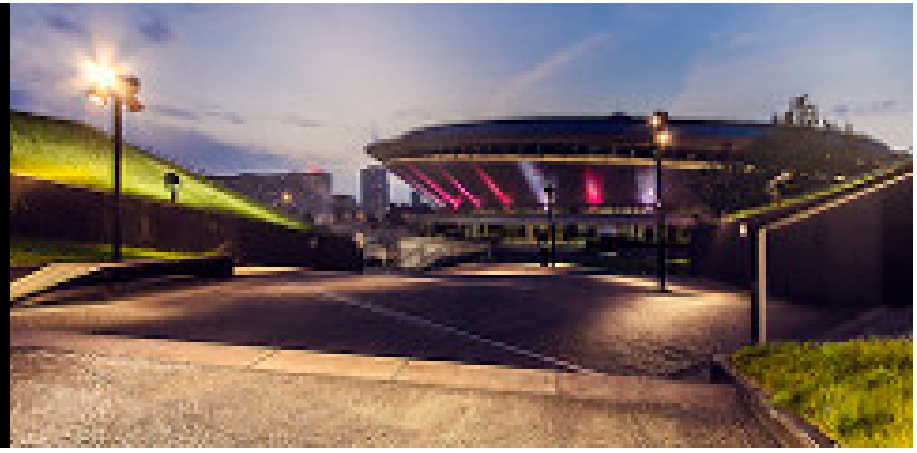

NNLOCAL: completely local subtractions for color-singlet production in hadron collisions

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based on V. Del Duca, C. Duhr, L. Fekésházy, F. Guadagni, P. Mukherjee, GS, F. Tramontano and S. Van Thurenhout, JHEP 05 (2025) 151, [arXiv:2412.21028 [hep-ph]]

18 September 2025, Katowice

Precision at colliders

The Standard Model (SM) of particle physics provides a very successful description of elementary particles and their interactions.

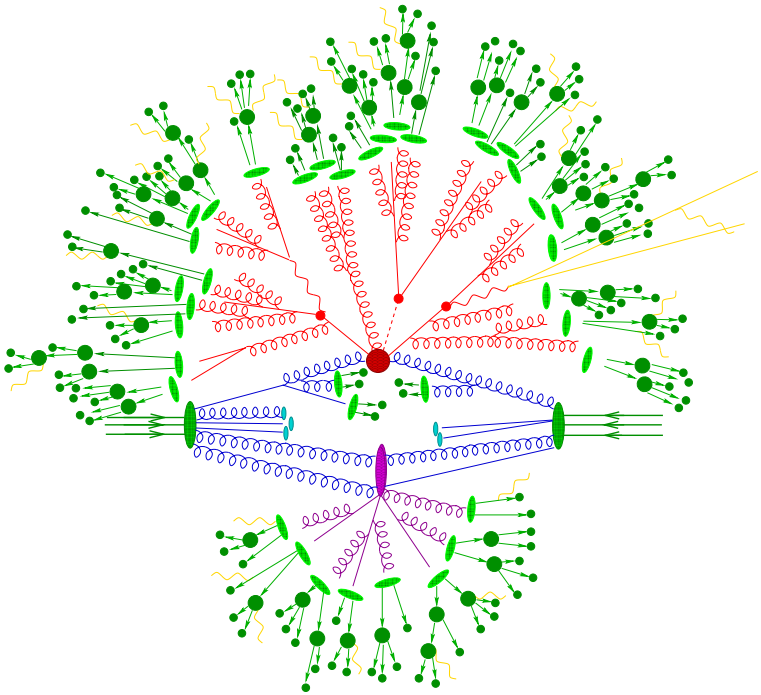
- Data collected at the LHC gives us a spectacular confirmation of the SM!
- Despite its successes, we know that the SM cannot be the ultimate theory of fundamental interactions (dark matter, baryogenesis, neutrino masses,...).
- However, no direct signals of physics beyond the SM have been seen at the LHC so far...
- Thus, high precision experiments (i.e., the high-luminosity phase of LHC) become crucial to test the validity of the SM.
- From a theoretical point of view, in order to fully exploit the physics potential of LHC, QCD (and EW) must be understood and modelled as best as possible.

First indications of New Physics could well be indirect: precision is key!

Precision at colliders

A hadron-hadron collision is a complicated affair: many aspects of precision

- Incoming protons are beams of partons: parton distribution functions (PDFs)
- Primary hard scattering: **fixed-order perturbation theory**, resummation
- Partonic evolution: parton showers
- Hadronization and heavy hadron decay
- Multiple parton interactions and underlying event

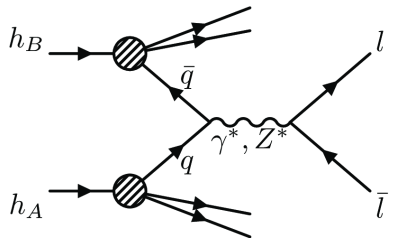


[A Sherpa artist]

A plethora of directions, will focus on higher orders in perturbation theory.

Cross sections for hadron collisions

Hadronic cross sections can be computed as convolutions of parton distribution functions (PDFs) with hard scattering partonic cross sections: **collinear factorization** theorem.



$$\sigma_{AB} = \sum_{a,b} \int \mathrm{d}x_a \int \mathrm{d}x_b \underbrace{f_{a/A}(x_a,\mu_F) f_{b/B}(x_b,\mu_F)}_{\text{PDF's}} \underbrace{\sigma_{ab}(x_a,x_b,\mu_F)}_{\text{partonic x-sect}} \left[1 + \mathcal{O}\left(\frac{\Lambda^n_{\text{QCD}}}{Q^n}\right)\right]$$

- $f_{a/A}(x_a,\mu_F)$, $f_{b/B}(x_b,\mu_F)$: non-perturbative PDFs, to be measured in experiment or computed on the lattice
- $d\sigma_{ab}(x_a,x_b,\mu_F)$: hard partonic cross section, can be computed in perturbation theory

$$\sigma_{ab}(x_a,x_b,\mu_F) = \alpha_s^\ell \sum_{k=0}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^k \sigma_{ab}^{\mathrm{N^kLO}}(x_a,x_b,\mu_R,\mu_F)$$

More accurate predictions $\equiv$ More terms in the perturbative series

QCD at NNLO

Beyond leading order, we must account for extra radiation, both real and virtual.

- Requires the evaluation of Feynman diagrams with extra legs and/or loops with respect to the Born process.
- Must deal with the problem of infrared (IR) singularities that arise at intermediate stages. The NNLO correction for a process with m jets at the Born level is a sum of three terms

$$\sigma_{ab}^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma_{ab}^{\mathrm{RR}} J_{m+2} + \int_{m+1} \left(\mathrm{d}\sigma_{ab}^{\mathrm{RV}} + \mathrm{d}\sigma_{ab}^{\mathrm{C}_1}\right) J_{m+1} + \int_{m} \left(\mathrm{d}\sigma_{ab}^{\mathrm{VV}} + \mathrm{d}\sigma_{ab}^{\mathrm{C}_2}\right) J_{m}$$

- While the sum of these contributions is finite for IR- and collinear-safe observables J, the three are separately divergent in limits in which one or more partons become unresolved.
- The sources of singularities are i) divergent phase space integrals due to unresolved real emission and ii) explicit IR ϵ-poles in loop integrals.
- We must regularize the divergences (dim. reg. in $d = 4 - 2\epsilon$ dimensions) and render each contribution finite in $d = 4$ before any numerical evaluation can take place.

The issue of IR divergences can be solved by local subtraction.

CoLoRFulNNLO

The basic idea of the subtraction method is to use appropriately chosen **subtraction terms** to reshuffle divergences between each contribution of the full higher-order correction in such a way that each contribution is finite after the reshuffling.

CoLoRFulNNLO is a particular realization of this idea. It is a local, analytic subtraction scheme for NNLO calculations. Aims:

- Exact perturbative result without slicing parameters
  (reduced source of numerical uncertainty)
- Point-by-point subtraction in phase space including spin and color correlations
  (no integrals that are finite in d dims., but undefined in 4 dims.)
- Analytic computation of integrated subtraction terms
  (rigorously show cancellation of virtual poles)
- Explicit and general expressions
  (coding, automation)

General subtraction procedure

In CoLoRFulNNLO, the complete NNLO cross section is written as

$$\sigma_{ab}^{\mathrm{NNLO}} = \int_{m+2} \left[\mathrm{d}\sigma_{ab}^{\mathrm{RR}} J_{m+2} - \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_1}} J_{m+1} - \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_2}} J_m + \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_{12}}} J_m \right]$$
$$+ \int_{m+1} \left\{ \left[\mathrm{d}\sigma_{ab}^{\mathrm{RV}} + \mathrm{d}\sigma_{ab}^{\mathrm{C_1}} + \int_1 \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_1}} \right] J_{m+1} - \left[\mathrm{d}\sigma_{ab}^{\mathrm{RV,A_1}} + \mathrm{d}\sigma_{ab}^{\mathrm{C_1,A_1}} + \left(\int_1 \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_1}} \right)^{\mathrm{A_1}} \right] J_m \right\}$$
$$+ \int_m \left\{ \mathrm{d}\sigma_{ab}^{\mathrm{VV}} + \mathrm{d}\sigma_{ab}^{\mathrm{C_2}} + \int_2 \left[\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_2}} - \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_{12}}} \right] + \int_1 \left[\mathrm{d}\sigma_{ab}^{\mathrm{RV,A_1}} + \mathrm{d}\sigma_{ab}^{\mathrm{C_1,A_1}} \right] + \int_1 \left(\int_1 \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_1}} \right)^{\mathrm{A_1}} \right\} J_m$$

- The phase space integral on each line is finite in $d = 4$ dimensions due to the presence of subtraction terms that remove non-integrable singularities.
- The integrated forms of subtraction terms cancel the explicit IR ϵ-poles in loop corrections, so each line is also free from ϵ-poles.
- Hence, each line can be computed using standard numerical methods.

General subtraction procedure

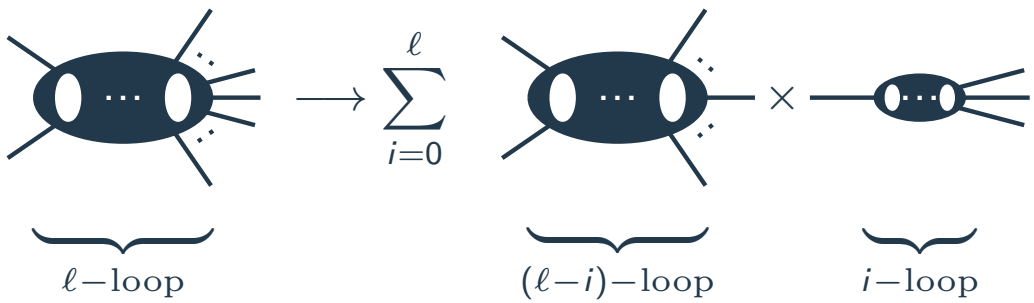
Role of individual terms

$$\left\{ \begin{array}{c} \mathrm{d}\sigma_{ab}^{\mathrm{RR},\mathrm{A}_1} \\ \mathrm{d}\sigma_{ab}^{\mathrm{RR},\mathrm{A}_2} \\ \mathrm{d}\sigma_{ab}^{\mathrm{RR},\mathrm{A}_{12}} \\ \mathrm{d}\sigma_{ab}^{\mathrm{RV},\mathrm{A}_1} \\ \mathrm{d}\sigma_{ab}^{\mathrm{C}_1,\mathrm{A}_1} \\ \left(\int_1 \mathrm{d}\sigma_{ab}^{\mathrm{RR},\mathrm{A}_1}\right)^{\mathrm{A}_1} \end{array} \right\}$$ regularizes $$\left\{ \begin{array}{c} \mathrm{d}\sigma_{ab}^{\mathrm{RR}} \\ \mathrm{d}\sigma_{ab}^{\mathrm{RR}} \\ \mathrm{d}\sigma_{ab}^{\mathrm{RR},\mathrm{A}_2}\,,\mathrm{d}\sigma_{ab}^{\mathrm{RR},\mathrm{A}_1} \\ \mathrm{d}\sigma_{ab}^{\mathrm{RV}} \\ \mathrm{d}\sigma_{ab}^{\mathrm{C}_1} \\ \int_1 \mathrm{d}\sigma_{ab}^{\mathrm{RR},\mathrm{A}_1} \end{array} \right\}$$ as $$\left\{ \begin{array}{c} 1 \\ 2 \\ 1\,,2 \\ 1 \\ 1 \\ 1 \end{array} \right\}$$ partons are unresolved

Construction of the subtraction terms

In the CoLoRFulNNLO scheme, subtraction terms are built from IR factorization formulae which capture the behaviour of QCD squared matrix elements in IR limits.

$$\boldsymbol{U}_j |\mathcal{M}_{ab,m+j}(\{p\}_{m+j})|^2_{\ell-\text{loop}} = \left(\frac{\alpha_s}{2\pi}\right)^j \sum_{i=0}^{\ell} \text{Sing}_j^{(i)} \times \underbrace{|\mathcal{M}_{\hat{a}\hat{b},m}(\{\hat{p}\}_m)|^2_{(\ell-i)-\text{loop}}}_{j \text{ partons removed}}$$



- $\boldsymbol{U}_j$: formal operator that takes some j-fold unresolved limit
- $\text{Sing}_j^{(i)}$: universal IR singular structure for i-loop, j-fold unresolved emission
- $|\mathcal{M}_{\hat{a}\hat{b},m}(\{\hat{p}\}_m)|^2_{(\ell-i)-\text{loop}}$: $(\ell - i)$-loop reduced matrix element with j partons removed

However, IR factorization formulae cannot directly be used as subtraction terms.

From limits to subtraction terms

Issues

1. Unresolved regions in phase space overlap $\Rightarrow$ care must be taken to **avoid multiple subtraction** in overlapping regions.
2. IR limit formulae are only well-defined in the strict limits $\Rightarrow$ **definitions must be carefully extended** over the full phase space away from the limits.
3. Subtraction terms must be **integrated** over the momenta of unresolved radiation $\Rightarrow$ many complicated phase space integrals.

From limits to subtraction terms

1. Overlapping singularities can be addressed by the **inclusion-exclusion principle**: subtract each limit once, add back pairwise overlaps, subtract triple overlaps and so on. E.g., at NLO

$$\mathbf{A}_1 = \sum_{ir} \mathbf{C}_{ir} + \sum_{r} \mathbf{S}_r - \sum_{\{ir,r\}} \mathbf{C}_{ir} \cap \mathbf{S}_r$$

2. In order to obtain true subtraction terms, two additional steps are needed

   - **Make** $|\mathcal{M}_{\hat{a}\hat{b},m}(\{\hat{p}\}_m)|^2_{(l-i)-\text{loop}}$ **well-defined** $\Rightarrow$ specify precisely the momenta entering the reduced matrix elements via momentum mappings that implement momentum conservation and mass-shell conditions

$$\{p\}_{m+j} \to \{\tilde{p}\}_m\,, \qquad j = 1,2$$

   - **Make** $\mathbf{Sing}^{(i)}_j$ **well-defined** $\Rightarrow$ the various quantities entering the singular structures such as momentum fractions, transverse momenta and eikonal factors need to be precisely specified.

   The adopted definitions must be such that they respect the structure of cancellations in all overlapping limits, which is a constraint for the entire construction.

From limits to subtraction terms

After these definitions are fixed, the **IR limit formula can be promoted** to a (sum of) **true subtraction term(s)** that is unambiguously defined at any point in phase space,

$$\boldsymbol{U}_j |\mathcal{M}_{ab,m+j}(\{p\}_{m+j})|^2_{l-\text{loop}} \to \sum_{i=0}^{l} \mathcal{U}_j^{(i,l-i)}$$

where

$$\mathcal{U}_j^{(i,l-i)} = \left(\frac{\alpha_s}{2\pi}\right)^j \widetilde{\text{Sing}}_j^{(i)} \times |\mathcal{M}_{\tilde{a}\tilde{b},m}(\{\tilde{p}\}_m)|^2_{(l-i)-\text{loop}}$$

- $\widetilde{\text{Sing}}_j^{(i)}$ represents the expression of the corresponding singular structure incorporating the precise definitions of momentum fractions, eikonal factors and so on.
- The matrix element is evaluated over the set of mapped momenta, $\{\tilde{p}\}_m$.

The subtractions are fully local in phase space and include all color and spin correlations. Cancellations can be tested point-by-point in phase space as any IR limit is approached.

CoLoRFulNNLO for color-singlet production: A_1

For color-singlet production in hadron-hadron collisions, $h_A + h_B \to X$, we have

$$\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_1}} = \mathrm{d}\phi_{X+2}(\{p\}_{X+2})\mathcal{A}_1^{(0)}$$

where

$$\mathcal{A}_1^{(0)} = \sum_{r\in F}\left[\mathcal{S}_r^{(0,0)} + \sum_{\substack{i\in F\\ i\neq r}}\left(\frac{1}{2}\mathcal{C}_{ir}^{FF(0,0)} - \mathcal{C}_{ir}^{FF}\mathcal{S}_r^{(0,0)}\right) + \sum_{c\in I}\left(\mathcal{C}_{cr}^{IF(0,0)} - \mathcal{C}_{cr}^{IF}\mathcal{S}_r^{(0,0)}\right)\right]$$

- I and F denote the sets of initial-state and final-state partons.
- The various subtraction terms correspond to the limit implied by the notation.
- The $(0,0)$ superscript signals that these terms originate from IR limit formulae that involve tree-level singular structures multiplying tree-level reduced matrix elements.
- Each term is explicitly defined and can be evaluated in any point of the double real emission phase space.

CoLoRFulNNLO for color-singlet production: A_2

For color-singlet production in hadron-hadron collisions, $h_A + h_B \to X$, we have

$$\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_2}} = \mathrm{d}\phi_{X+2}(\{p\}_{X+2})\mathcal{A}_2^{(0)}$$

where

$$\mathcal{A}_2^{(0)} = \frac{1}{2}\sum_{r\in F}\sum_{\substack{s\in F\\ s\neq r}}\left\{\mathcal{S}_{rs}^{(0,0)} + \sum_{c\in I}\left[\mathcal{C}_{crs}^{IFF(0,0)} - \mathcal{C}_{crs}^{IFF}\mathcal{S}_{rs}^{(0,0)} + \sum_{\substack{d\in I\\ d\neq c}}\left(\mathcal{C}_{cr,ds}^{IF,IF(0,0)} + \mathcal{C}_{cr,ds}^{IF,IF}\mathcal{S}_{rs}^{(0,0)}\right)\right]\right\}$$

- Again, various subtraction terms correspond to the limit implied by the notation.
- Notice no triple collinear-double collinear overlap: general feature at NNLO.
- Notice no soft-collinear terms: these cancel for color-singlet production due to the precise definitions we adopt.
- Each term is explicitly defined and can be evaluated in any point of the double real emission phase space.

CoLoRFulNNLO for color-singlet production: A_{12}

For color-singlet production, we have $\mathrm{d}\sigma_{ab}^{\mathrm{RR,A_{12}}} = \mathrm{d}\phi_{X+2}(\{p\}_{X+2})\mathcal{A}_{12}^{(0)}$ where

$$\mathcal{A}_{12}^{(0)} = \sum_{s\in F}\left[\mathcal{A}_2^{(0)}\,\mathcal{S}_s + \sum_{\substack{r\in F\\ r\neq s}}\left(\frac{1}{2}\mathcal{A}_2^{(0)}\,\mathcal{C}_{rs}^{FF} - \mathcal{A}_2^{(0)}\,\mathcal{C}_{rs}^{FF}\mathcal{S}_s\right) + \sum_{c\in I}\left(\mathcal{A}_2^{(0)}\,\mathcal{C}_{cs}^{IF} - \mathcal{A}_2^{(0)}\,\mathcal{C}_{cs}^{IF}\mathcal{S}_s\right)\right]$$

with

$$\mathcal{A}_2^{(0)}\,\mathcal{S}_s = \sum_{\substack{r\in F\\ r\neq s}}\left[\mathcal{S}_{rs}^{(0,0)}\,\mathcal{S}_s + \sum_{c\in I}\left(\mathcal{C}_{crs}^{IFF(0,0)}\,\mathcal{S}_s - \mathcal{C}_{crs}^{IFF}\,\mathcal{S}_{rs}^{(0,0)}\,\mathcal{S}_s\right)\right]$$

$$\mathcal{A}_2^{(0)}\,\mathcal{C}_{rs}^{FF} = \mathcal{S}_{rs}^{(0,0)}\,\mathcal{C}_{rs}^{FF} + \sum_{c\in I}\left(\mathcal{C}_{crs}^{IFF(0,0)}\,\mathcal{C}_{rs}^{FF} - \mathcal{C}_{crs}^{IFF}\,\mathcal{S}_{rs}^{(0,0)}\,\mathcal{C}_{rs}^{FF}\right)$$

$$\mathcal{A}_2^{(0)}\,\mathcal{C}_{rs}^{FF}\mathcal{S}_s = \sum_{c\in I}\mathcal{C}_{crs}^{IFF(0,0)}\,\mathcal{C}_{rs}^{FF}\mathcal{S}_s$$

$$\mathcal{A}_2^{(0)}\,\mathcal{C}_{cs}^{IF} = \sum_{\substack{r\in F\\ r\neq s}}\left(\mathcal{C}_{csr}^{IFF(0,0)}\,\mathcal{C}_{cs}^{IF} + \sum_{\substack{d\in I\\ d\neq c}}\mathcal{C}_{cs,dr}^{IF,IF(0,0)}\,\mathcal{C}_{cs}^{IF}\right)$$

$$\mathcal{A}_2^{(0)}\,\mathcal{C}_{cs}^{IF}\mathcal{S}_s = \sum_{\substack{r\in F\\ r\neq s}}\left(\mathcal{S}_{rs}^{(0,0)}\,\mathcal{C}_{cs}^{IF}\mathcal{S}_s + \mathcal{C}_{csr}^{IFF(0,0)}\,\mathcal{C}_{cs}^{IF}\mathcal{S}_s - \mathcal{C}_{csr}^{IFF}\,\mathcal{S}_{rs}^{(0,0)}\,\mathcal{C}_{cs}^{IF}\mathcal{S}_s\right)$$

Integrating the subtraction terms

In order to finish the definition of the scheme, all **subtraction terms must be integrated** over the momenta of unresolved emission(s).

Integrated subtraction terms are not simple functions of kinematics but rather **distributions** in the momentum fractions of incoming momenta.

The required phase space integrals were computed with a **combination of methods**

- IBP reduction + DEs: $\int_2 \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_2}}$
- Direct integration of parametric representations: $\int_2 \mathrm{d}\sigma_{ab}^{\mathrm{RR,A_{12}}}$
- MB methods: elements of the distributional expansions

Results are **fully analytic**

- All integrated subtraction terms can be expressed in terms of GPLs
- Algebraic letters are present (square roots) in finite parts
- Can check cancellation of double virtual poles analytically

The right tool for the job

Choosing the **correct method can make a massive difference**!

- Among many others, we must evaluate the parametric integral

$$
\begin{aligned}
I(\xi_a, \xi_b) = {} & 2^{-1-3\epsilon} \xi_a^{-1-3\epsilon} (1-\xi_a)^{-1-3\epsilon} (1+\xi_a)^{\epsilon} \\
& \times \int_0^1 dx_1 \int_0^1 dx_2 \, x_1^{-\epsilon} \, (1-x_1)^{-1-3\epsilon} x_2^{-1-2\epsilon} \, (1-x_2)^{-1+\epsilon} \, (2\xi_a + x_1 - x_1\xi_a)^{\epsilon} \\
& \times \left(2x_1x_2\xi_a^2 - 2x_2\xi_a^2 - x_1^2\xi_a + 2x_1\xi_a + x_1^2x_2\xi_a - 4x_1x_2\xi_a + 2x_2\xi_a + x_1^2 - x_1^2x_2\right)^{\epsilon}
\end{aligned}
$$

- Singular in ϵ with overlapping singularities in x_1 and x_2 (last factor)
- The last factor is quadratic in x_1
- Resolving the ϵ-poles by sector decomposition will make this factor quadratic also in x_2 $\Rightarrow$ direct integration is very painful, if at all possible

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$$I(\xi_a,\xi_b) = 2^{-1-3\epsilon}\xi_a^{-1-3\epsilon}(1-\xi_a)^{-1-3\epsilon}(1+\xi_a)^{\epsilon}$$
$$\times \int_0^1 dx_1 \int_0^1 dx_2\, x_1^{-\epsilon}\,(1-x_1)^{-1-3\epsilon} x_2^{-1-2\epsilon}\,(1-x_2)^{-1+\epsilon}\,(2\xi_a + x_1 - x_1\xi_a)^{\epsilon}$$
$$\times \left(2x_1x_2\xi_a^2 - 2x_2\xi_a^2 - x_1^2\xi_a + 2x_1\xi_a + x_1^2x_2\xi_a - 4x_1x_2\xi_a + 2x_2\xi_a + x_1^2 - x_1^2x_2\right)^{\epsilon}$$

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- The last factor is quadratic in x_1
- Resolving the ϵ-poles by sector decomposition will make this factor quadratic also in x_2 $\Rightarrow$ direct integration is very painful, if at all possible

The right tool for the job

However, the calculation is essentially trivial with MB methods!

- Can find an MB representation that is only 2-dimensional

$$I = (1+\xi_a)^{\epsilon} \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{dz_2}{2\pi i}\, 2^{-1-2\epsilon-z_1+z_2}\, \xi_a^{-1-2\epsilon-z_1+z_2} (1-\xi_a)^{-1-z_2-\epsilon-1}$$
$$\times \frac{\Gamma(-z_1)\,\Gamma(1+2z_1-z_2)\,\Gamma(-z_2)\,\Gamma(-2\epsilon-z_1)\,\Gamma(-\epsilon+z_1)\,\Gamma(-\epsilon-z_1+z_2)\,\Gamma(\epsilon+z_1)}{\Gamma(-\epsilon)\Gamma(1-2\epsilon+z_1-z_2)}$$

- Requires auxiliary regularization to find straight line contours (notice $\Gamma(-z_1)$, $\Gamma(-\epsilon)$ and $\Gamma(\epsilon+z_1)$ all appear in the integrand): e.g., introduce $(1-x_2)^{\delta}$ into the orignial integral
- The usual steps: straight line contours at $\epsilon \neq 0$ and $\delta \neq 0 \Rightarrow$ analytically continue to $\epsilon = 0 \Rightarrow$ resolve ϵ poles $\Rightarrow$ analytically continue to $\delta = 0$
- The result is a 0-dimensional MB integral $\Rightarrow$ we obtain the result without having to perform any integration at all! We have

$$I = 2^{-1-\epsilon} \xi_a^{-1-\epsilon} (1-\xi_a)^{-1-\epsilon} (1+\xi_a)^{\epsilon} \frac{\Gamma(1-2\epsilon)\Gamma(-2\epsilon)\Gamma(\epsilon)}{\Gamma(1-3\epsilon)}$$

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$$I = (1+\xi_a)^{\epsilon} \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{dz_2}{2\pi i} 2^{-1-2\epsilon-z_1+z_2} \xi_a^{-1-2\epsilon-z_1+z_2} (1-\xi_a)^{-1-z_2-\epsilon-1}$$
$$\times \frac{\Gamma(-z_1)\Gamma(1+2z_1-z_2)\Gamma(-z_2)\Gamma(-2\epsilon-z_1)\Gamma(-\epsilon+z_1)\Gamma(-\epsilon-z_1+z_2)\Gamma(\epsilon+z_1)}{\Gamma(-\epsilon)\Gamma(1-2\epsilon+z_1-z_2)}$$

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The NNLOCAL code

```
~/research/colorsinglet/runs_for_nnlocalgg/nnlocal/bin/testrun-H
› ../nnlocal
************************* version  beta  ************************
*                                                               *
* ##    ## ##    ## ##       ######   #####     ###    ##       *
* ###   ## ###   ## ##      ##     ## ##   ##   ## ##   ##       *
* ####  ## ####  ## ##      ##     ## ##        ##  ##  ##       *
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* ##    ## ##    ## ######## ######   #####  ##     ## ######## *
*                                                               *
************************ December 20th, 2024 ************************
*                                                               *
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*                                                               *
* https://github.com/nnlocal/nnlocal.git                        *
*                                                               *
*****************************************************************
* Using input file named input.DAT
```

The NNLOCAL code

NNLOCAL: a proof-of-concept **Monte Carlo program** implementing the described scheme

- Architecture based on MCFM-4.0 (note current version is 10.3) written in Fortran77. (Have ensured that this is tolerated by the authors!)
- Publicly available, only external dependency is LHAPDF.

https://github.com/nnlocal/nnlocal.git

Proof-of-concept code:

- Gluon fusion Higgs production in HEFT ($m_t \to \infty$) with no light quarks ($n_f = 0$): not a restriction on the structure of the subtraction scheme, on the contrary, all possible IR singularities for color-singlet production are present.
- Minimal optimisation only (especially in VV part): some issues with numerical cancellations in integrated subtraction terms handled by dynamical switching to quadruple precision.
- Many features to support checking/validation.

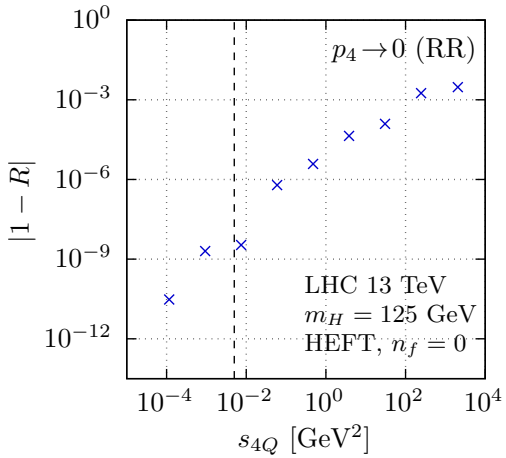
The NNLOCAL code

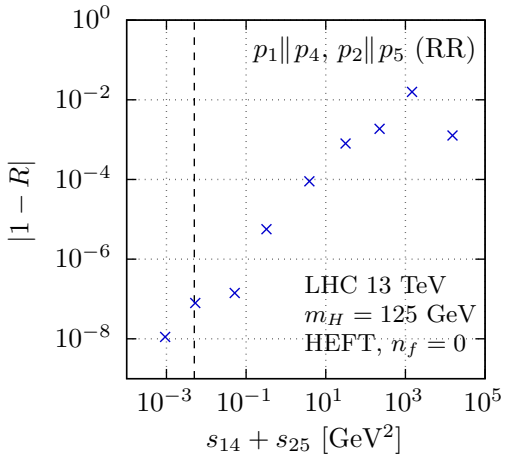
Main features

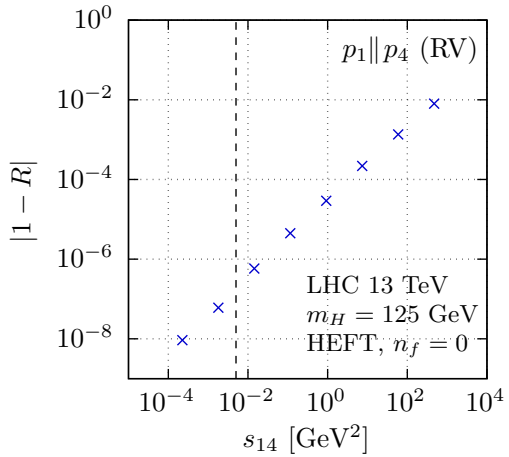
- Publicly available local analytic subtraction code at NNLO.
- Computation of any infrared and collinear-safe observables via user defined analysis routine.
- Support for parallel running if desired (managed by shell scripts).
- Support for efficient building and monitoring of Monte Carlo integration grids (via parallel running and scripts for visualisation).
- Support for checking cancellation of kinematical singularities through dedicated phase space routines.
- Support for checking cancellation of ε-poles order by order.

Code validation

Validate **cancellation** of kinematical singularities in double real and real-virtual emission ($R = \sum \mathcal{A}/|\mathcal{M}|^2$)







Check **cancellation** of ε-poles

```
na = 0.755605220795
nb = 0.458650112152
                                    1/eps^4              1/eps^3              1/eps^2               1/eps                eps^0
-------------------------------------------------------------------------------------------------------------------------------------
ICT  f(xa/na)*f(xb/nb) =       0.000000000000       0.000000000000      -0.000000000001      -0.000000001763       5835.169801514889
ICT  f(xa/na)*f(xb)    =       0.000000000000      -0.000000000000       0.000000000002      -0.000000000002      -3532.639553984512
ICT  f(xa)*f(xb/nb)    =       0.000000000000       0.000000000000       0.000000000000      -0.000000000001      -4741.126731759623
ICT  f(xa)*f(xb)       =     -18.000000000000     -90.750000000000     162.250675918787     440.936804453876       3724.953259726909
VV/B f(xa)*f(xb)       =      18.000000000000      90.750000000000    -162.250675918791    -440.936804453905       1358.713586569218
```

Code validation

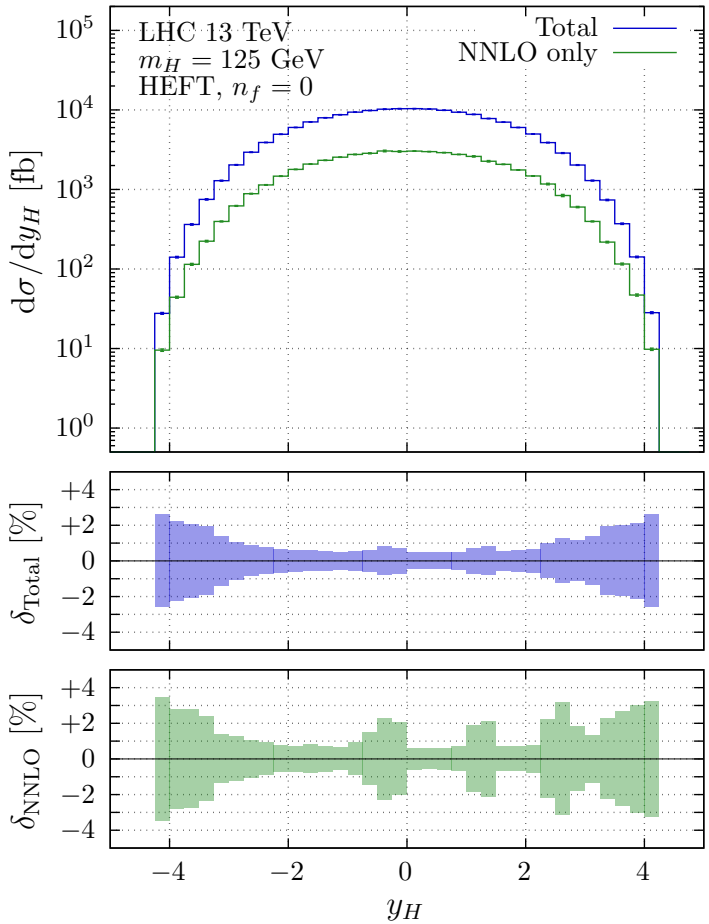
Tuned comparison of **inclusive cross section** to `n3loxs` [Baglio, Duhr, Mistlberger, Szafron 2022] (modified to exclude quark channels and use same α_s running as `NNLOCAL`)

m_H [GeV]	`n3loxs` (gg)	`NNLOCAL` (gg)
100	65.72 pb	65.74(4) pb
125	42.94 pb	42.94(2) pb
250	9.730 pb	9.733(5) pb
500	1.626 pb	1.626(1) pb
1000	173.7 fb	173.7(1) fb
2000	8.794 fb	8.790(5) fb

- Several choices of μ_R, μ_F checked, shown values are for $\mu_R = \mu_F = m_H$.
- Runtime per mass value: $\sim$ 20 mins. on a MacBook Pro M2 with 8 CPUs.

Differential result

Rapidity distribution of a Higgs boson of mass $m_H = 125$ GeV at the 13 TeV LHC



- Error shown is estimated Monte Carlo integration uncertainty.
- Runtime $\sim$ 1 hr 15 mins. on a MacBook Pro M2 with 8 CPUs.

Beyond gluons only

Full set of subtracted double real partonic subprocesses for $pp \to H$ at NNLO

Subprocess	$\sigma^{\mathrm{RR,reg.}}$ (fb)	%
gg	180.7 ± 3.6	40.06
gq	166.7 ± 3.4	36.97
$g\bar{q}$	55.87 ± 0.48	12.39
qq	27.49 ± 0.01	6.09
$q\bar{q}$	18.17 ± 0.01	4.03
$\bar{q}\bar{q}$	2.121 ± 0.001	0.74
$\sum$	451.1 ± 7.5	100

- Recall the total gluons-only cross section is $\sigma = 42943$ fb
- The double real contribution is $\sim 1\%$ of the total cross section

Conclusions and outlook

Precision is key

- Understand subtle features of the Standard Model.
- First signs of New Physics might very well be indirect.
- Exact higher-order calculations required beyond NLO.

CoLoRFulNNLO scheme for regularizing initial-state singularities

- Completely local subtraction terms.
- Fully analytic integrated subtraction terms.

Implemented in `NNLOCAL`

- Gluon fusion Higgs production in HEFT has very simple matrix elements but otherwise all essential features.
- Quark channels in double real radiation complete, full result will be available in the near future.

Conclusions and outlook

Outlook

- `NNLOCAL`: from proof-of-concept to useful tool.
  (all channels, more color-singlet processes, optimisation)
- The inclusion of final-state jets and heavy quarks in hadron collisions is feasible in our methodology.
- Extension to next order (N^3LO) for color-singlet appears conceivable.

Conclusions and outlook

Outlook

- `NNLOCAL`: from proof-of-concept to useful tool.
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Thank you for your attention!

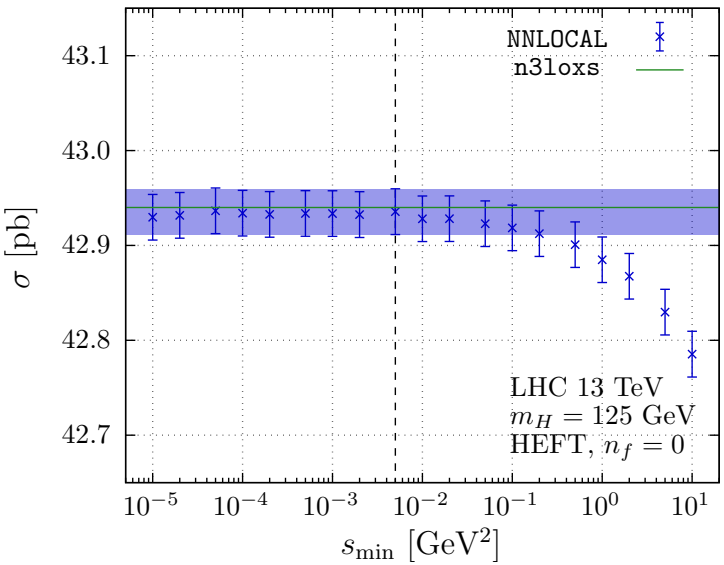
Acknowledgments

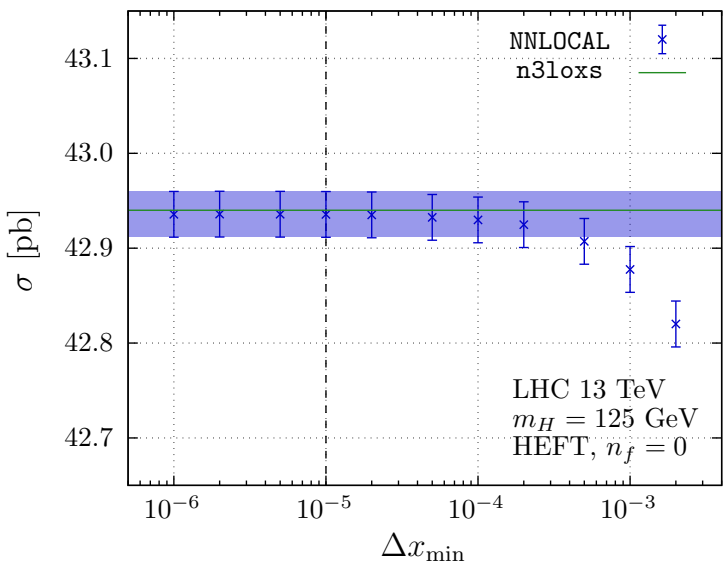
This work was supported by grant K 143451 of the National Research, Development and Innovation Fund in Hungary and by the Bolyai Fellowship programme of the Hungarian Academy of Sciences.

Backup material

Technical cuts

Due to finite precision arithmetics, instabilities eventually set in as IR limits are approached $\Rightarrow$ technical cuts





- $s_{\min}$: smallest allowed 2-particle invariant in real emission
- $\Delta x_{\min}$: closest approach to critical ("0/0") lines in integrated subtraction terms

The pure NNLO contribution

The pure NNLO contribution for the total cross section

m_H [GeV]	NNLO only	double virtual	real-virtual	double real
100	19.76(5) pb	20.76(5) pb	−1.318(1) pb	0.321(9) pb
125	12.64(1) pb	13.16(1) pb	−0.7492(4) pb	0.235(3) pb
250	2.724(2) pb	2.781(2) pb	−0.10475(4) pb	0.0474(2) pb
500	0.4420(3) pb	0.4460(3) pb	−0.009645(4) pb	0.00564(1) pb
1000	47.23(3) fb	47.30(3) fb	−0.4699(2) fb	0.3982(8) fb
2000	2.496(2) fb	2.492(2) fb	−0.007597(2) fb	0.01220(3) fb

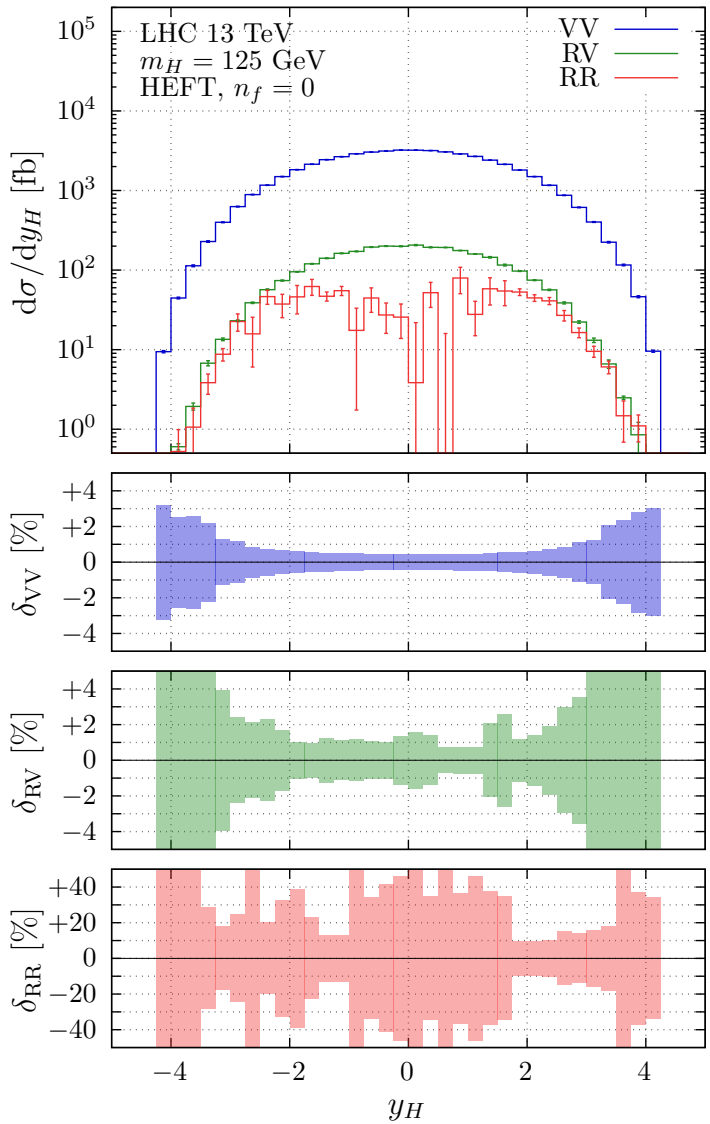
Recall the total NNLO gluons only cross section

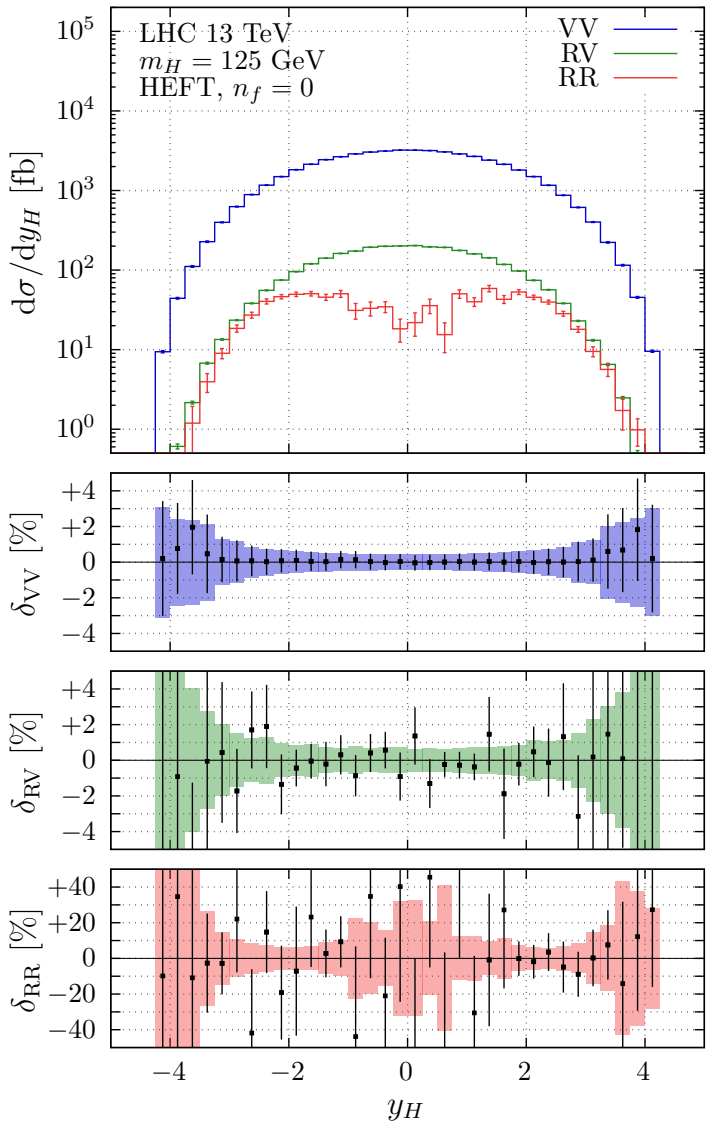
m_H [GeV]	`n3loxs` (gg)	`NNLOCAL` (gg)
100	65.72 pb	65.74(4) pb
125	42.94 pb	42.94(2) pb
250	9.730 pb	9.733(5) pb
500	1.626 pb	1.626(1) pb
1000	173.7 fb	173.7(1) fb
2000	8.794 fb	8.790(5) fb

So VV $\sim \mathcal{O}(1)$, RV $\sim \mathcal{O}(0.1)$ and RR $\sim \mathcal{O}(0.01)$ of the total NNLO cross section

The pure NNLO contribution

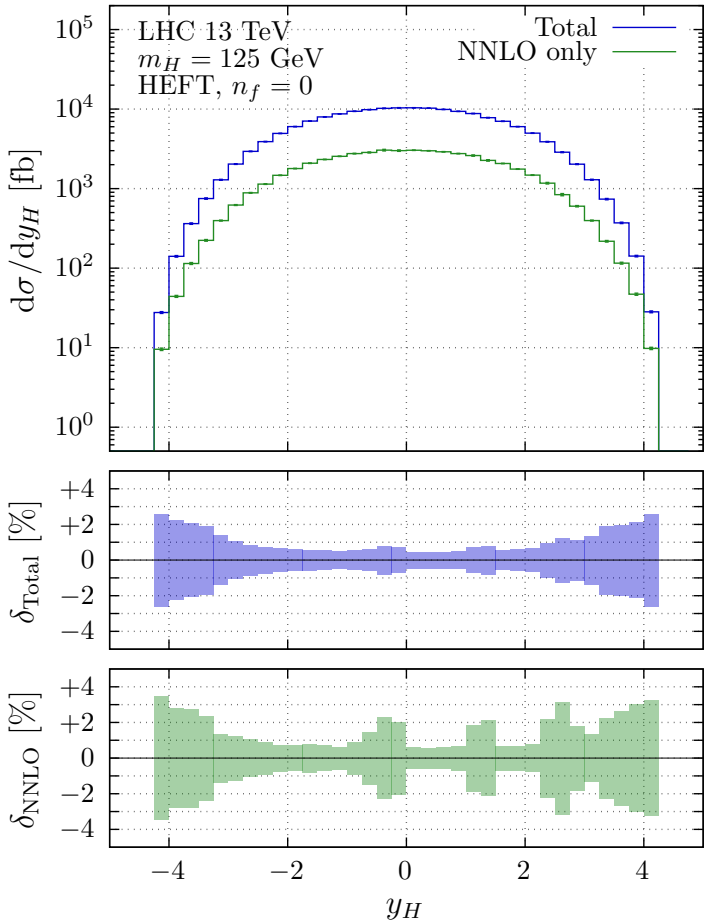
The untrimmed (left) and trimmed (right) VV, RV and RR parts of the NNLO correction

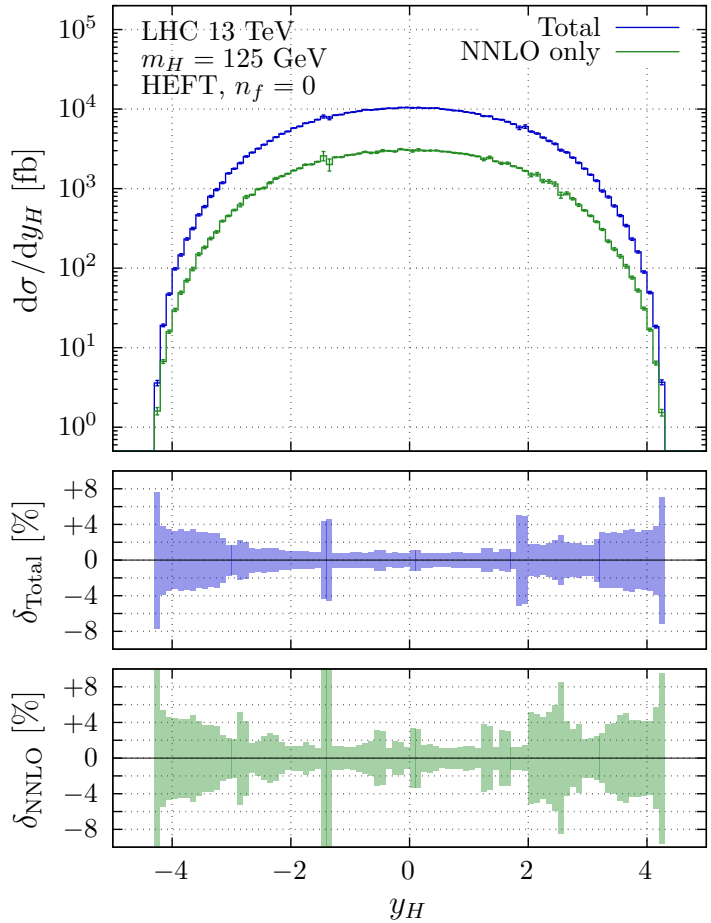




Mitigating misbinning with robust statistics

Rapidity distribution of a Higgs boson of mass $m_H = 125$ GeV at the 13 TeV LHC





- Finer binning to test numerical stability under more demanding conditions
- Overall good stability and convergence, but some spikes appear: **misbinning**

Mitigating misbinning with robust statistics

Misbinning: well-known and more or less unavoidable consequence of using a subtraction scheme to compute binned distributions.

- The weights of the double real matrix element and the subtraction end up in different bins.
- This leads to an apparent uncanceled singularity as far as the bin is concerned.
- Misbinning introduces unphysical outliers into the set of bin value estimates.
- A single extreme value can change the arithmetic mean by a large amount.
- So use an estimator that is robust to such extreme data.

Mitigating misbinning with robust statistics

Robust statistics: the study of statistical methods that remain reliable and accurate even when data contain outliers or deviate from standard assumptions.

- One of the simplest methods for building robust estimators is trimming. E.g., the α-trimmed mean is defined as

$$\bar{x}_\alpha = \frac{1}{n-2m} \sum_{i=m+1}^{n-m} x_{(i)}$$

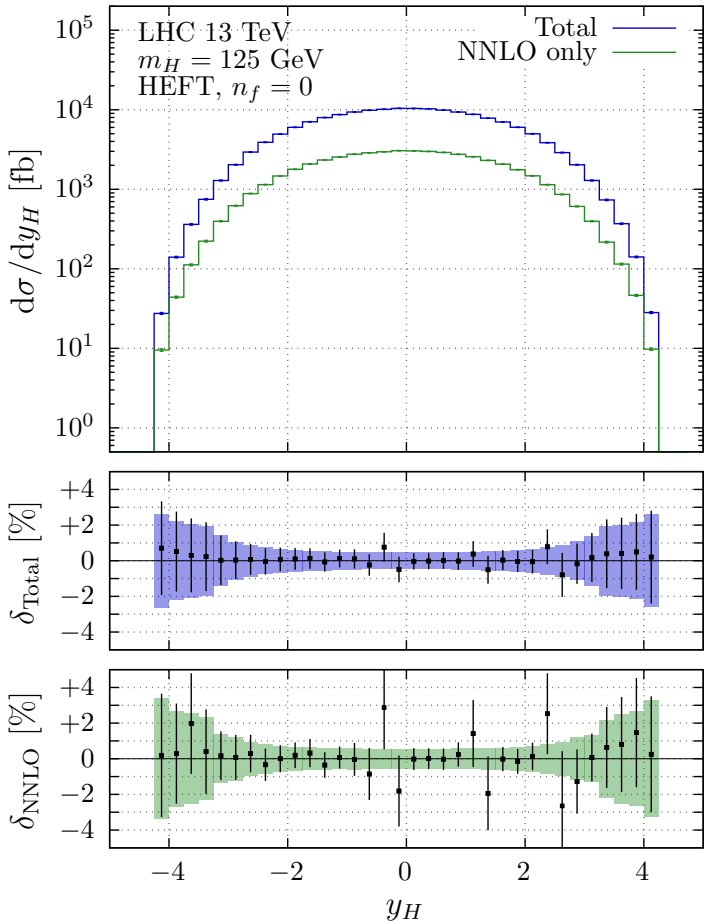
  where $m = [n\alpha]$ (here $[\ldots]$ denotes the integer part) and $x_{(i)}$ is the order statistics $\{x_{(1)}, x_{(1)}, \ldots, x_{(n)}\}$, obtained by sorting the n numbers such that

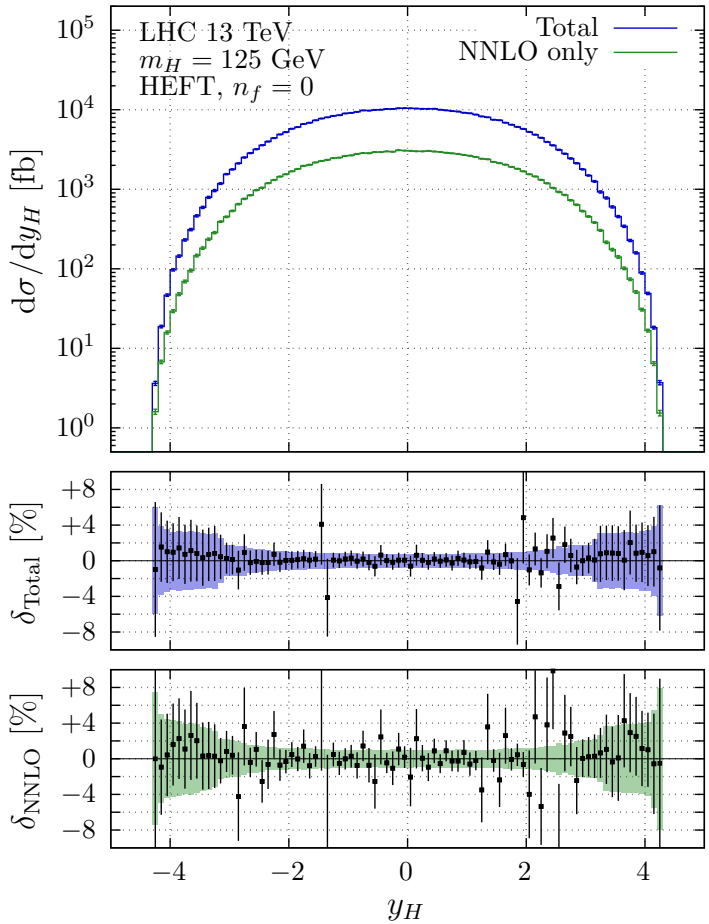
$$x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$$

- In words, it is the arithmetic mean of the $(n-2m)$ numbers obtained by removing the smallest and largest $m = [n\alpha]$ entries.
- Typically results are obtained by combining several statistically independent runs by computing bin-by-bin averages of bin value estimates from the individual runs. It is trivial to implement trimming on these bin value estimates.

Mitigating misbinning with robust statistics

Trimmed rapidity distribution





- α-trimming with $\alpha = 0.015$ used on data containing 256 separate runs