\delta x^4 |\mathcal{P}(r)|}\left(\delta_{r,0} - \delta_{r,\delta x}\right)$$

  - ▷ Lattice: $|\mathcal{P}(r)|$ counts nr. of neighbors
  - ▷ Lattice: $\delta_{r,a}$ Kronecker-delta

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$$\xi_{\text{iso}}(r) = \frac{3}{\delta x^4 |\mathcal{P}(r)|}\big(\delta_{r,0} - \delta_{r,\delta x}\big) \qquad \rightarrow \qquad \boxed{\xi^{(\text{c})}_{\text{iso}}(r) = \frac{3}{4\pi r^2 \delta x}\big(\delta(r) - \delta(r - \delta x)\big)}$$

  - ▷ Lattice: $|\mathcal{P}(r)|$ counts nr. of neighbors $\rightarrow$ Continuum: $|\mathcal{P}(r)| \rightarrow 4\pi r^2/\delta x^2$
  - ▷ Lattice: $\delta_{r,a}$ Kronecker-delta $\rightarrow$ Continuum: $\delta_{r,a} \rightarrow \delta x\, \delta(r-a)$

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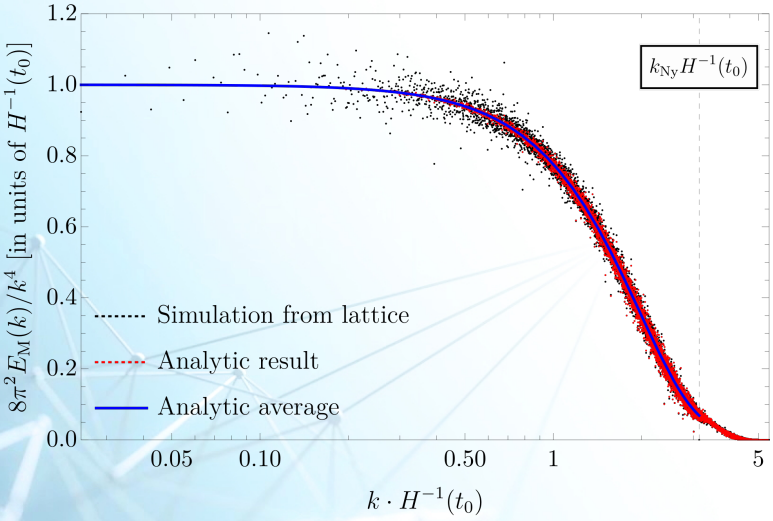
- The energy spectrum is the Hänkel transform of $\xi^{(\text{c})}_{\text{iso}}(r)$:

$$E_{\text{M}}(k) = \frac{k^2}{\pi}\int_0^\infty \mathrm{d}r\, r^2 \xi^{(\text{c})}_{\text{iso}}(r) j_0(kr)$$

$$\boxed{8\pi^2 \frac{E_{\text{M}}(k)}{k^4} = \frac{6}{k^2 \delta x}\big(1 - j_0(k\delta x)\big) \quad \rightarrow \quad E_{\text{M}}(k) \propto k^4}$$

Simulation vs Analytic solution



Continuous fields and spectra

Going further: Interpolated Fields

- There remain some issues with the above lattice approach:
  - Crudeness of derivatives (link variables)
  - Insensitivity for scales below the lattice spacing
  - Regularity of the cubic lattice

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- There remain some issues with the above lattice approach:
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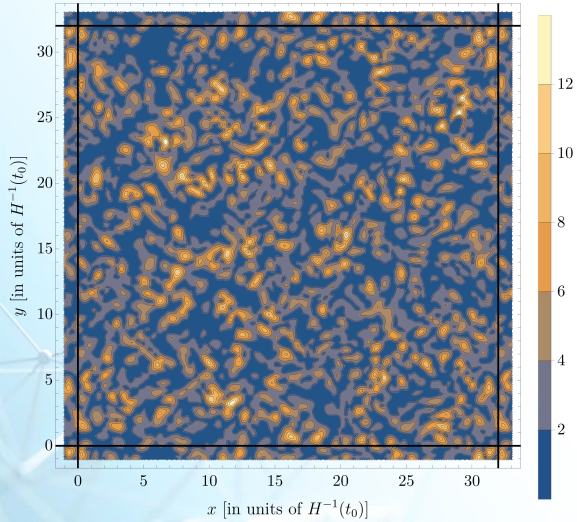
  - ▷ Insensitivity for scales below the lattice spacing
  - ▷ Regularity of the cubic lattice

- We attempt to solve all of these with the following changes:
  - ▷ Irregular lattice setup
  - ▷ Continuous interpolation of the lattice field
    → Using “radial basis function (RBF)” interpolation due to irregularity
  - ▷ Sampling of the continuous field on a dense grid → DFT as before...
    → Technically, direct FT is possible but very slow

Example Slice of the Continuous Magnetic Field



Causal correlation functions

- Due to the interpolation $\longrightarrow$ nearby lattice sites are not independent
- Need "reasonable" analytic formulae to fit for the correlation function and the power spectrum

- Important detail: **Causal correlation functions** [Durrer et al., 2003]
  - ▷ Mathematics: Provided that $\xi(r) = 0$ for $r > \ell$ then $E_{\mathrm{M}}(k) \propto k^4$ for small k (actually, exponentially vanishing is enough)
  - ▷ Our above analytic calculation confirmed this explicitly
  - ▷ If $\xi(r)$ is exponentially small for large r then so is $E_{\mathrm{M}}(k)$ for large k
  - ▷ Empirical formula for the k^4 energy spectrum with an exponential cutoff:

$$\mathcal{E}(k; \{b_a\}, \sigma) \propto k^4 \left(1 + \sum_{a=1}^{n/2} b_a k^{2a}\right) \exp\left(-\frac{2k^2}{\sigma^2}\right)$$

Admissible Correlation Functions

- What specific form should a causal correlation function take to give $\mathcal{E}(k) \propto k^4$?

$$\mathcal{C}(r) = \sum_{n=0}^{n_{\max}} p_n r^n \exp(-ar^2), \quad \text{with } p_{\text{odd}} = 0$$

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- Energy spectrum $\rightarrow$ Hänkel transform of $\mathcal{C}(r)$: [Fubini's theorem]

$$\mathcal{E}(k) = \frac{1}{\pi} \sum_{n=0}^{n_{\max}} p_n k^{-(n+1)} \int_0^\infty \mathrm{d}q\, q^{n+2} \exp(-aq^2/k^2) j_0(q)$$

Admissible Correlation Functions

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$$\mathcal{C}(r) = \sum_{n=0}^{n_{\max}} p_n r^n \exp(-ar^2)\,, \quad \text{with } p_{\text{odd}} = 0$$

- Energy spectrum $\rightarrow$ Hänkel transform of $\mathcal{C}(r)$:

$$\mathcal{E}(k) = \frac{1}{2\pi} \sum_{n=0}^{n_{\max}} p_n \frac{k^2}{a^{m_n}} \Gamma(m_n)\, {}_1\mathrm{F}_1\left(m_n; m_0; -\frac{k^2}{4a}\right), \quad \text{where } m_n = \frac{3+n}{2}$$

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- Compare with the empirical formula:

$$\mathcal{E}(k) \propto k^4 \Big(1 + \sum_{a=1}^{n/2} b_a k^{2a}\Big) \exp\Big(-\frac{2k^2}{\sigma^2}\Big)$$

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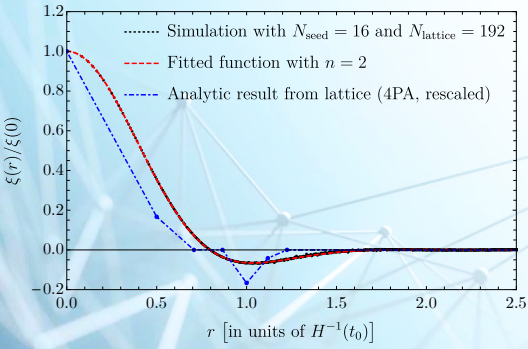
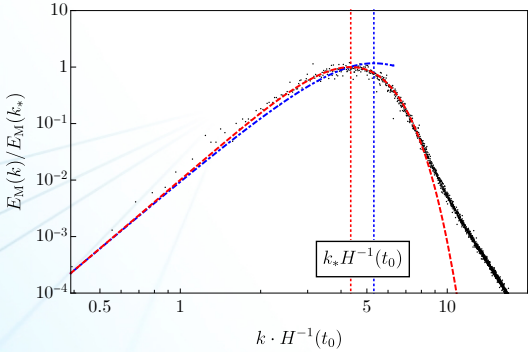
- The k^2 term $\leftrightarrow$ the $j=0$ part of the expression has to cancel:

$$\boxed{\sum_{n=0}^{n_{\max}} p_n a^{-n/2} \Gamma\Big(\frac{3+n}{2}\Big) = 0}$$

Spectrum of the Primordial Magnetic Field

- The spatial correlation function and the energy spectrum (normalized to the spectral peak at k_*):





Conclusions and Outlook

- **Conclusions:**
  - We investigated the spectrum of the magnetic field that could be produced around the time of the EWPT
  - We showed that the spectrum may be directly derived without the need of numerical simulations
  - We confirmed that the spectrum is "causal" and that it scales as $E_{\mathrm{M}} \propto k^4$
  - We derived a general formula for the isotropic spectra
  - We showed that the general formula fits the results of the "continuous" simulations
- **Outlook:**
  - The spectrum is evolved using MHD equations to compare to the magnetic field strength of $\mathcal{O}(10^{-16})$ G at $\lambda = 1$ Mpc scales
  - This simulation does not generate helical fields thus the obtained fields today are too weak
  - Introduce parity violation initially and get a non-zero helicity spectrum