# Towards NNLO Precision in $e^+e^- \to \pi^+\pi^-\gamma$ with PHOKHARA

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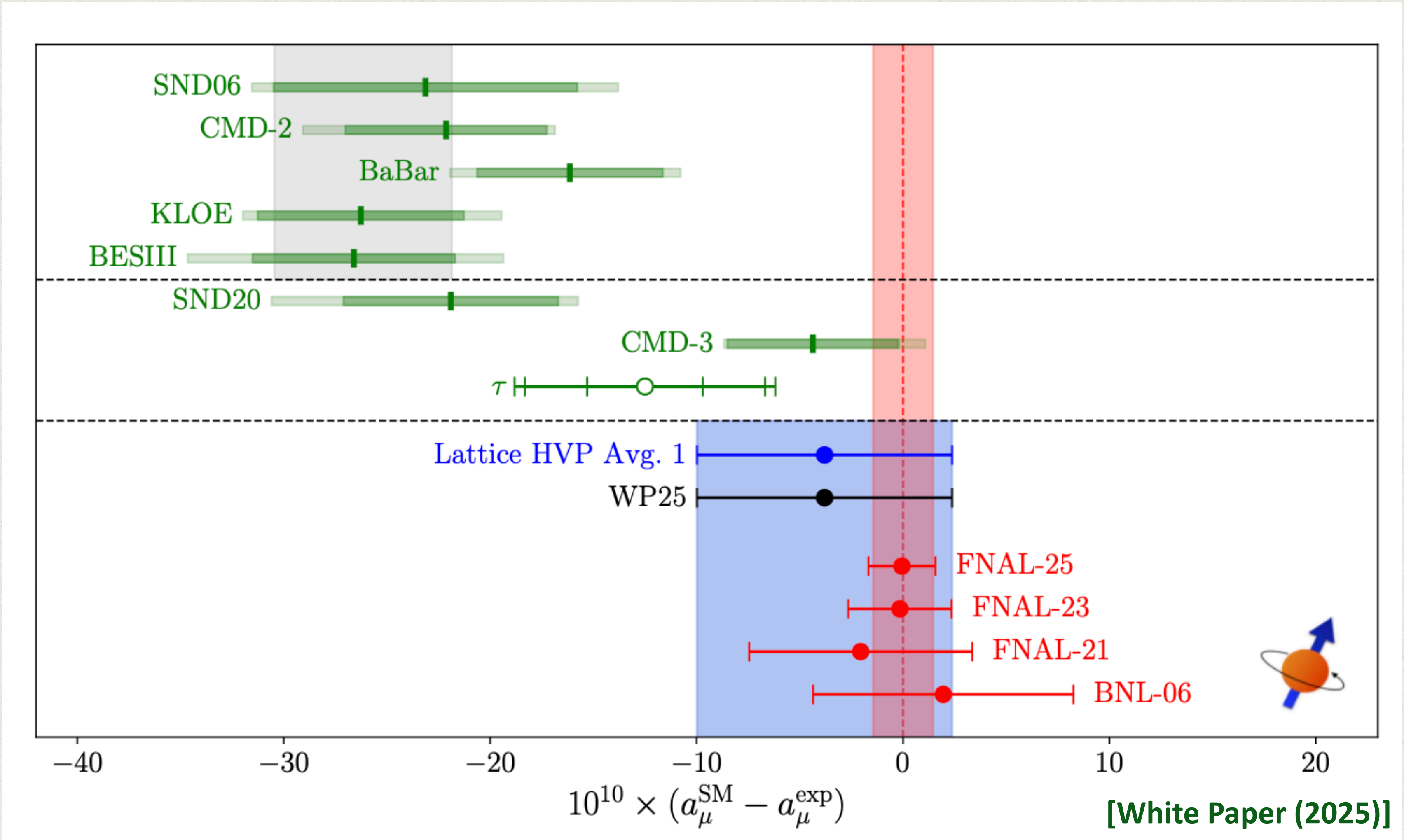
*University of Liverpool*

In collaboration w/ Tom Dave, Pau Petit Rosas, Jérémy Paltrinieri, Mattia Pozzoli

*Matter To The Deepest*
*spinPLACE, Katowice, Poland*
*September 15—19, 2025*

# *Current SM prediction for $a_\mu$ in comparison to experiment*

<<Fred's talks




[White Paper (2025)]

$$a_\mu^{\mathrm{SM}} = 116592033(62) \times 10^{-11} \longleftrightarrow a_\mu^{\mathrm{exp}} = 116592071.5(14.5) \times 10^{-11}$$

$$\Delta a_\mu = a_\mu^{exp} - a_\mu^{SM} = 38(63) \times 10^{-11}$$

☑ No tension between the SM prediction and the experimental world average

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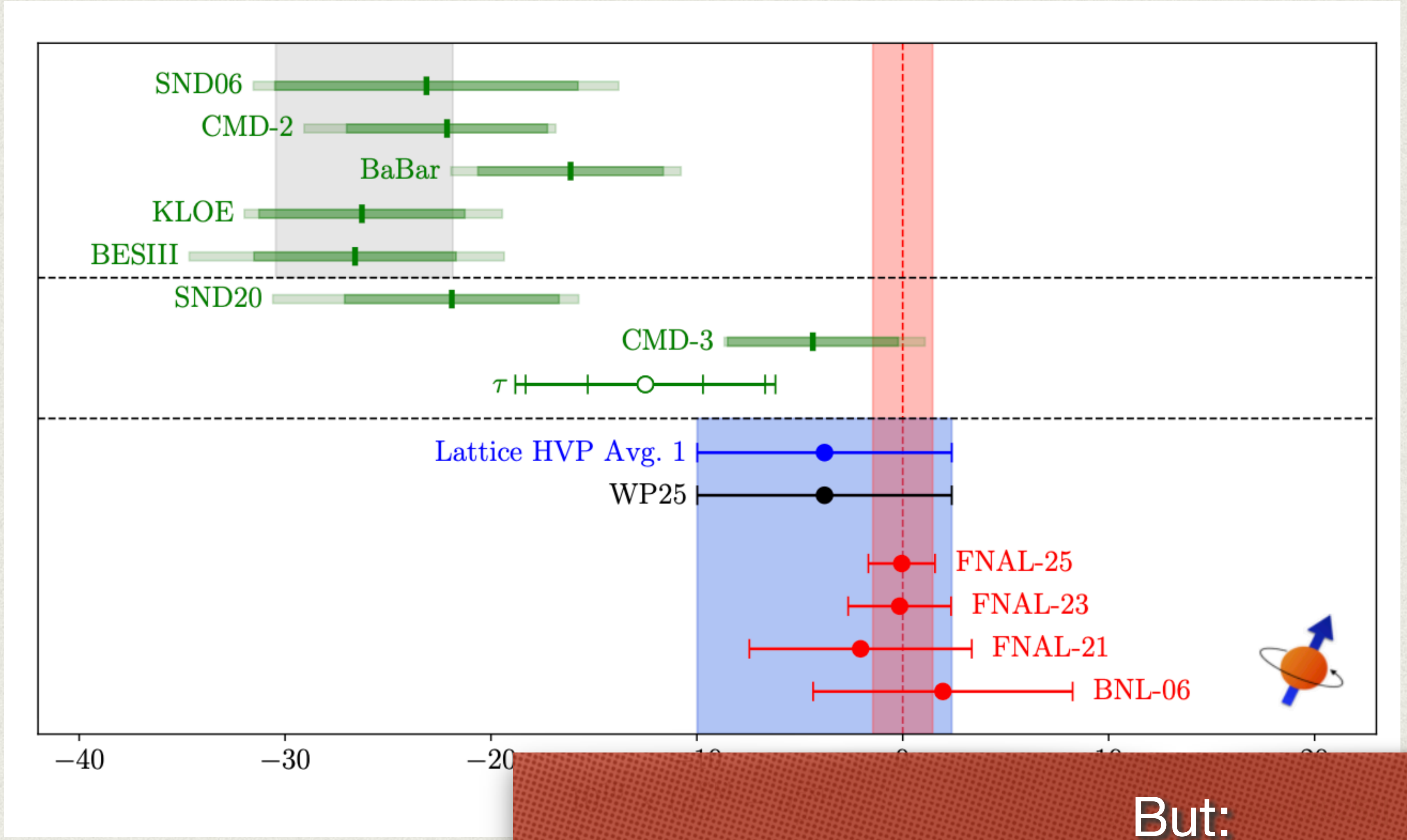


But:

- Disagreement between data-driven & Lattice QCD
- Tension between data of various $e^+e^-$ experiments
- Improve theoretical predictions

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$$\Delta a_\mu = a_\mu^{exp} - a_\mu^{SM} = 38(63) \times 10^{-11}$$

☑ No tension between the SM prediction and the experimental world average

# *And beyond $a_\mu$*

- ☑ Test SM at low energy w/ high precision

Eur. Phys. J. C (2010) 66: 585–686
DOI 10.1140/epjc/s10052-010-1251-4

THE EUROPEAN PHYSICAL JOURNAL C

Review

**Quest for precision in hadronic cross sections at low energy: Monte Carlo tools vs. experimental data**

**Working Group on Radiative Corrections and Monte Carlo Generators for Low Energies**

SciPost Phys. Comm. Rep. 9 (2025)

**Radiative corrections and Monte Carlo tools for low-energy hadronic cross sections in $e^+e^-$ collisions**

Riccardo Aliberti[1], Paolo Beltrame[2], Ettore Budassi[3,4], Carlo M. Carloni Calame[4], Gilberto Colangelo[5], Lorenzo Cotrozzi[2], Achim Denig[1], Anna Driutti[6,7], Tim Engel[8], Lois Flower[2,9], Andrea Gurgone[3,6,7], Martin Hoferichter[5], Fedor Ignatov[2], Sophie Kollatzsch[10,11], Bastian Kubis[12], Andrzej Kupść[13,14]*, Fabian Lange[10,11], Alberto Lusiani[7,15], Stefan E. Müller[16], Jérémy Paltrinieri[2], Pau Petit Rosàs[2], Fulvio Piccinini[4], Alan Price[17], Lorenzo Punzi[7,15], Marco Rocco[10,18], Olga Shekhovtsova[19,20], Andrzej Siódmok[17], Adrian Signer[10,11]*, Giovanni Stagnitto[21], Peter Stoffer[10,11], Thomas Teubner[2], William J. Torres Bobadilla[2], Francesco P. Ucci[3,4], Yannick Ulrich[2,5]* and Graziano Venanzoni[2,7]*
(RadioMonteCarLow 2 working group)

- ☑ Provide state-of-the-art predictions for:
  - $e^+e^- \to \mu^+\mu^-(\gamma)$
  - $e^+e^- \to e^+e^-(\gamma)$
  - $e^+e^- \to \pi^+\pi^-(\gamma)$

RadioMonteCarlow 2 is a community effort aimed at improving the description of electron-positron collisions at low energy ($\sqrt{s} \leq$ few GeV)

Phase I: comparison w/ Monte Carlo tools:

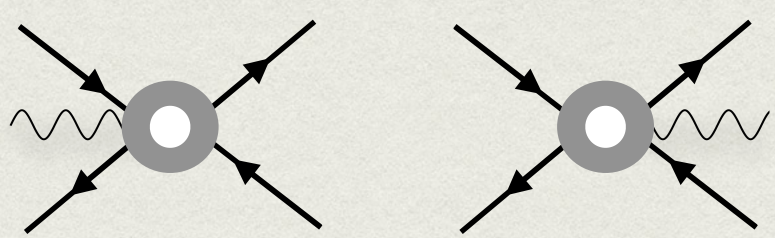
- AfkQed
- BabaYaga@NLO
- KKMC
- Sherpa.
- MCGPJ
- McMule
- **Phokhara**

# *A shameless introduction to Phokhara*

- Phokhara is a Monte Carlo event generator for low energy $e^+e^-$ colliders, **with +20 years of development** ([https://looptreeduality.csic.es/phokhara/](https://looptreeduality.csic.es/phokhara/))

- LO & NLO contributions to radiative return processes

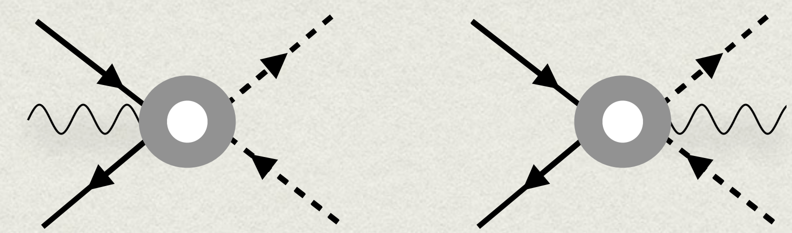
$e^+e^- \to \mu^+\mu^-\gamma$



ISR, FSR & Mixed

+ real radiations

$e^+e^- \to \pi^+\pi^-\gamma$



**Use of F×sQED**

- Phokhara also contains a variety of hadronic production channels

| $e^+e^- \to$ | Order | VP | VFF | Extras |
|---|---|---|---|---|
| $\mu^+\mu^-$ | LO | `alphaQED`, or NSK | - | Narrow resonances of $J/\psi$ and $\psi(2S)$ |
| $\mu^+\mu^-\gamma$ | NLO with full mass dependence | | | |
| $\pi^+\pi^-$ | LO | `alphaQED`, or NSK | F×sQED choice of 3 VFF | Narrow resonances of $J/\psi$ and $\psi(2S)$ Radiative $\phi$ decays |
| $\pi^+\pi^-\gamma$ | NLO with full mass dependence | | | |
| $X$ | $X \in 2\pi^0\pi^+\pi^-$, $2\pi^+2\pi^-$, $p\bar{p}$, $n\bar{n}$, $K^+K^-$, $K^0\bar{K}^0$, $\pi^+\pi^-\pi^0$, $\Lambda(\to \pi^-p)\bar{\Lambda}(\to \pi^+\bar{p})$, $\eta\pi^+\pi^-$, $\pi^0\gamma$, $\eta\gamma$, $\eta'\gamma$, $\chi_{c1} \to J/\psi(\to \mu^+\mu^-)\gamma$, $\chi_{c2} \to J/\psi(\to \mu^+\mu^-)\gamma$ | | | |

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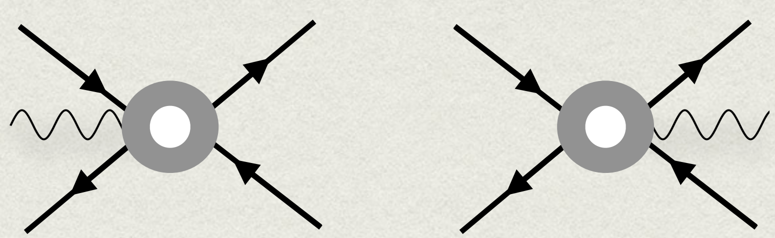

ISR, FSR & Mixed

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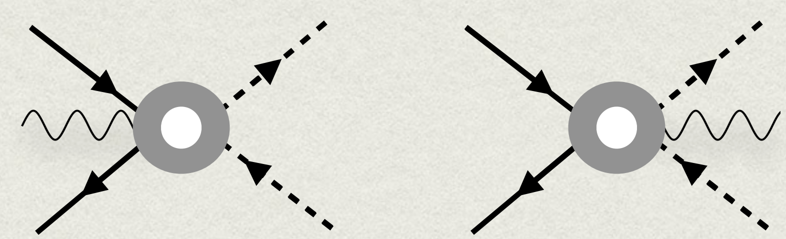

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| $\mu^+\mu^-\gamma$ | NLO with full mass dependence | or NSK | | of $J/\psi$ and $\psi$(2 |
| $\pi^+\pi^-$ | LO | alphaQED, | F×sQED | Narrow resonances |
| $\pi^+\pi^-\gamma$ | NLO with full mass dependence | or NSK | choice of 3 VFF | of $J/\psi$ and $\psi(2S)$ Radiative $\phi$ decays |
| $X$ | $X \in 2\pi^0\pi^+\pi^-$, $2\pi^+2\pi^-$, $p\bar{p}$, $n\bar{n}$, $K^+K^-$, $K^0\bar{K}^0$, $\pi^+\pi^-\pi^0$, $\Lambda(\to\pi^-p)\bar{\Lambda}(\to\pi^+\bar{p})$, $\eta\pi^+\pi^-$, $\pi^0\gamma$, $\eta\gamma$, $\eta'\gamma$, $\chi_{c1}\to J/\psi(\to\mu^+\mu^-)\gamma$, $\chi_{c2}\to J/\psi(\to\mu^+\mu^-)\gamma$ | | | |

**Outlook**

No further development of the code is planned

[H Czyż (STRONG 2020 Virtual Workshop 25.11.2021)]

This talk: I plan to present a status of the higher-order corrections required in `Phokhara` at NNLO.

| | $\mu^+\mu^-\gamma$ | $\pi^+\pi^-\gamma$ | |
|---|---|---|---|
| Phokhara | NNLO | NNLO | exponentiation, F×sQED, GVMD, FsQED |

- Fixed order NLO + soft photon resummation
- GVMD (NLO) and $F \times$ sQED (NNLO) within Phokhara
- **Fixed order NNLO**

# *Outline*

- Validation of Phokhara 10 & current developments
- Fast numerical evaluation of scattering amplitudes
- Progress on $e^+e^- \to \mu^+\mu^-\gamma$ at two loops
- Conclusions & Outlook

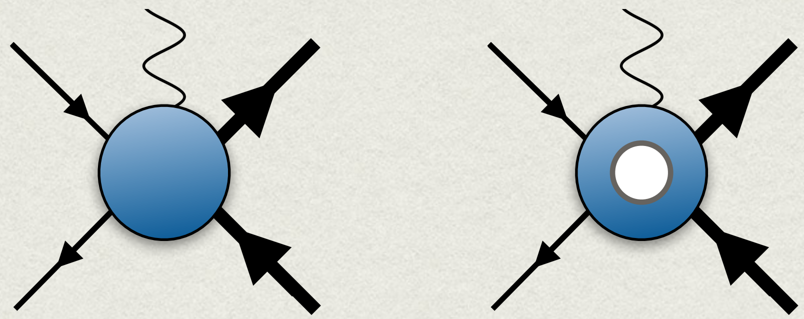
# Validation of `Phokhara`

# *Validation @ NLO*

- Evaluation of five and six-point amplitudes

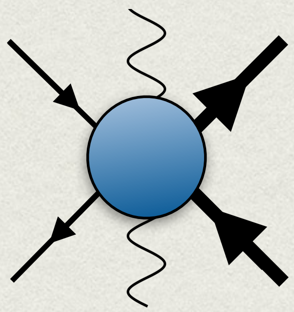
Born & virtual contribution

**Five** kinematic scales $s_{ij}$ + **two** masses



Real contribution

**Eight** kinematic scales $s_{ij}$ + **two** masses



- Automated tools in Amplitudes' methodology:
  — Qgraph/FeynArts, Form/FeynCalc (construction of amplitudes)
  — QCDLoop, LoopTools, Collier, … (numerical evaluation of Feynman integrals)
  — GoSam, OpenLoops, Recola, Helac, … (numerical evaluation of amplitudes)

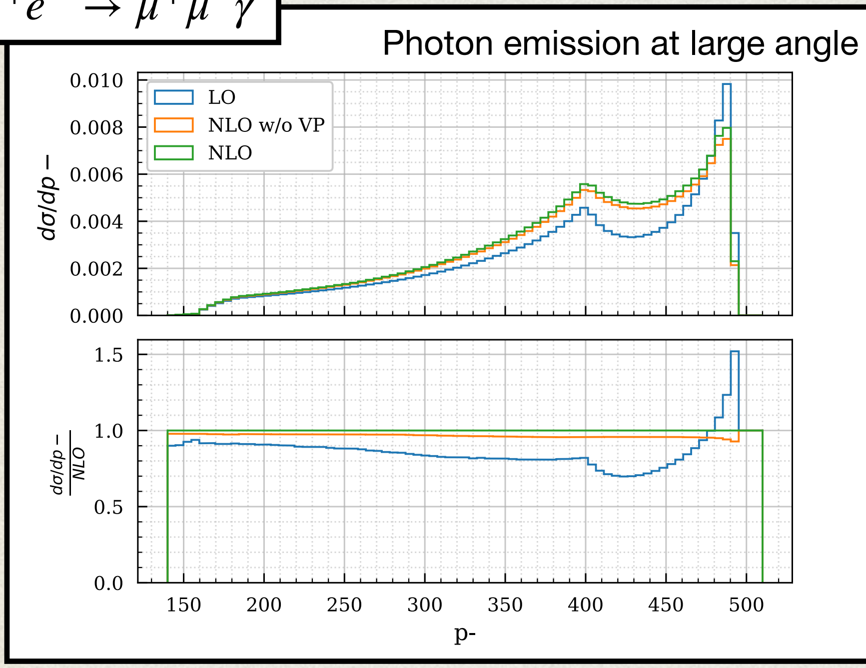
- Dimensionally regulated one-loop amplitude ($D = 4 - 2\epsilon$)

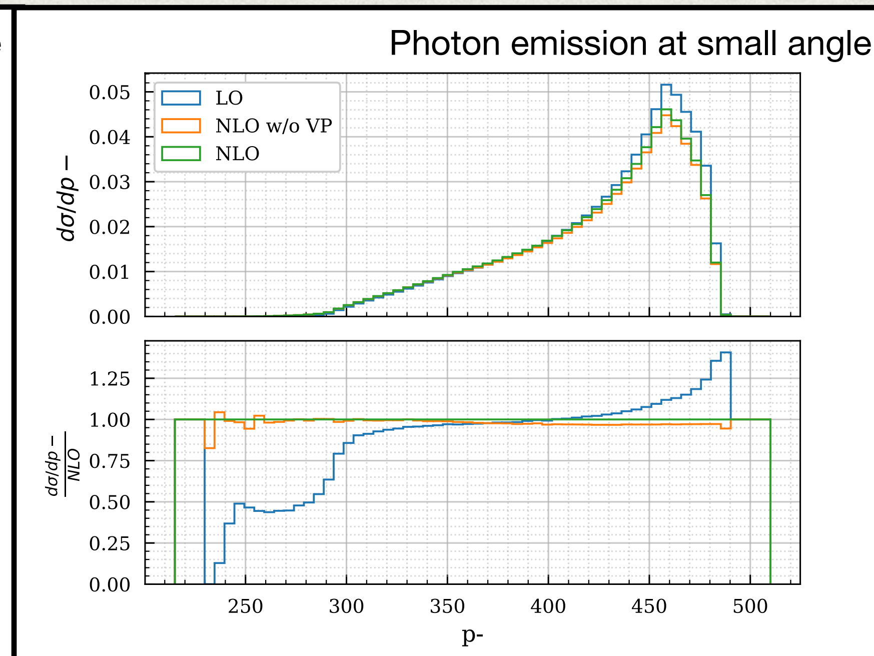
$$\mathscr{A}^{(1)}\left(e^+e^- \to \mu^+\mu^-\gamma\right) = \frac{c_{-1}}{\epsilon} + c_0$$

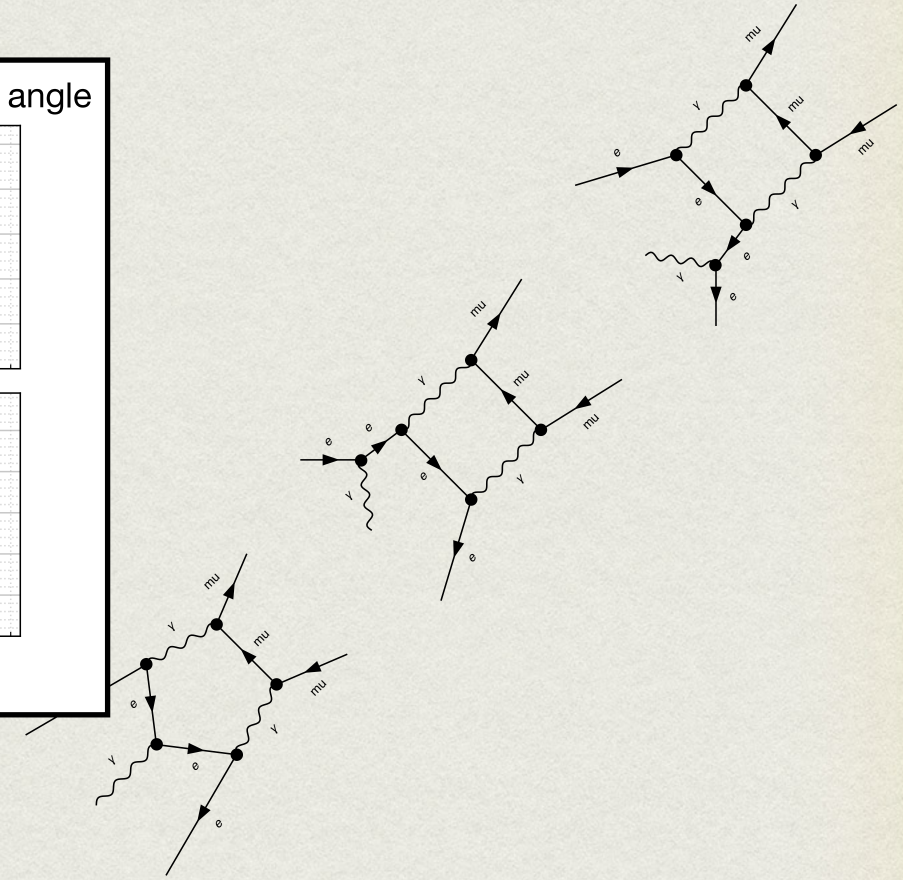
(validated with `Phokhara`)

# *Validation @ NLO*

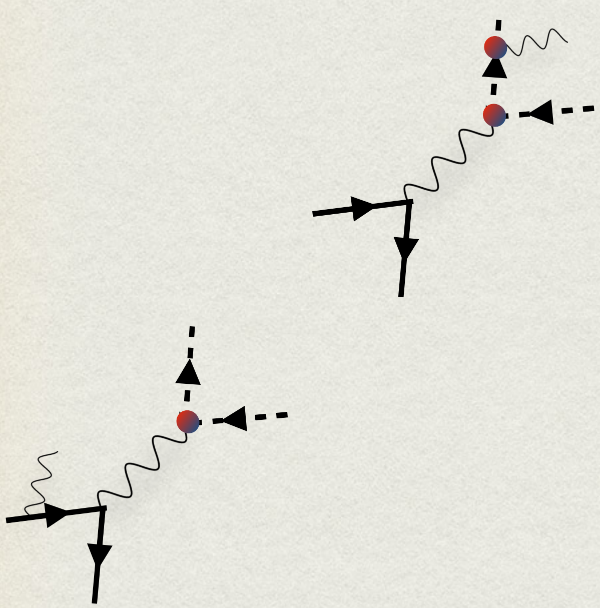
$e^+e^- \to \mu^+\mu^-\gamma$

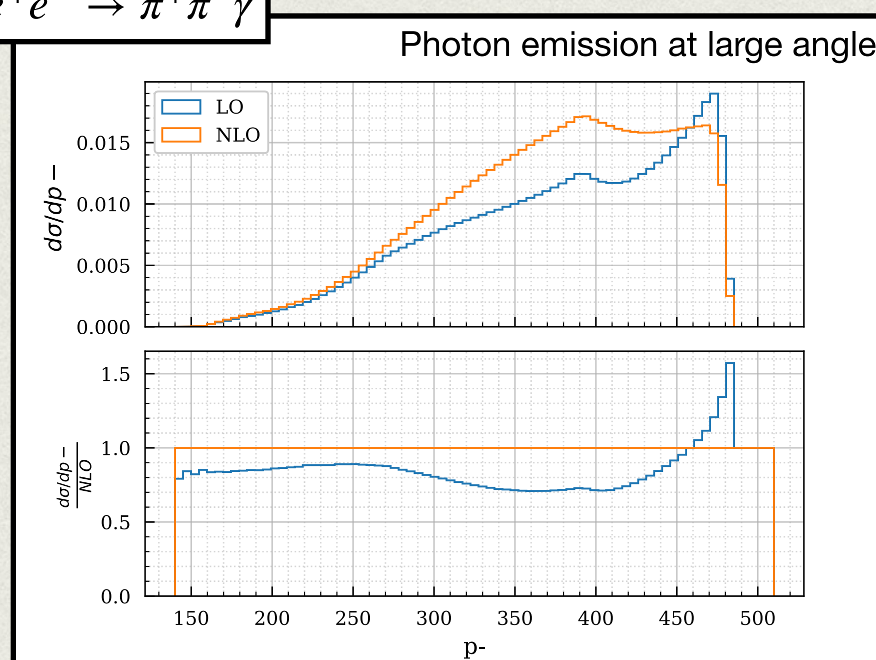


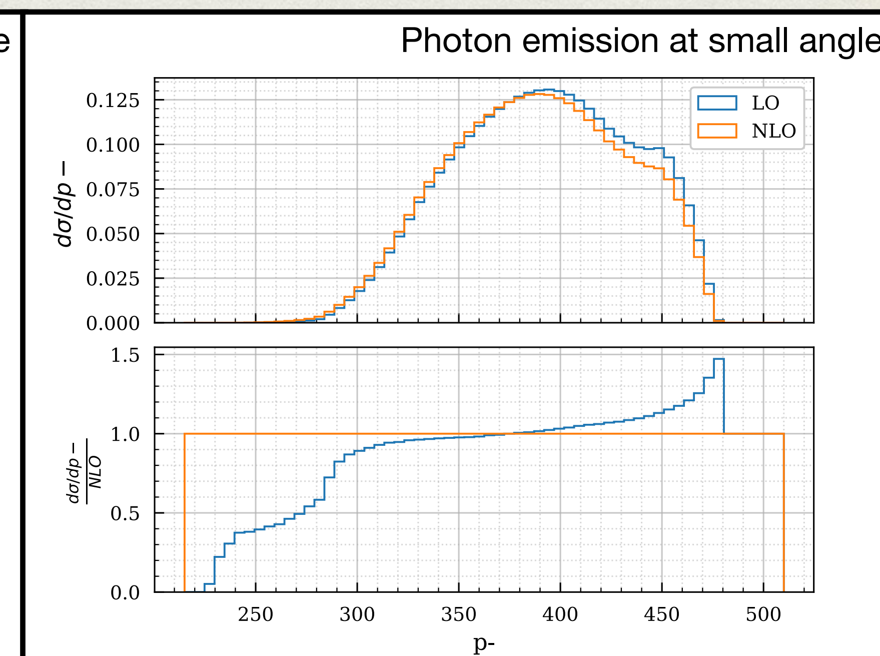



$e^+e^- \to \pi^+\pi^-\gamma$



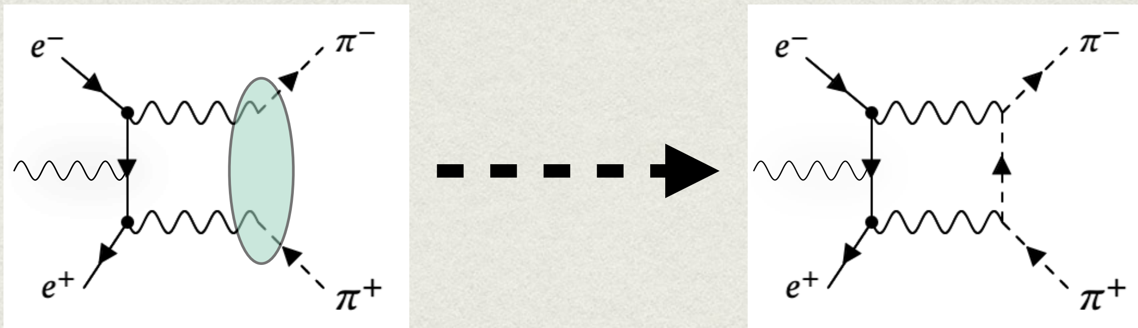


# *NLO+ within* `Phokhara`

[Work in progress]

Improve interaction between pions and photons

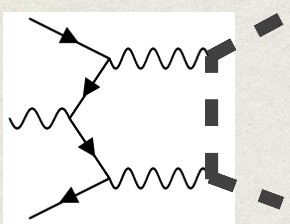



$\times \left[F(M_{\pi\pi}) + F(q_1^2)F(q_2^2) - F(M_{\pi\pi})\right]$

$q_2$

- $F \times$ sQED:
  Scalar QED multiplied by form factors for external virtualities

[Tracz Ph.D. thesis (2018)]

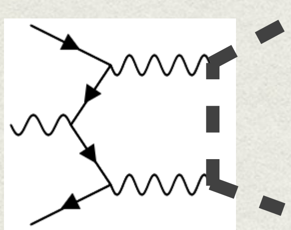


$$+\text{boxes} + \text{crossed diagrams} = \frac{c_{-1}}{\epsilon} + c_0 + c_1\epsilon + \mathcal{O}(\epsilon^2)$$

(need at higher loop orders)

- Generalised vector-meson dominance (GVMD):
  photon-pion coupling modified by a sum of vector-meson Breit–Wigner propagators

[Lee, Ignatov (2022)]



$$+\text{boxes} + \text{crossed diagrams} = \tilde{c}_0 + \mathcal{O}(\epsilon)$$

(to be included in `Phokhara`)



[Petit Rosàs, WJT (to appear)]

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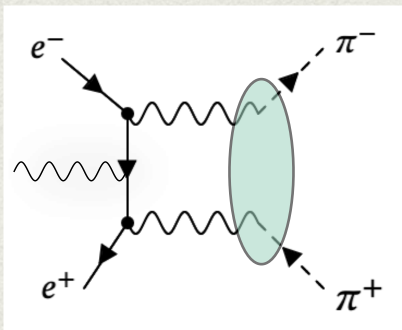


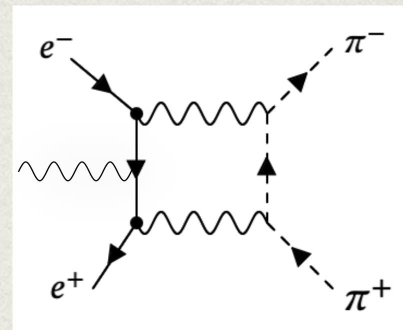


$$\times \left[F(M_{\pi\pi})+F(q_1^2)F(q_2^2) - F(M_{\pi\pi})\right]$$

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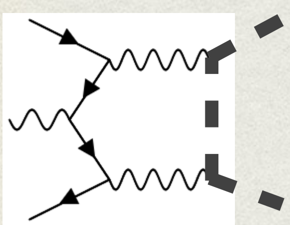

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(need at higher loop orders)

$$F(q^2) = \sum_{v=1}^{n} a_v \frac{\Lambda_n}{\Lambda_n - q^2}, \qquad \text{with } \sum_{v=1}^{n} a_v = 1\,.$$

- Generalised vector-meson dominance (GVMD):
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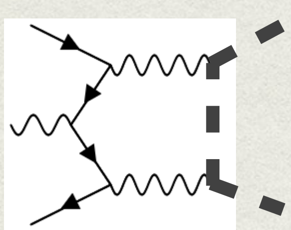
**[Lee, Ignatov (2022)]**



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(to be included in `Phokhara`)



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# *NLO+ within* `Phokhara`

[Work in progress]

- Resummation of soft photons in `Phokhara`

Build on Yennie, Frautschi, and Suura (YFS) and Coherent Exclusive Exponentiation (CEEX)

<<Alan's, Zbigniew's, and Bennie's talks

$$d\sigma = \sum_{n_\gamma=0}^{\infty} \frac{e^{2\alpha(B+\tilde{B}(\Omega))}}{n_\gamma!} d\Phi_Q \left[ \prod_{i=1}^{n_\gamma} d\Phi_i^\gamma \tilde{S}(k_i) \Theta(k_i, \Omega) \right] \left( \tilde{\beta}_0 + \sum_{j=1}^{n_\gamma} \frac{\beta_1(k_j)}{\tilde{S}(k_j)} + \cdots \right)$$

YFS form factor      Eikonal factors      Finite contributions

Resummed cross sections accounting for leading soft-photon effects at all order in the QED coupling

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YFS form factor | Eikonal factors | Finite contributions

Resummed cross sections accounting for leading soft-photon effects at all order in the QED coupling

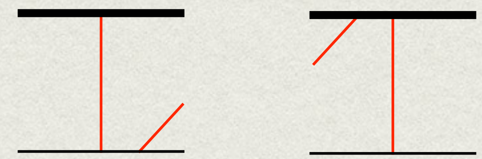
**Channels under study**

- $e^+e^- \to \mu^+\mu^-$
- $e^+e^- \to \pi^+\pi^-$
- $e^+e^- \to \mu^+\mu^-\gamma$
- $e^+e^- \to \pi^+\pi^-\gamma$

# $e^+e^- \to F^+F^-\gamma$ @ NNLO

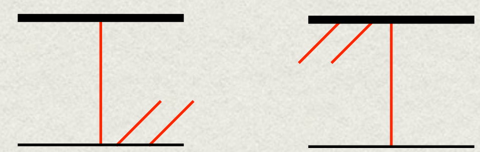
## Anatomy @ LO

- Born matrix element
  tree-level & n-pt process

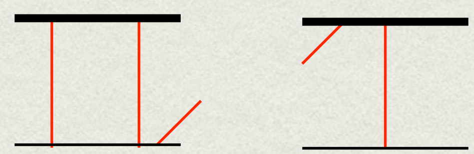


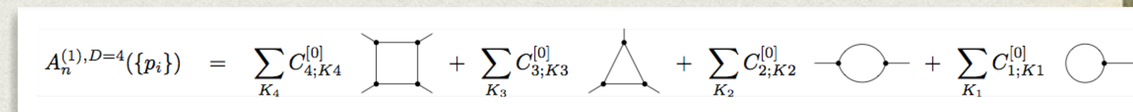
## Anatomy @ NLO

- Real contribution tree-level $(n+1)$-particles



- Virtual Contribution one-loop $(n+1)$-particles



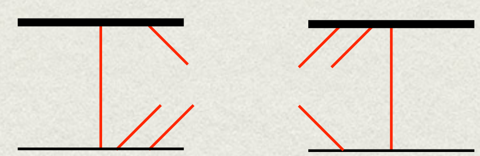


$$A_n^{(1),D=4}(\{p_i\}) = \sum_{K_4} C^{[0]}_{4;K4} + \sum_{K_3} C^{[0]}_{3;K3} + \sum_{K_2} C^{[0]}_{2;K2} + \sum_{K_1} C^{[0]}_{1;K1}$$

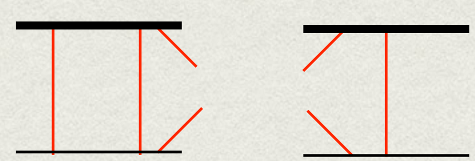
☑ Automated one-loop Feynman integral & phase-space evaluation
☑ IR subtraction schemes under control
☑ Efficient numerical evaluation (MC friendly)

## Anatomy @ NNLO

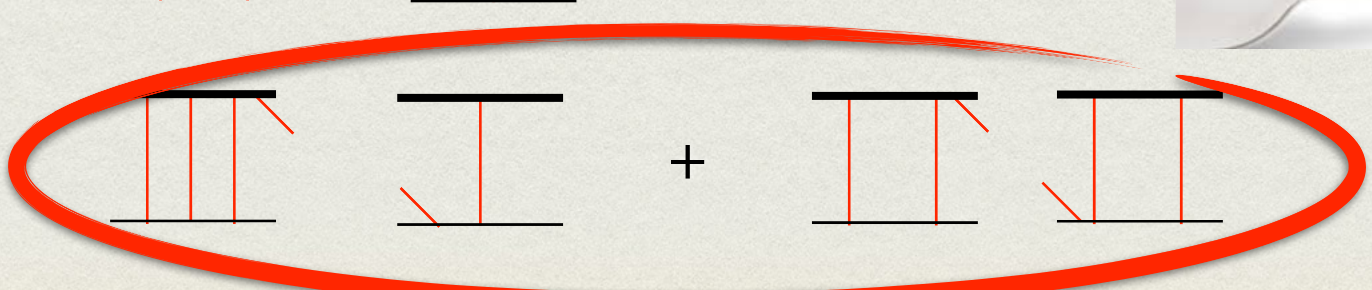
- Real-Real contribution
  Tree-level $(n+2)$-particles



- Real-Virtual Contribution
  one-loop $(n+1)$-particles



- Virtual-Virtual Contribution
  two-loop $n$-particles




☐ Harder (but doable) phase-space integration
☐ Extend numerical evaluation of one-loop Feynman integrals
☐ Basis of two-loop Feynman integral not known

# $e^+e^- \to \pi\pi\gamma$ @ 2L

Amplitude generation

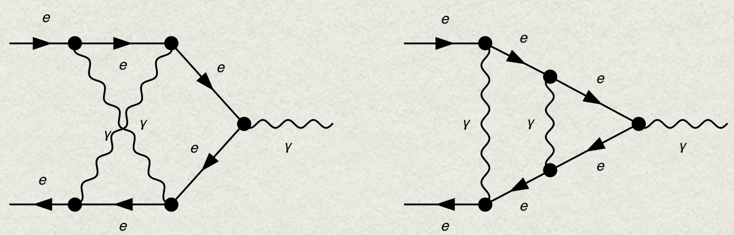
Algebraic decomposition

Loop integral evaluation

MC input

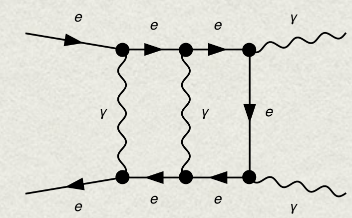
- Two-loop gauge invariant pieces

$f^+f^- \to \gamma^* \to F^+F^- + \gamma$



☑ Easy $(m_f^2, s)$

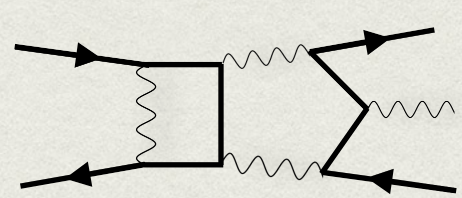
$f^+f^- \to \gamma\,\gamma^* \to F^+F^-$



$+\ldots$

☐ "Normal" $(s, t, m_e^2, q^2)$

$f^+f^- \to F^+F^-\gamma$



$+\ldots$

☐ Hard $(s_{12}, s_{23}, s_{34}, s_{45}, s_{51}, m_f^2, m_F^2)$

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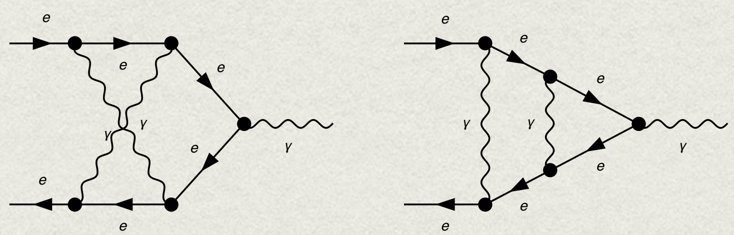
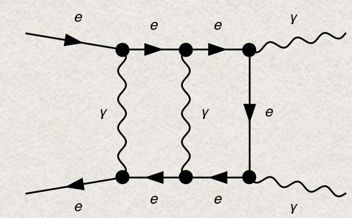


☑ Easy $(m_f^2, s)$

$$f^+f^- \to \gamma\,\gamma^* \to F^+F^-$$

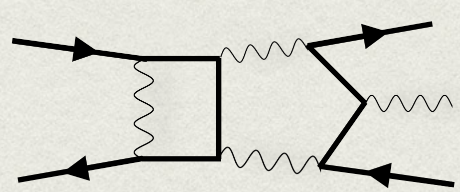


$+\ldots$

☐ "Normal" $(s, t, m_e^2, q^2)$

Progress on these Feynman integrals

$$f^+f^- \to F^+F^-\gamma$$



$+\ldots$

☐ Hard $(s_{12}, s_{23}, s_{34}, s_{45}, s_{51}, m_f^2, m_F^2)$

# Efficient evaluation of Scattering Amplitudes

# *Numerical evaluation*

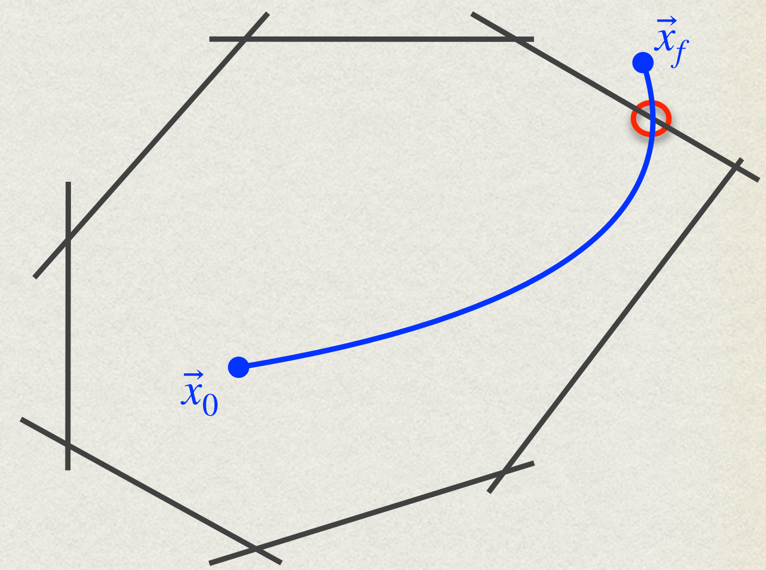
- Evaluation of Feynman integrals by the method of differential equations

$$\partial_x \vec{I}(\vec{x};\epsilon) = A_x(\vec{x};\epsilon)\,\vec{I}(\vec{x};\epsilon)$$

$\vec{x}$ (kinematic invariants)

$$\frac{\partial \vec{J}}{\partial x} = \sum_{k=0}^{2} \epsilon^k A_k \vec{J}$$

- Canonical
- Non-canonical but polynomial in $\epsilon$




Available `Mathematica` implementations (DiffExp, SeaSyde, …)
but slow for MC evaluations and costly to generate grids

- Combine physical & mathematical insights

  - ☑ When possible find a canonical basis $\vec{J} = R\vec{I}$ **[Henn 2013]**
  - ☑ Solve DEQ along the path **[Moriello 2019]**
  - ☑ Get boundary constants $\vec{J}_0$ analytically or numerically
  - ☑ Account for analytic continuations when crossing regions

# *Numerical evaluation*

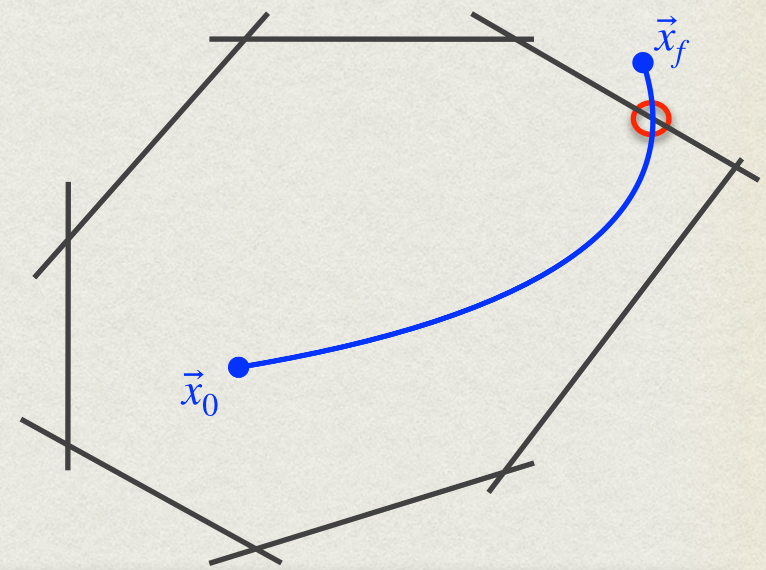
- Evaluation of Feynman integrals by the method of differential equations

$$\partial_x \vec{I}(\vec{x};\epsilon) = A_x(\vec{x};\epsilon)\,\vec{I}(\vec{x};\epsilon)$$

$\vec{x}$ (kinematic invariants)

$$\frac{\partial \vec{J}}{\partial x} = \sum_{k=0}^{2} \epsilon^k A_k \vec{J}$$

- Canonical
- Non-canonical but polynomial in $\epsilon$

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Prepared for submission to JHEP

**Fast evaluation of Feynman integrals for Monte Carlo generators**


**Pau Petit Rosàs[a] and William J. Torres Bobadilla[a]**

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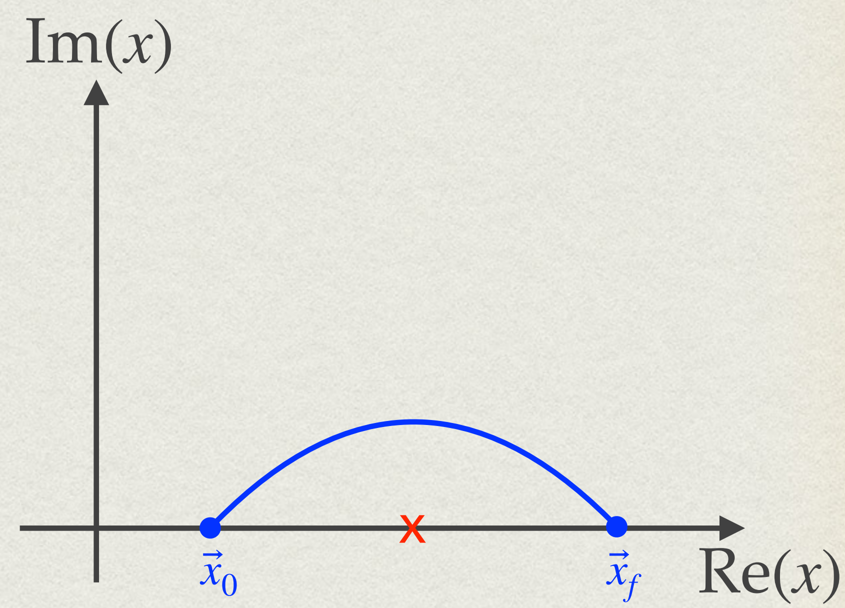
Abstract: Building on the idea of numerically integrating differential equations satisfied by Feynman integrals, we propose a novel strategy for handling branch cuts within a numerical framework. We develop an integrator capable of evaluating a basis of integrals in both double and quadruple precision, achieving significantly reduced computational times compared to existing tools. We demonstrate the performance of our integrator by evaluating one- and two-loop five-point Feynman integrals with up to nine complex kinematic scales. In particular, we apply our method to the radiative return process of massive electron-positron annihilation into pions plus an energetic photon within scalar QED, for which we also build the differential equation, and extend it to the case where virtual photons acquire an auxiliary complex mass under the Generalised Vector-Meson Dominance model. Furthermore, we validate our approach on two integral families relevant for the two-loop production of $t\bar{t}$ + jet. The integrator achieves, in double precision, execution times of the order of milliseconds for one-loop topologies and hundreds of milliseconds for the two-loop families, enabling for on-the-fly computation of Feynman integrals in Monte Carlo generators and a more efficient generation of grids for the topologies with prohibitive computational costs.

# *Numerical evaluation for MC tools*

- Fast integrator in a low-level language :: C++
- Precise, but no need for 50 significant figures!
- Ideally, fast enough to not use grids




- Treatment of branch cuts

$$r = \sqrt{K^n(\vec{x})} \longrightarrow \prod_{i=1}^{n} \sqrt{x - e_i}$$

Branch cuts :: $\prod_{i=1}^{n} \sqrt{x - e_i} - \rho$ vs $\prod_{i=1}^{n} \left( \sqrt{x - e_i} - \rho_i \right)$

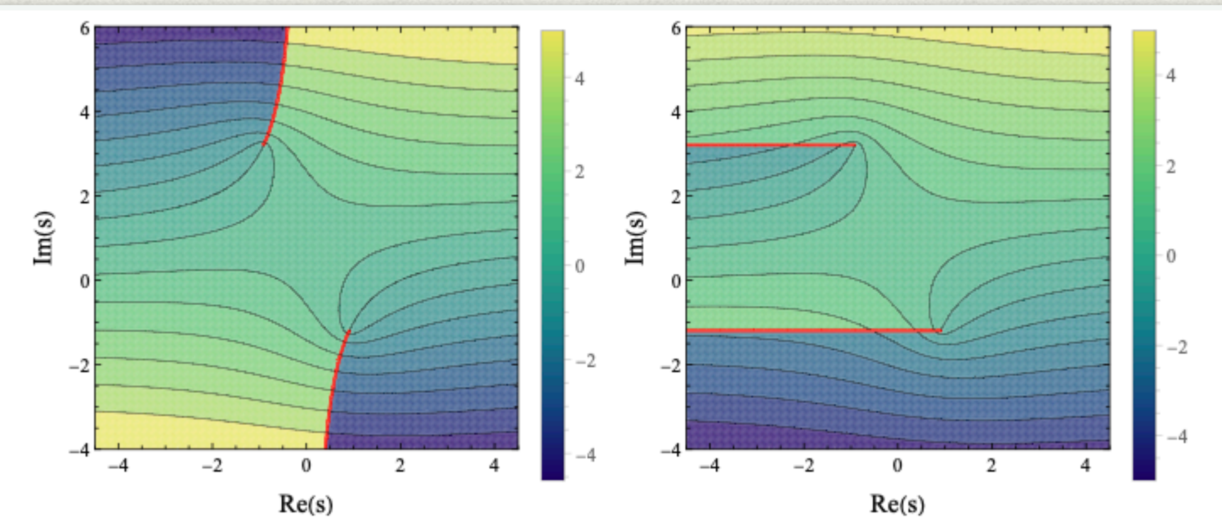
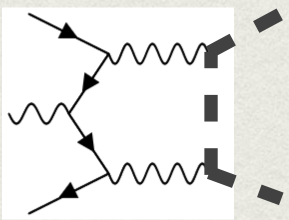



**Figure 1**: Value of the imaginary part of $\sqrt{\lambda_K(s,t,m)}$ written as $r_i = \sqrt{K_i^n(\vec{x})}$ (left) and $r_i = \Pi_{j=1}^{n} \sqrt{x - e_j}$ (right) in the complex $s$ plane. The branch cuts are depicted with light red lines.

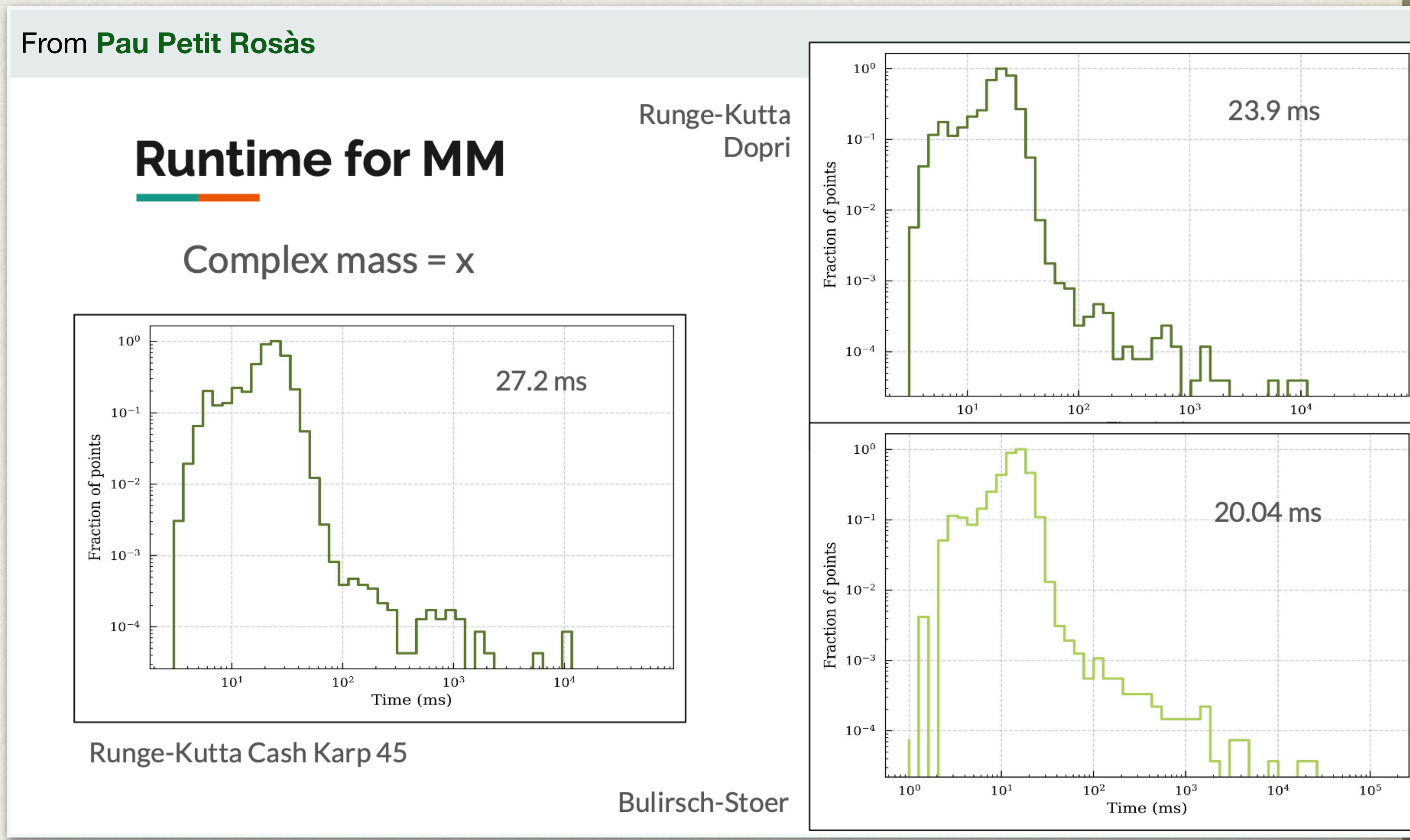
# *Numerical evaluation for $e^+e^- \to \pi^+\pi^-\gamma$ @ 1L*

Evaluation of integral families with complex kinematics

First contributions at NNLO



$$+\text{boxes} + \text{crossed diagrams} = \frac{c_{-1}}{\epsilon} + c_0 + c_1\epsilon + c_2\epsilon^2$$



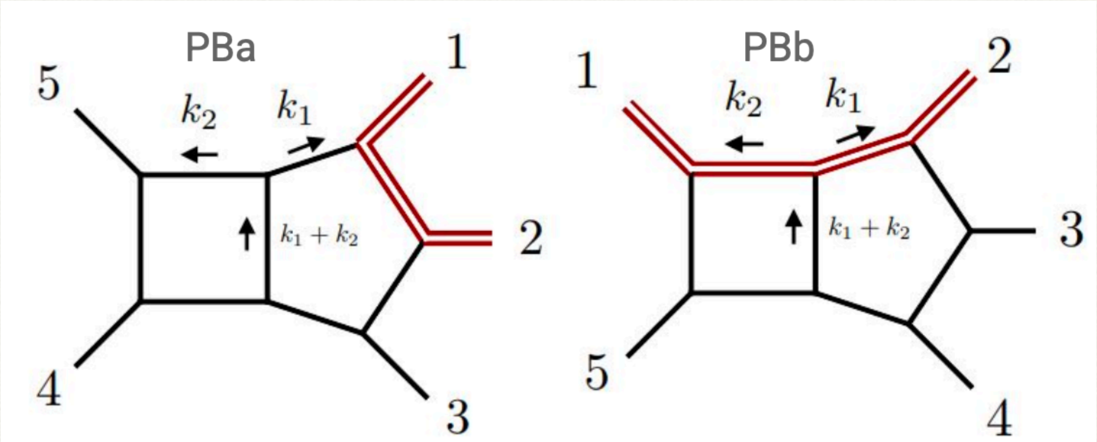

Successful comparison against:

- ☑ Collier
- ☑ AMFLow
- ☑ Dimension-changing transformation

# *Towards* $e^+e^- \to \pi\pi\gamma$ @ 2L

- Two-loop integrals present in $pp \to t\bar{t}j$

  PB$_A$ :: 88 MIs :: canonical DEQ
  PB$_B$ :: 121 MIs :: No canonical form




[Badger, Becchetti, Giraudo, Zoia 2024]

- Fast numerical evaluation in double and in quad precision

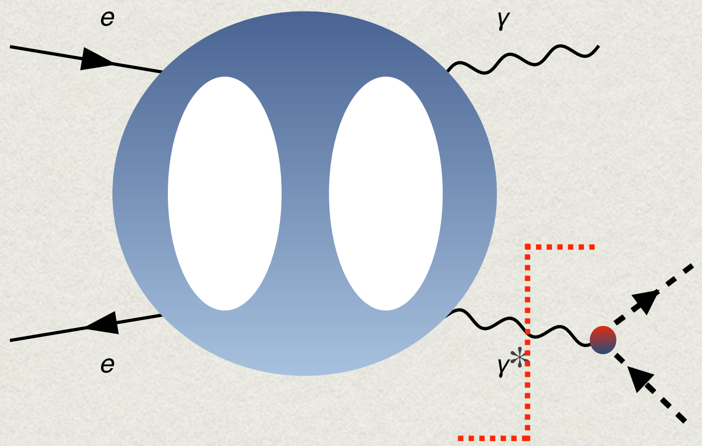
| | | $\mathcal{T}_A$, $\mathcal{T}_R$ | $\mathcal{R}$ | $\langle\tau\rangle$ [s] | DiffExp $\langle\tau\rangle$ [s] |
|---|---|---|---|---|---|
| PB$_A$ | double | $10^{-12}$, $10^{-12}$ | 10 | 0.0881 | 580.85 |
| | quad | $10^{-28}$, $10^{-28}$ | 27 | 51.588 | 795.516 |
| PB$_B$ | double | $10^{-12}$, $10^{-12}$ | 10 | 0.100 | 555.438 |
| | quad | $10^{-28}$, $10^{-28}$ | 27 | 89.088 | 826.219 |

# Progress on $e^+e^- \to \gamma\gamma^*$ at two loops
# — "initial state radiation"

# *Two-loop Scattering Amplitude for $e^+e^- \to \pi\pi\gamma$*

[Work in progress]

- Let's assume that the energetic photon is emitted only from electron lines




$$s = (k_1 + k_2)^2\,, \quad t = (k_2 - p_3)^2\,, \quad u = (k_1 - p_3)^2\,,$$
$$s + t + u = 2m_e^2 + q^2$$

4-point scattering process depending on 4 scales

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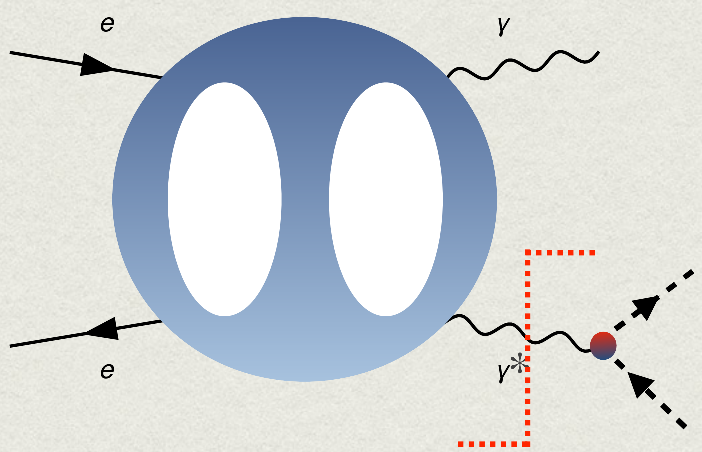


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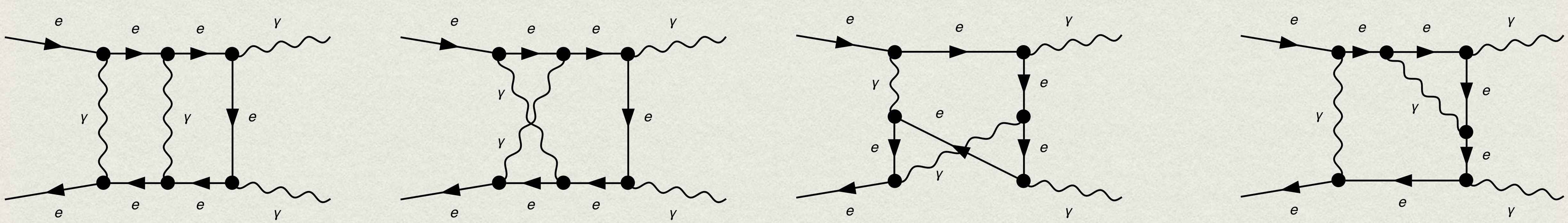


- Diagrammatic approach to construct Feynman diagrams & integrands :: **done!**
- Integration-by-parts identities :: **generated w/ `Kira 3.0`**
- Reduction/reconstructions :: in the making

# *Two-loop Feynman integrals for QED processes*

- Insights from “massive” calculations of Feynman integrals

  - $e^+e^- \to e^+e^-$ with $m_e^2 \neq 0$ [Henn, Smirnov (2013), Duhr et al (2021, 2023) ]
  - $e^+e^- \to \mu^+\mu^-$ with $m_e^2 \neq m_\mu^2 \neq 0$ [Heller (2021)]

- Recall our practical approach

  - Use canonical DEQs as much as possible
  - Find integrals that obeys partial DEQs with the form

$$\frac{\partial \vec{J}}{\partial x} = \sum_{k=0}^{2} \epsilon^k A_k \vec{J}$$

(move ‘difficult’ integrals to very late stages)

  - Make use of our C++ implementation

# *Two-loop Feynman integrals for QED processes*

- Easy example w/ 8 MIs

Only one elliptic sector w/ 2 MIs

Differential equation

$$d\vec{J} = \left( \epsilon \sum_{i=1}^{3} \mathbb{A}_i \, d\log\alpha_i + \sum_{i=0}^{2} \epsilon^i \, d\mathbb{B}_i \right) \vec{J} \,, \qquad \text{w/ } \vec{\alpha} = \{ m^2 \,, s \,, m^2 - s \,, 9m^2 - s \} \,,$$

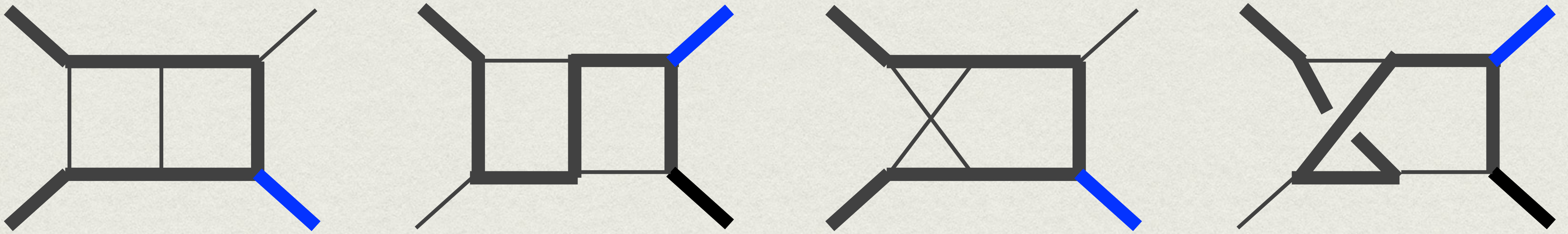
$\mathbb{B}$ only couples elliptic integrals and in terms of logarithmic forms

and the one-form: $W_1 = \dfrac{m^2}{s}\left(1 - \dfrac{3}{2}\dfrac{m^2}{s}\right) .$

☑ Fully suitable for our C++ integrator!

# *Two-loop Feynman integrals for $e^+e^- \to \pi\pi\gamma$*

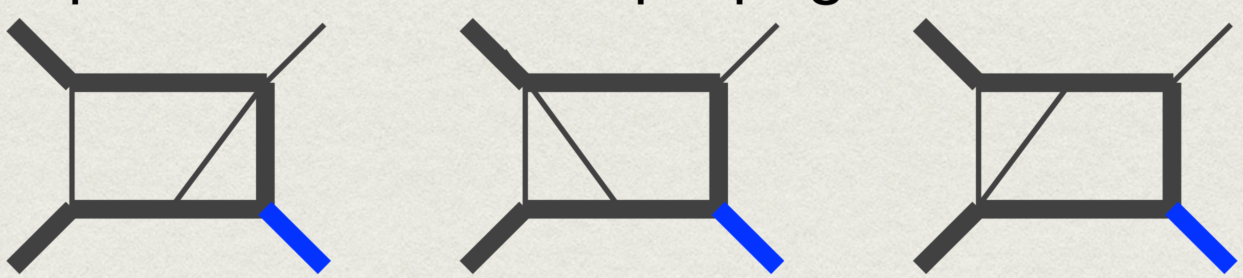
- Main focus on ISC @ 2L $(f^+f^- \to \gamma\,\gamma^*)$



$+\ldots$

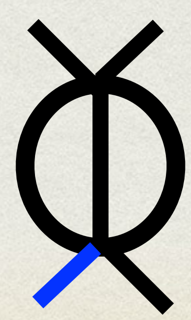
PL1 :: 68 MIs　　　　PL2 :: 74 MIs

- PL1
  * 50 MIs obey canonical DEQ w/ 3 square roots
  * Elliptic sectors with 6 propagators



[Pozzoli, WJT (to appear)]

- PL2
  * 23 MIs obey canonical DEQ w/ 3 square roots
  * Simple elliptic sectors (e.g., equal-mass sunrise diagrams)



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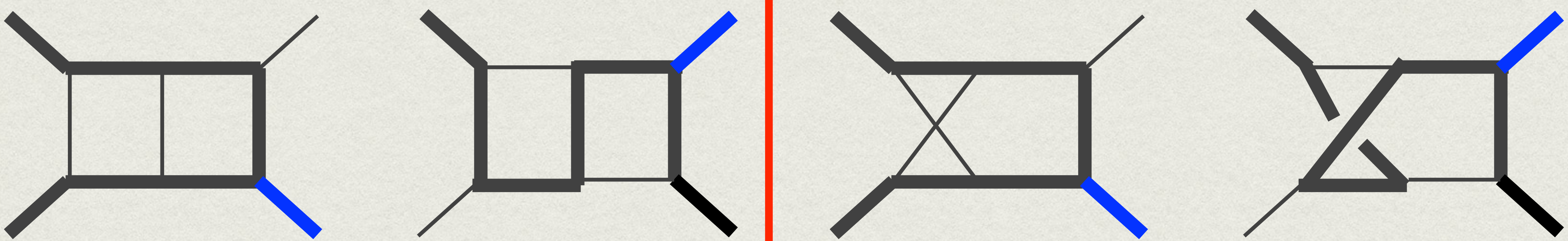

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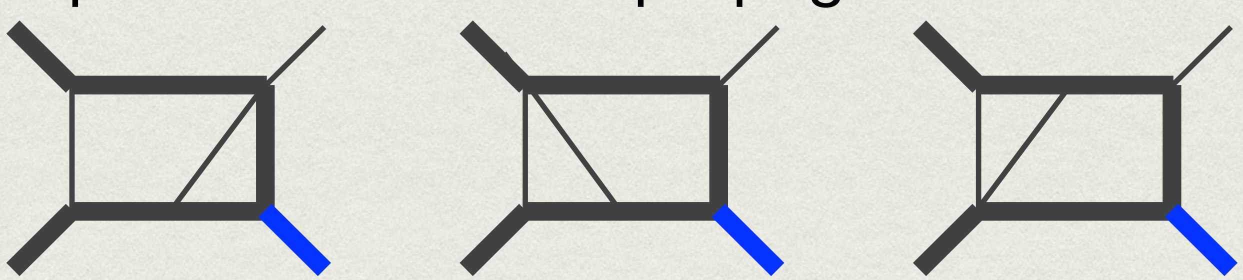

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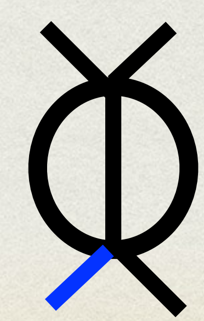

# *Conclusions*

- We have reached:

  - ☑ First improvements and validation of the `Phokhara` generator
  - ☑ Unravel crucial differences between MC generators (Strong2020)
  - ☑ First look at the evaluation of two-loop Feynman integrals for $e^+e^- \to \gamma\gamma^*$

- Open questions & future directions

  - ☐ Release GVMD within Phokhara
  - ☐ Get DEQs for non-planar integrals of $e^+e^- \to \gamma\gamma^*$
  - ☐ Include emission of energetic photons from hadrons ("final-state radiation")

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*Thanks*

# Backup

# *Feynman integrals in terms of graded functions*

$$d\vec{J} = \epsilon \, d\tilde{A} \, \vec{J}$$

$J_i^{(k)}$ are redundant order-by-order in $\epsilon$

[Henn (2013)]

$$\vec{W} = \mathbb{R} \, \vec{J}$$

$W_{i_k}$ are independent integrals order-by-order in $\epsilon$

[Chicherin, Sotnikov, Zoia (2021)]

[Gehrmann, Henn, Jakubčík, Lim, Mella, Syrrakos, Tancredi, WJT (2024)]

$$W_{i_k}\left(\vec{x};\epsilon\right) = \sum_{k'=k}^{4} \epsilon^{k'} \, w_{i_k}^{(k')}\left(\vec{x}\right)$$

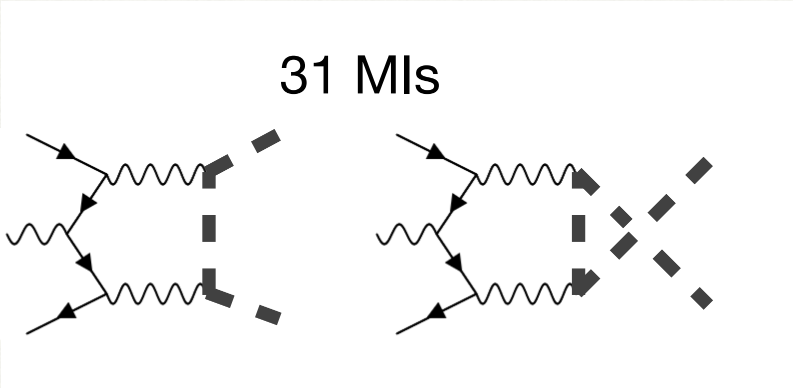
$w_{i_k}^{(k')}$ are independent transcendental functions

$$dw_{i_k}^{(k')}\left(\vec{x}\right) = \sum_{l=0}^{4} d\Omega_{i_k\, i_l} \, w_{i_l}^{(k'-1)}$$

Differential equations independent of $\epsilon$

[Henn, Caron-Huot (2014)]

# *Gauge invariant combination of pentagon w/ boxes*




Presence of 13 square roots

$$d\vec{J} = \epsilon\, d\tilde{A}\, \vec{J}$$

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[Henn (2013)]

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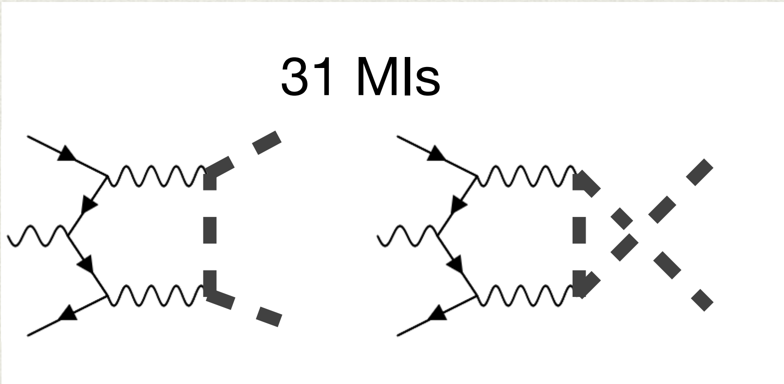


Presence of 13 square roots

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$J_i^{(k)}$ are redundant

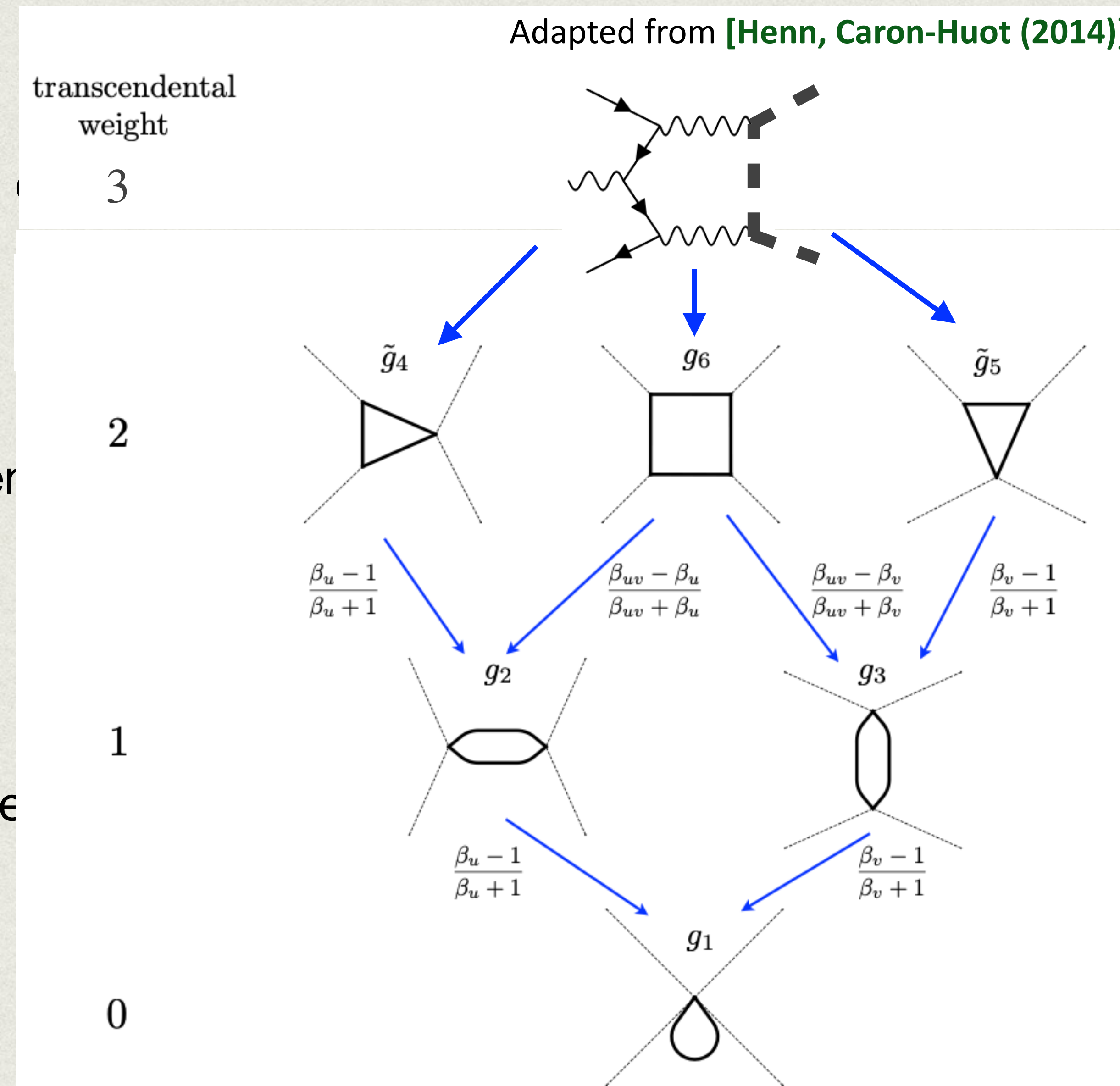
$$\overrightarrow{W} = \mathbb{R}\, \vec{J}$$

$W_{i_k}$ are independen

$$W_{i_k}\left(\vec{x};\epsilon\right) = \sum_{k'=k}^{4} \epsilon^{k'}\, w_{i_k}^{(k')}\left(\vec{x}\right)$$

$w_{i_k}^{(k')}$ are

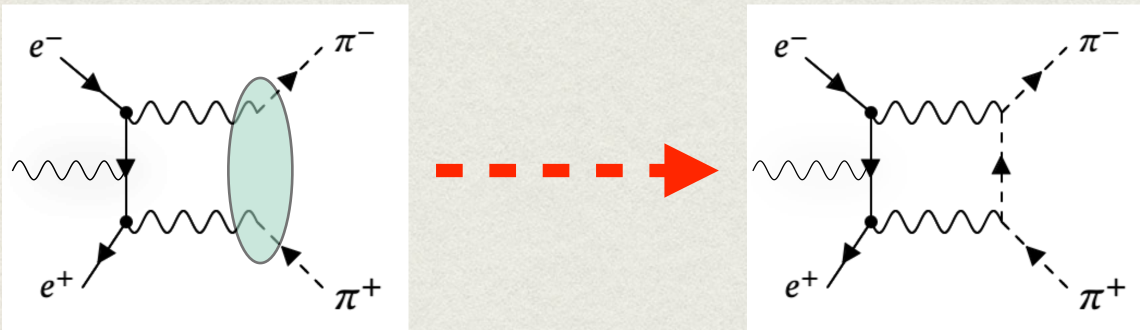
$$dw_{i_k}^{(k')}\left(\vec{x}\right) = \sum_{l=0}^{4} d\Omega_{i_k i_l}\, w_{i_l}^{(k'-1)}$$




Differential equations independent of $\epsilon$

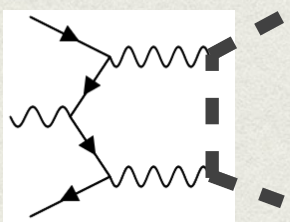
[Henn, Caron-Huot (2014)]

# *Radiative return processes @ NNLO*




$$\times \left[F(M_{\pi\pi})+F(q_1^2)F(q_2^2) - F(M_{\pi\pi})\right]$$

- $F \times$ sQED



$$+\text{boxes} + \text{crossed diagrams} = \frac{c_{-1}}{\epsilon} + c_0 + c_1\epsilon + \mathcal{O}(\epsilon^2)$$

$$c_{-1} = \left|A_{\text{ISC}}^{(0)}\right|^2 \left[-\frac{2w_1^{(1)}\left(m_e^2+m_\pi^2-s_{15}\right)}{r_8} - \frac{2w_2^{(1)}\left(m_e^2+m_\pi^2-s_{23}\right)}{r_9} + (3 \leftrightarrow 5)\right]$$

$$c_{0|1} = \sum_{ij} r_{ij}\, w_i^{(\text{j})}$$

31 functions present in $c_0$

54 functions present in $c_1$

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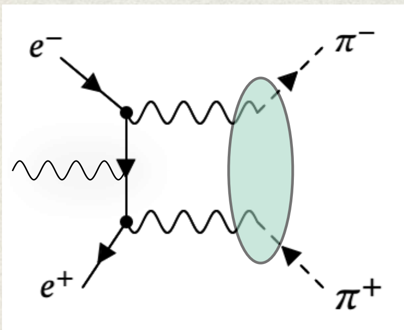
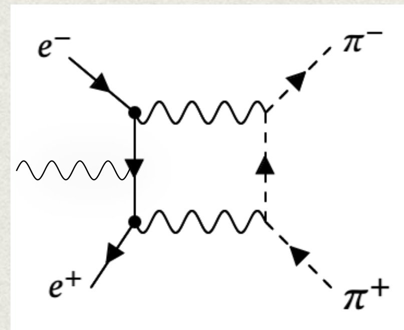
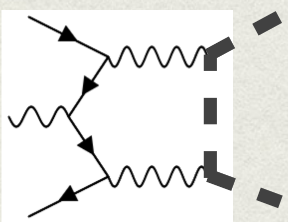



$$\times \left[F(M_{\pi\pi}) + F(q_1^2)F(q_2^2) - F(M_{\pi\pi})\right]$$

- $F \times$ sQED



$$+\text{boxes} + \text{crossed diagrams} = \frac{c_{-1}}{\epsilon} + c_0 + c_1\epsilon + \mathcal{O}(\epsilon^2)$$

Log's :: exactly match IR pole prediction

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