

Confronting neutrino mixing-schemes with correlations of neutrino oscillation data

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[arXiv:2310.20681](#) PPNP (2024), [arXiv:2403.13966](#) APPB (2024),

arXiv:25xx.xxxxx.
work in progress
with Janusz Gluza, Biswajit Karmakar

Matter To The Deepest 2025

Katowice, September 18, 2025

3 ν oscillations and masses, $m_0 = m_{lightest}$

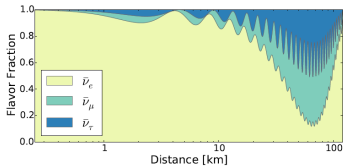
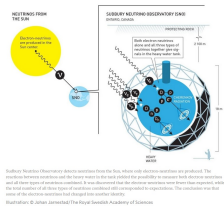


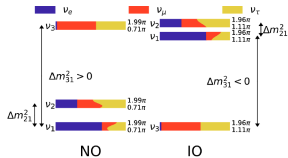
FIG. 1. **Illustration of neutrino oscillations.** The expected flavor composition of the reactor neutrino flux, for neutrinos of 4 MeV energy used as an example, is plotted as a function of distance to the reactor cores. [arXiv:1503.01059](#)

Normal mass ordering (NO)

$$\begin{aligned} m_1 &= m_0, \\ m_2 &= \sqrt{m_0^2 + \Delta m_{21}^2}, \\ m_3 &= \sqrt{m_0^2 + \Delta m_{31}^2}, \end{aligned}$$

Inverted mass ordering (IO)

$$\begin{aligned} m_1 &= \sqrt{m_0^2 - \Delta m_{21}^2 - \Delta m_{32}^2}, \\ m_2 &= \sqrt{m_0^2 - \Delta m_{32}^2}, \\ m_3 &= m_0, \end{aligned}$$



Taken from <https://globalfit.astroparticles.es>,
updated arXiv:1806.11051, Figure 1

$$\Delta m_{21}^2 > 0$$

$$|\Delta m_{31(32)}^2| > 0$$

NO or IO ?

$$\Sigma m_j < 0.12 \text{ eV at 95\% CL from Planck (CMB)}$$

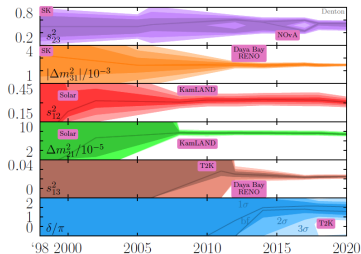
Neutrino flavor and mass eigenstates are related by

$$|\nu_\alpha\rangle = U_{\alpha i} |\nu_i\rangle$$

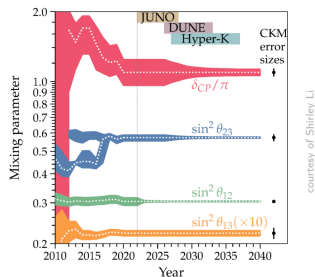
Pontecorvo–Maki–Nakagawa–Sakata parametrization of mixing matrix

$$U_{\text{PMNS}} = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{bmatrix} \begin{bmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $c_{ij} \equiv \cos \theta_{ij}$, $s_{ij} \equiv \sin \theta_{ij}$, $\delta \equiv \delta_{\text{CP}}$.



Peter B. Denton talk, [link here](#).



[arXiv:2204.08668](#), Figure 2.1

courtesy of Shirley Li

Dirac or Majorana ? $(3 + n)\nu$, $n=0$?

NO or IO ? $m_{lightest} \neq 0$?

flavor symmetry ? mixing scheme ?

θ_{23} octant ?

$\delta_{CP} \neq 0$?

$\Sigma m_i < 0.12$ eV $\leftarrow$ this we know from Planck at 95% CL.

$\theta_{13} \neq 0$ $\leftarrow$ this we know from reactor experiments, e.g. Daya Bay.

Experimental data indicate that $\delta_{CP} \neq 0$ and NO are favored.

$\theta_{13} \neq 0$, Daya Bay, RENO (2012)

BM, TB, GR, HG disfavored by non-zero θ_{13} .

$$U_{\text{PMNS}} = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{bmatrix}$$

$$\theta_{13} \neq 0^\circ$$

$\Downarrow$

$$\theta_{23} = 45^\circ$$

$$U_0 = \begin{bmatrix} c_{12} & s_{12} & 0 \\ -\frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{s_{12}}{\sqrt{2}} & -\frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\Downarrow$

$$\theta_{12} = 45^\circ, s_{12} = 1/\sqrt{2}$$

$$\theta_{12} = 35, 26^\circ, s_{12} = 1/\sqrt{3}$$

$$\theta_{12} = 31, 7^\circ$$

$$\theta_{12} = 30^\circ, s_{12} = 1/2$$

Bimaximal Mixing

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Tribimaximal Mixing

$$\begin{bmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Golden Ratio Mixing

$$\begin{bmatrix} \frac{1}{\sqrt{2+\varphi}} & \frac{1}{\sqrt{2+\varphi}} & 0 \\ \frac{-1}{\sqrt{4+2\varphi}} & \frac{\varphi}{\sqrt{4+2\varphi}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{4+2\varphi}} & \frac{-\varphi}{\sqrt{4+2\varphi}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Hexagonal Mixing

$$\begin{bmatrix} \sqrt{\frac{3}{4}} & \frac{1}{2} & 0 \\ -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Golden Ratio Mixing: $\tan \theta_{12} = 1/\varphi$, $\varphi = (1 + \sqrt{5})/2$ being the golden ratio.

Non-zero θ_{13} : Successors of tribimaximal mixing

$$U_{\text{TBM}} = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad U_{\text{PMNS}} \simeq \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0.15 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

```
In[688]:= Sin[8.6*Pi/180]
Sin[8.6*Pi/180]^2
Out[688]= 0.149535
Out[689]= 0.0223608
```

$$|U_{\text{TM}_1}| = \begin{bmatrix} \frac{2}{\sqrt{6}} & * & * \\ \frac{1}{\sqrt{6}} & * & * \\ \frac{1}{\sqrt{6}} & * & * \end{bmatrix}, \quad |U_{\text{TM}_2}| = \begin{bmatrix} * & \frac{1}{\sqrt{3}} & * \\ * & \frac{1}{\sqrt{3}} & * \\ * & \frac{1}{\sqrt{3}} & * \end{bmatrix},$$

$$U_{\text{TM}_1} = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{c_\theta}{\sqrt{3}} & \frac{s_\theta}{\sqrt{3}} e^{-i\gamma} \\ -\frac{1}{\sqrt{6}} & \frac{c_\theta}{\sqrt{3}} - \frac{s_\theta}{\sqrt{2}} e^{i\gamma} & -\frac{s_\theta}{\sqrt{3}} e^{-i\gamma} - \frac{c_\theta}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{c_\theta}{\sqrt{3}} - \frac{s_\theta}{\sqrt{2}} e^{i\gamma} & -\frac{s_\theta}{\sqrt{3}} e^{-i\gamma} + \frac{c_\theta}{\sqrt{2}} \end{bmatrix},$$

$$U_{\text{TM}_2} = \begin{bmatrix} \frac{2c_\theta}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{2s_\theta}{\sqrt{6}} e^{-i\gamma} \\ -\frac{c_\theta}{\sqrt{6}} + \frac{s_\theta}{\sqrt{2}} e^{i\gamma} & \frac{1}{\sqrt{3}} & -\frac{s_\theta}{\sqrt{3}} e^{-i\gamma} - \frac{c_\theta}{\sqrt{2}} \\ -\frac{c_\theta}{\sqrt{6}} + \frac{s_\theta}{\sqrt{2}} e^{i\gamma} & \frac{1}{\sqrt{3}} & -\frac{s_\theta}{\sqrt{3}} e^{-i\gamma} + \frac{c_\theta}{\sqrt{2}} \end{bmatrix}.$$

Mixing-schemes structure: TM_1 and TM_2 , partial μ - τ

Flavor and mass mixing:

$$\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix}.$$

TM_1 and TM_2 mixing-schemes structure

$$|U_{\text{TM}_1}| = \begin{bmatrix} \frac{2}{\sqrt{6}} & * & * \\ \frac{1}{\sqrt{6}} & * & * \\ \frac{1}{\sqrt{6}} & * & * \end{bmatrix}, \quad |U_{\text{TM}_2}| = \begin{bmatrix} * & \frac{1}{\sqrt{3}} & * \\ * & \frac{1}{\sqrt{3}} & * \\ * & \frac{1}{\sqrt{3}} & * \end{bmatrix}.$$

Partial μ - τ ($\mu 1$ - $\tau 1$, $\mu 2$ - $\tau 2$) reflection symmetry mixing structure:

$$|U_{\mu 1}| = |U_{\tau 1}|, \quad |U_{\mu 2}| = |U_{\tau 2}|.$$

TM_1 variant of more general $\mu 1$ - $\tau 1$

TM_2 variant of more general $\mu 2$ - $\tau 2$

TM₁ and TM₂ mixing schemes, partial $\mu - \tau$ symmetry

Comparing the corresponding elements of the first column of U_{PMNS} and U_{TM_1} .

$$|U_{e1}|^2 = c_{12}^2 c_{13}^2 = 2/3 \quad : \quad \theta_{12}^{\text{TM}_1}(\theta_{13}) \quad : \quad s_{12}^2 = \frac{1 - 3s_{13}^2}{3 - 3s_{13}^2},$$

$$|U_{\mu 1}|^2 = |U_{\tau 1}|^2 = 1/6 \quad : \quad \delta_{\text{CP}}^{\text{TM}_1}(\theta_{13}, \theta_{23}) \quad : \quad \cos \delta_{\text{CP}} = \frac{(1 - 5s_{13}^2)(2s_{23}^2 - 1)}{4s_{13}s_{23}\sqrt{2(1 - 3s_{13}^2)(1 - s_{23}^2)}}.$$

Comparing the corresponding elements of the second column of U_{PMNS} and U_{TM_2} .

$$|U_{e2}|^2 = s_{12}^2 c_{13}^2 = 1/3 \quad : \quad \theta_{12}^{\text{TM}_2}(\theta_{13}) \quad : \quad s_{12}^2 = \frac{1}{3 - 3s_{13}^2},$$

$$|U_{\mu 2}|^2 = |U_{\tau 2}|^2 = 1/3 \quad : \quad \delta_{\text{CP}}^{\text{TM}_2}(\theta_{13}, \theta_{23}) \quad : \quad \cos \delta_{\text{CP}} = -\frac{(2 - 4s_{13}^2)(2s_{23}^2 - 1)}{4s_{13}s_{23}\sqrt{(2 - 3s_{13}^2)(1 - s_{23}^2)}}.$$

In partial μ - τ reflection symmetry, the mixing matrix symmetry is given by:

$$|U_{\mu 1}| = |U_{\tau 1}| \quad : \quad \delta_{\text{CP}}^{\mu 1 - \tau 1}(\theta_{13}, \theta_{12}, \theta_{23}) \quad : \quad \cos \delta_{\text{CP}} = \frac{(c_{23}^2 - s_{23}^2)(c_{12}^2 s_{13}^2 - s_{12}^2)}{4c_{12}s_{12}c_{23}s_{23}s_{13}},$$

$$|U_{\mu 2}| = |U_{\tau 2}| \quad : \quad \delta_{\text{CP}}^{\mu 2 - \tau 2}(\theta_{13}, \theta_{12}, \theta_{23}) \quad : \quad \cos \delta_{\text{CP}} = \frac{(c_{23}^2 - s_{23}^2)(c_{12}^2 - s_{12}^2 s_{13}^2)}{4c_{12}s_{12}c_{23}s_{23}s_{13}}.$$

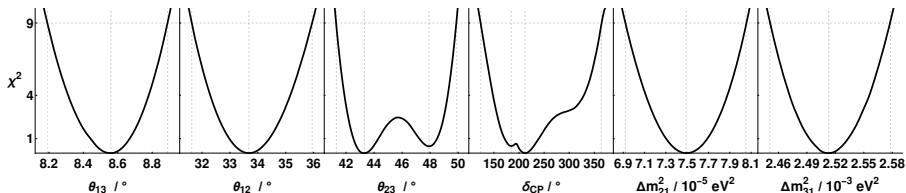
Summary of relations arising from mixing-schemes structure

TM ₁	TM ₂
$s_{12}^2 = \frac{1-3s_{13}^2}{3-3s_{13}^2}$	$s_{12}^2 = \frac{1}{3-3s_{13}^2}$
$\cos \delta_{\text{CP}} = \frac{(1-5s_{13}^2)(2s_{23}^2-1)}{4s_{13}s_{23}\sqrt{2(1-3s_{13}^2)(1-s_{23}^2)}}$	$\cos \delta_{\text{CP}} = -\frac{(2-4s_{13}^2)(2s_{23}^2-1)}{4s_{13}s_{23}\sqrt{(2-3s_{13}^2)(1-s_{23}^2)}}$
$\mu 1 - \tau 1$	$\mu 2 - \tau 2$
$\cos \delta_{\text{CP}} = \frac{(c_{23}^2-s_{23}^2)(c_{12}^2s_{13}^2-s_{12}^2)}{4c_{12}s_{12}c_{23}s_{23}s_{13}}$	$\cos \delta_{\text{CP}} = \frac{(c_{23}^2-s_{23}^2)(c_{12}^2-s_{12}^2s_{13}^2)}{4c_{12}s_{12}c_{23}s_{23}s_{13}}.$

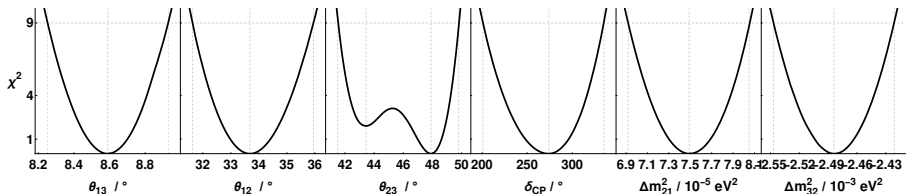
How to confront the mixing schemes with experimental data ?

Oscillation parameters, 3σ ranges, b-fs, NuFIT v6.0 (2025), "model independent"

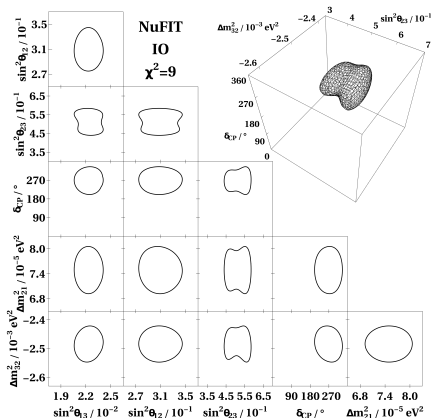
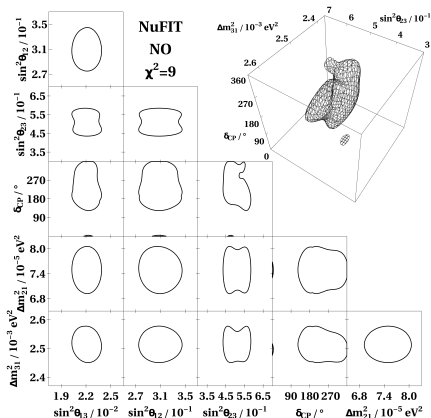
NO



IO



Oscillation parameters, correlations, NuFIT v6.0 (2025), "model independent"



Currently, up to 3D datasets are available.

Mathematica output based on (interpolated) NuFIT v6.0 datasets (correlations).

```
In[686]:= {{{"", "sin2θ23=0.57", "Δm312=0.00255 eV2", "δCP=290°"},  
           {"χ2", deltachisq["NO"] [{"T23"}][5.7], deltachisq["NO"] [{"DMA"}][2.55], deltachisq["NO"] [{"DCP"}][290]}} // TableForm
```

Out[686]//TableForm=

	sin ² θ ₂₃ =0.57	Δm ₃₁ ² =0.00255 eV ²	δ _{CP} =290°	← treated independently, each $\chi^2 \sim 3$
χ ²	2.71635	3.07873	2.82341	

```
In[687]:= {{{"", "sin2θ23=0.57 ∧ Δm312=0.00255 eV2"}, {"χ2", deltachisq["NO"] [{"T23", "DMA"}][5.7, 2.55]}} // TableForm
```

Out[687]//TableForm=

	sin ² θ ₂₃ =0.57 ∧ Δm ₃₁ ² =0.00255 eV ²	←
χ ²	4.68546	

```
In[688]:= {{{"", "Δm312=0.00255 eV2 ∧ δCP=290°"}, {"χ2", deltachisq["NO"] [{"DMA", "DCP"}][2.55, 290]}} // TableForm
```

Out[688]//TableForm=

	Δm ₃₁ ² =0.00255 eV ² ∧ δ _{CP} =290°	← correlations of two, $\chi^2 \sim 5$ to 13
χ ²	7.56033	

```
In[689]:= {{{"", "sin2θ23=0.57 ∧ δCP=290°"}, {"χ2", deltachisq["NO"] [{"T23", "DCP"}][5.7, 290]}} // TableForm
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	sin ² θ ₂₃ =0.57 ∧ δ _{CP} =290°	←
χ ²	13.454	

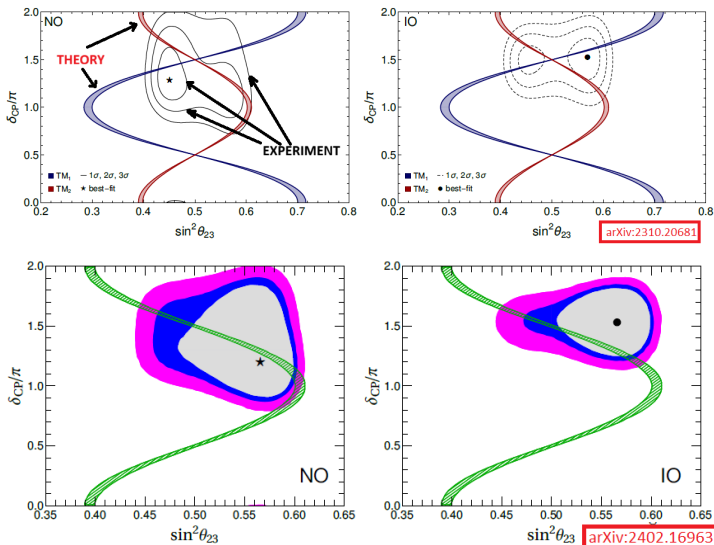
```
In[690]:= {{{"", "sin2θ23=0.57 ∧ Δm312=0.00255 eV2 ∧ δCP=290°"}, {"χ2", deltachisq["NO"] [{"T23", "DMA", "DCP"}][5.7, 2.55, 290]}} // TableForm
```

Out[690]//TableForm=

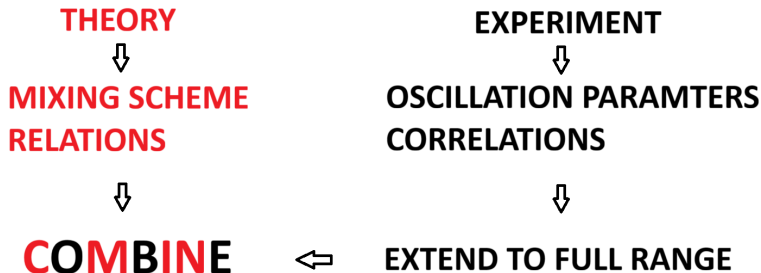
	sin ² θ ₂₃ =0.57 ∧ Δm ₃₁ ² =0.00255 eV ² ∧ δ _{CP} =290°	← correlation of three, $\chi^2 \sim 17$
χ ²	17.4936	

Correlations improve the probability output through the χ^2 value.

Confronting mixing schemes with oscillation data (examples)



⇒ Next, we will combine theory with experiment which are flashed independently on the plots.



Extend to full range..

We assume, that for random variables (X_i) with corresponding best-fits (bf_i) and standard deviations (σ_i), the relation holds:

$$\chi^2(X_1, \dots, X_i) = \sum_i \left(\frac{X_i - bf_i}{\sigma_i} \right)^2.$$

Extending the number of random variables, we have:

$$\chi^2(X_1, \dots, X_i) \geq \max \left\{ \chi^2(X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_i), \quad k \in \{1 \dots i\} \right\},$$

since for any $k \in \{1 \dots i\}$ **no risk of excluding allowed region because of " $\geq$ "**

$$\chi^2(X_1, \dots, X_i) = \sum_i \left(\frac{X_i - bf_i}{\sigma_i} \right)^2 \geq \sum_{i \neq k} \left(\frac{X_i - bf_i}{\sigma_i} \right)^2 = \chi^2(X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_i),$$

and in the context of neutrino parameters e.g.

$$\chi^2(\theta_{23}, \Delta m_{31}^2, \delta_{CP}) \geq \max \left\{ \chi^2(\Delta m_{31}^2, \delta_{CP}), \chi^2(\theta_{23}, \delta_{CP}), \chi^2(\theta_{23}, \Delta m_{31}^2) \right\}$$

INDEED: $17.5 \geq \max\{7.6, 13.5, 4.7\}$ $\leftarrow$ numbers taken from slide 12

..and reduce (when needed).

To reduce the number of random variables, we have:

$$\begin{aligned}\chi^2(X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_i) &= \sum_{i \neq k} \left(\frac{X_i - bf_i}{\sigma_i} \right)^2 = \sum_{i \neq k} \left(\frac{X_i - bf_i}{\sigma_i} \right)^2 + \left(\frac{bf_k - bf_k}{\sigma_k} \right)^2 \\ &= \chi^2(X_1, \dots, X_k = bf_k, \dots, X_i),\end{aligned}$$

and in the context of neutrino parameters e.g.

$$\chi^2(\theta_{23}, \delta_{\text{CP}}) = \chi^2(\theta_{23}, \theta_{13}^{bf}, \delta_{\text{CP}}).$$

A full (6D) correlations directly from experiment would further constrain the results.

The extension formula may still be useful due to numerical complexity and memory requirements of 6D data sets.

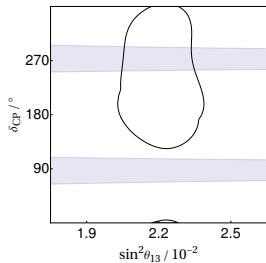
TM₁ mixing relations + extended to full range correlations, example:

$$\begin{array}{c} \sin^2 \theta_{12} = \frac{1-3\sin^2 \theta_{13}}{3-3\sin^2 \theta_{13}} \\ \downarrow \\ \chi^2(\theta_{13}, \theta_{12}, \theta_{23}, \delta_{\text{CP}}, \Delta m_{21}^2, \Delta m_{31}^2) \\ \uparrow \\ \cos \delta_{\text{CP}} = \frac{(1-5\sin^2 \theta_{13})(2\sin_{23}^2-1)}{4\sin_{13}\sin\theta_{23}\sqrt{2(1-3\sin^2\theta_{13})(1-\sin^2\theta_{23})}} \end{array}$$

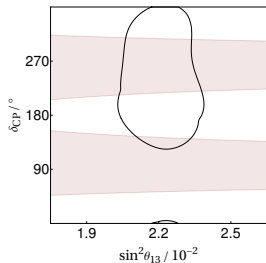
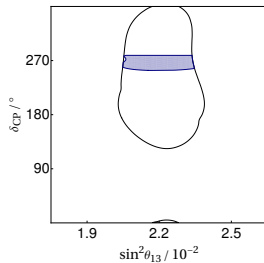
For any choice of parameters, the equations for these parameters must be solved first.

NO and IO need to be treated separately.

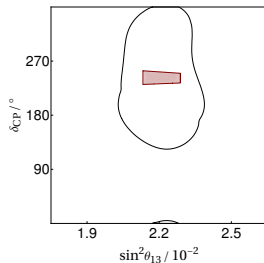
Impact of complete correlations on model's predictions (NuFIT v6.0)



THEORY AND EXPERIMENT
 FLASHED
 INDEPENDENTLY
TM₁
COMBINED →

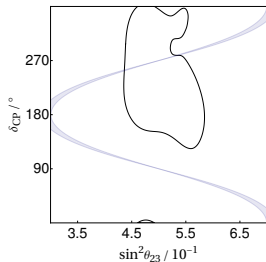


THEORY AND EXPERIMENT
 FLASHED
 INDEPENDENTLY
TM₂
COMBINED →

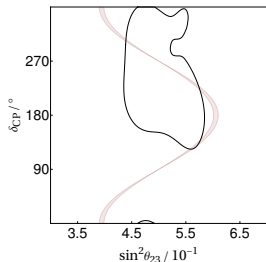
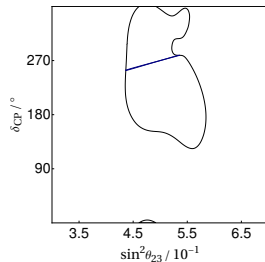


NO, $\chi^2 < 9$

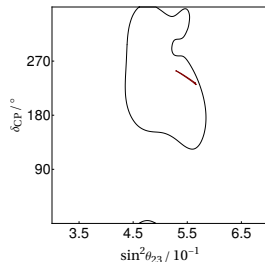
Impact of complete correlations on model's predictions (NuFIT v6.0)



THEORY AND EXPERIMENT
 FLASHED
 INDEPENDENTLY
 TM_1
 COMBINED

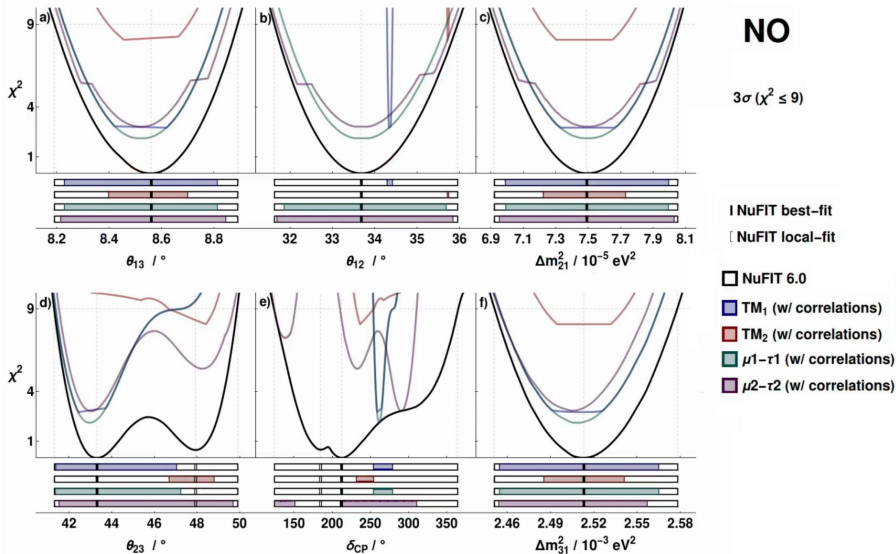


THEORY AND EXPERIMENT
 FLASHED
 INDEPENDENTLY
 TM_2
 COMBINED



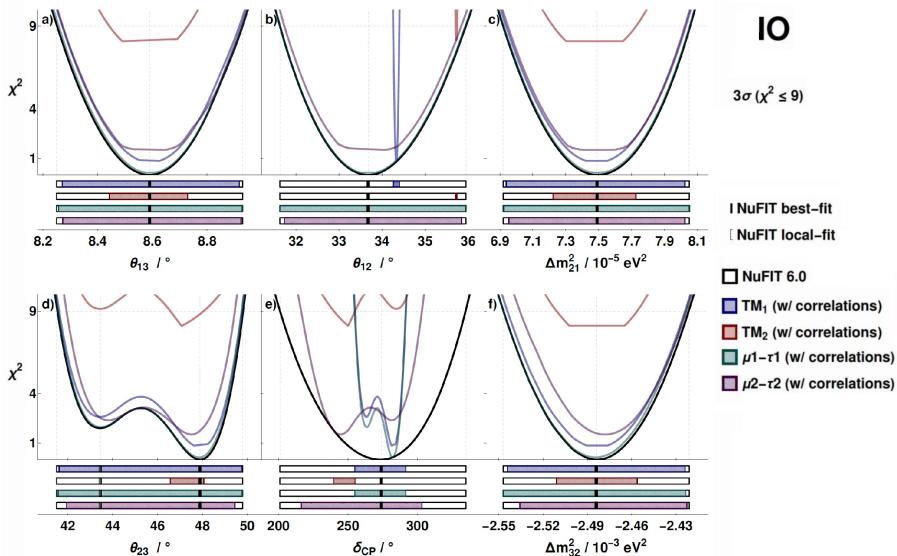
NO, $\chi^2 < 9$

Extracted ranges of model's parameters from complete correlations (NuFIT v6.0)



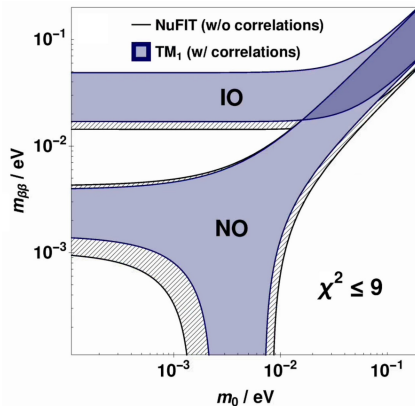
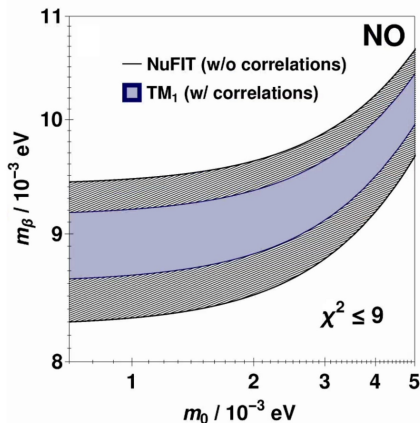
χ^2_{min} : TM₁ = 2.8, **TM₂ = 8.1**, $\mu 1 - \tau 1 = 2.1$, $\mu 2 - \tau 2 = 2.8$

Extracted ranges of model's parameters from complete correlations (NuFIT v6.0)



$$\chi^2_{min}: TM_1 = 0.9, TM_2 = 8.1, \mu 1 - \tau 1 = 0.1, \mu 2 - \tau 2 = 1.5$$

Phenomenological predictions: m_β and $m_{\beta\beta}$ (NuFIT v6.0)



Multidimensional correlation data on neutrino oscillation parameters analysis allows for better theoretical model predictions

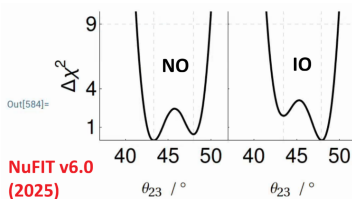
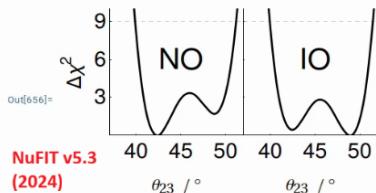
With the presented approach, we

- test TM_1 , TM_2 and partial μ - τ mixing-schemes over the full range of parameters at a given χ^2 level
- some parameters are exceptionally constrained
 - θ_{12} for TM_1 and TM_2 ,
 - θ_{23} for TM_2 ,
 - δ_{CP} for all considered mixing-schemes,
 - Δm_{21}^2 and $\Delta m_{31(32)}^2$ for TM_2
- find tighter, model-dependent constraints on the effective neutrino masses

The method is general and can be applied to any model providing mixing-scheme relations in the form analytical expressions involving the neutrino oscillation parameters.

Thank you
for your attention.

3 ν mixing



```

In[658]:= {deltachisq[#][{"T13"}][0] & /@ {"NO", "IO"},
           deltachisq[#][{"DCP"}][0] & /@ {"NO", "IO"}}
Out[658]:= {{1674.27, 1664.84}, {11.0731, 14.4343}}

```

```

In[585]:= {deltachisq[#][{"T13"}][0] & /@ {"NO", "IO"},
           deltachisq[#][{"DCP"}][0] & /@ {"NO", "IO"}}
Out[585]:= {{1774.37, 1768.02}, {8.03189, 19.3677}}

```

Left: Mathematica output based on NuFIT v5.3 data (2024). **Right:** Mathematica output based on NuFIT v6.0 data (2025).

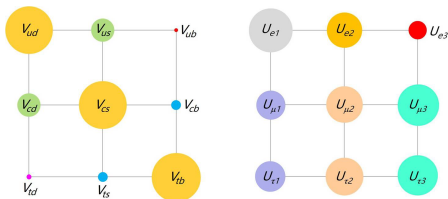
NO

0.800 → 0.843	0.519 → 0.580	0.143 → 0.155
0.354 → 0.461	0.533 → 0.620	0.635 → 0.769
0.277 → 0.485	0.589 → 0.616	0.620 → 0.759

IO

0.800 → 0.843	0.519 → 0.580	0.143 → 0.156
0.288 → 0.479	0.567 → 0.606	0.635 → 0.772
0.246 → 0.527	0.585 → 0.603	0.616 → 0.759

Left: Mixing matrix ($|U_{\text{PMNS}}|$) ranges in 3σ , NuFIT v5.3.

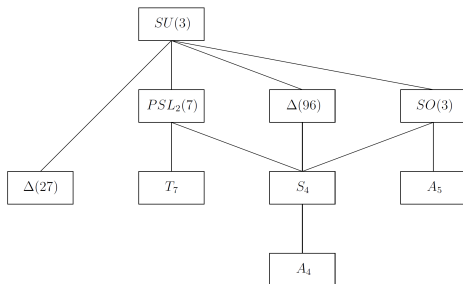


(a) CKM quark flavor mixing

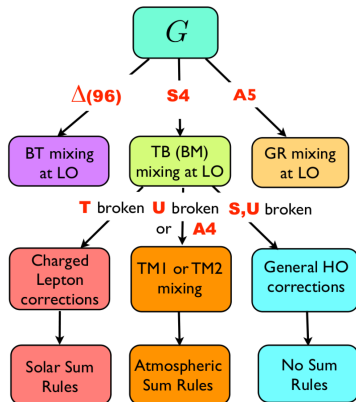
(b) PMNS lepton flavor mixing

Right: Taken from [arXiv:2210.11922](https://arxiv.org/abs/2210.11922), Figure 2.

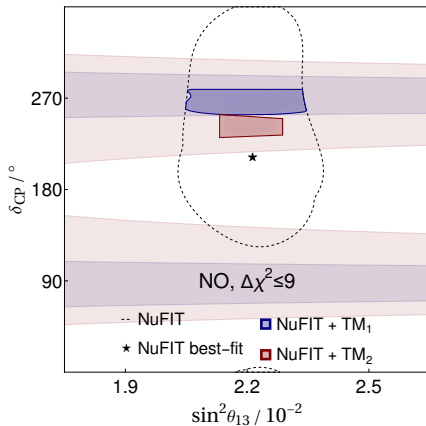
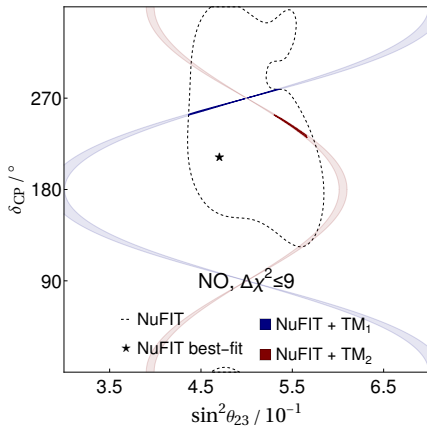
Flavor symmetries and mixing-schemes



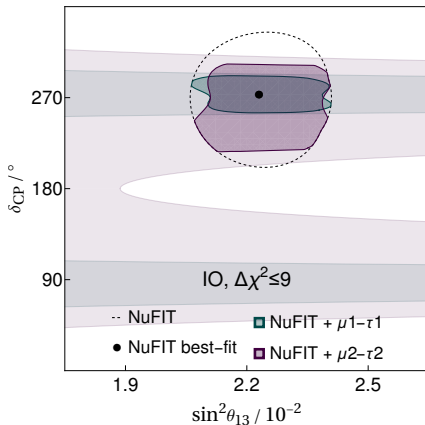
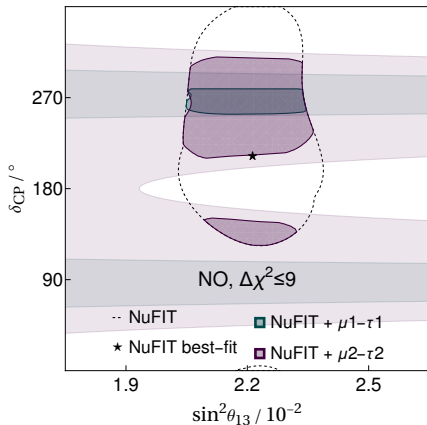
arXiv:1301.1340 S.F. King, C. Luhn



Experiment / Theory / Combined at $\chi^2 < 9$ (NuFIT v6.0)



Experiment / Theory / Combined at $\chi^2 < 9$ (NuFIT v6.0)



Effective neutrino mass

Effective electron neutrino mass:

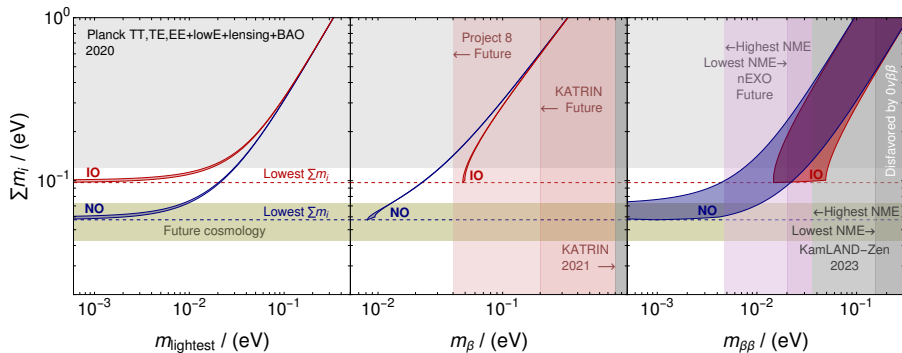
$$m_{\beta}^2 = c_{13}^2 c_{12}^2 m_1^2 + c_{13}^2 s_{12}^2 m_2^2 + s_{13}^2 m_3^2 = \begin{cases} \text{NO:} & m_0^2 + \Delta m_{21}^2 c_{13}^2 s_{12}^2 + \Delta m_{3\ell}^2 s_{13}^2, \\ \text{IO:} & m_0^2 - \Delta m_{21}^2 c_{13}^2 c_{12}^2 - \Delta m_{3\ell}^2 c_{13}^2. \end{cases}$$

Effective Majorana mass:

$$m_{\beta\beta} = \left| \sum_i m_i U_{ei}^2 \right| = \left| m_1 c_{13}^2 c_{12}^2 e^{i2\alpha_1} + m_2 c_{13}^2 s_{12}^2 e^{i2\alpha_2} + m_3 s_{13}^2 e^{-i2\delta_{\text{CP}}} \right|$$
$$= \begin{cases} \text{NO:} & m_0 \left| c_{13}^2 c_{12}^2 e^{i2(\alpha_1 - \delta_{\text{CP}})} + \sqrt{1 + \frac{\Delta m_{21}^2}{m_0^2}} c_{13}^2 s_{12}^2 e^{i2(\alpha_2 - \delta_{\text{CP}})} + \sqrt{1 + \frac{\Delta m_{3\ell}^2}{m_0^2}} s_{13}^2 \right| \\ \text{IO:} & m_0 \left| \sqrt{1 - \frac{\Delta m_{3\ell}^2 + \Delta m_{21}^2}{m_0^2}} c_{13}^2 c_{12}^2 e^{i2(\alpha_1 - \delta_{\text{CP}})} + \sqrt{1 - \frac{\Delta m_{3\ell}^2}{m_0^2}} c_{13}^2 s_{12}^2 e^{i2(\alpha_2 - \delta_{\text{CP}})} + s_{13}^2 \right|. \end{cases}$$

Here $m_0 = m_1$ (m_3) and $\ell = 1$ (2) in $\Delta m_{3\ell}^2$ for NO (IO).

Neutrino mass



Planck: $\sum_i m_i < 0.12 \text{ eV}$, 95% CL, arXiv:1807.06209

KATRIN: $m_\beta < 0.8 \text{ eV}$, 90% CL, arXiv:2105.08533

KamLAND-Zen: $m_{\beta\beta} < (0.036 \text{ to}) 0.156 \text{ eV}$, 90% CL, arXiv:2203.02139