Soft limit of QCD amplitudes with external massive quarks

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General situation

- Each collision at the LHC involves interactions of quarks and gluons
  $\hookrightarrow$ **Understanding of strong interactions is critical to fully exploit potential of the LHC**
- Stringent limits on BSM have been set. So far, no new physics
  $\hookrightarrow$ **This calls for even more precise theoretical predictions**

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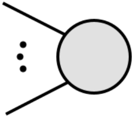
Predictions in perturbative QCD

- In the region where the strong coupling $\alpha_s \ll 1$, fixed-order perturbative expansions is expected to work well

$$\sigma = \underbrace{\sigma_0}_{\text{LO}} + \underbrace{\alpha_s \sigma_1}_{\text{NLO}} + \underbrace{\alpha_s^2 \sigma_2}_{\text{NNLO}} + \underbrace{\alpha_s^3 \sigma_3}_{\text{N}^3\text{LO}} + \cdots$$

Building blocks of N3LO amplitudes

- Born level



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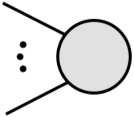

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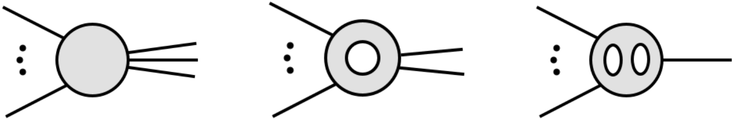

Kinematic regions of gluon emissions

Gluons' momenta in light-cone coordinates

$$k_i^\mu = \left(k_i^+, k_i^-, \boldsymbol{k}_i^\perp\right) \qquad \text{where} \qquad k^\pm = k^0 \pm k^3$$

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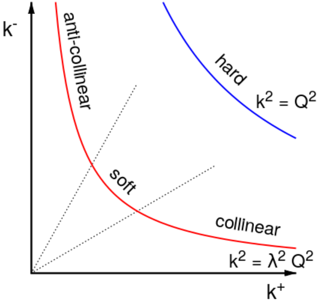
collinear	$k_i^\mu \sim (1, \lambda^2, \lambda)\, Q^2$
anti-collinear	$k_i^\mu \sim (\lambda^2, 1, \lambda)\, Q^2$
hard	$k_i^\mu \sim (1, 1, 1)\, Q^2$
soft	$k_i^\mu \sim (\lambda, \lambda, \lambda)\, Q^2$



where $\lambda \ll 1$ and $Q^2 \sim \mathcal{O}(1)$

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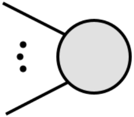

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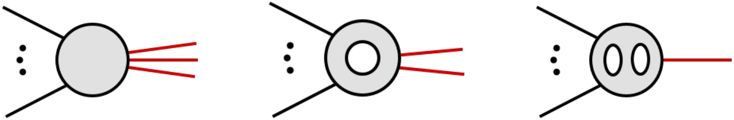

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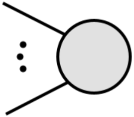

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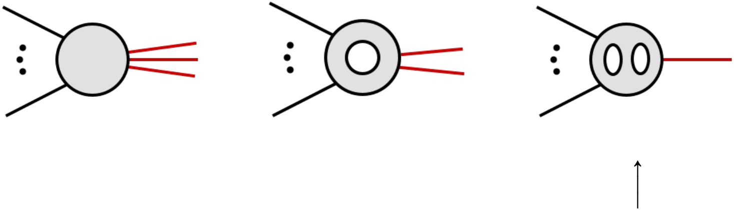
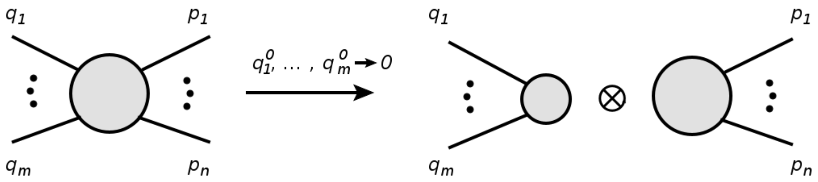


single soft limit at two loops

Soft factorization in QCD: tree level



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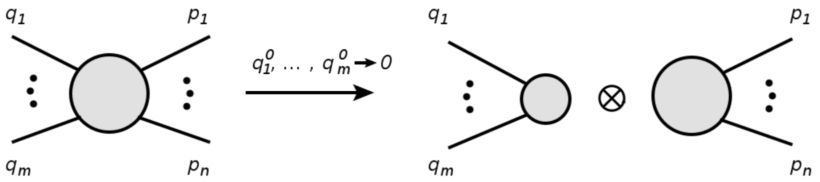

$$|\mathcal{M}^{(0)}(q_1,\dots,q_m,p_1,\dots,p_n)\rangle \xrightarrow{q_1^0,\dots,q_m^0\to 0} \boldsymbol{J}^{(0)}(q_1,\dots,q_m)\,|\mathcal{M}^{(0)}(p_1\dots,p_n)\rangle$$

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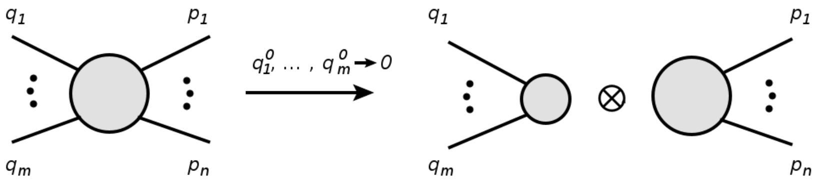

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- $\boldsymbol{J}^{(0)}(q_1,\ldots,q_m)$ is the soft current at tree level

Soft factorization in QCD: higher orders

One loop

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Two loops

$$|\mathcal{M}^{(2)}(q_1,\dots,q_m,p_1,\dots,p_n)\rangle \xrightarrow{q_1^0,\dots,q_m^0\to 0} \boldsymbol{J}^{(2)}(q_1,\dots,q_m)\,|\mathcal{M}^{(0)}(p_1\dots,p_n)\rangle + \boldsymbol{J}^{(1)}(q_1,\dots,q_m)\,|\mathcal{M}^{(1)}(p_1\dots,p_n)\rangle + \boldsymbol{J}^{(0)}(q_1,\dots,q_m)\,|\mathcal{M}^{(2)}(p_1\dots,p_n)\rangle$$

Soft current

In general

$$\boldsymbol{J} = \boldsymbol{J}^{(0)} + \boldsymbol{J}^{(1)} + \boldsymbol{J}^{(2)} + \cdots$$

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The tree level result, for massive ($p_i^2 > 0$) and massless ($p_i^2 = 0$) hard partons, takes the simple form

$$\boldsymbol{J}_a^{\mu(0)} = \sum_{i=1}^{n} \boldsymbol{T}_i^a \frac{p_i^\mu}{p_i^\mu \cdot q} \,,$$

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while at two loops both from dipole and tripole emissions

$$\boldsymbol{J}_a^{\mu(1)} = \sum_{i \neq j} \boldsymbol{T}_i^{a_i} \boldsymbol{T}_j^{a_j} S_{a,ij}(p_i, p_j, \{q_m\}) + \sum_{i \neq j \neq k} \boldsymbol{T}_i^{a_i} \boldsymbol{T}_j^{a_j} \boldsymbol{T}_k^{a_k} S_{a,ijk}(p_i, p_j, p_k, \{q_m\}).$$

Soft current - state of the art

Massless fermions

- one loop
  [Catani, Grazzini '00] *exact in* ϵ
- two loops
  [Li and Zhu '13] *dipole* $\mathcal{O}\left(\epsilon^2\right)$
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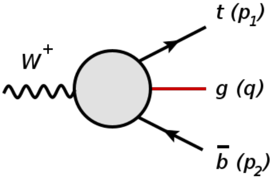
Massive fermions

- one loop
  [Bierenbaum, Czakon, Mitov '12, Czakon, Mitov '18] *dipole* $\mathcal{O}(\epsilon)$

Our aim is to get the massive soft current at two loops to $\mathcal{O}(\epsilon)$

Kinematics



- Five invariants:

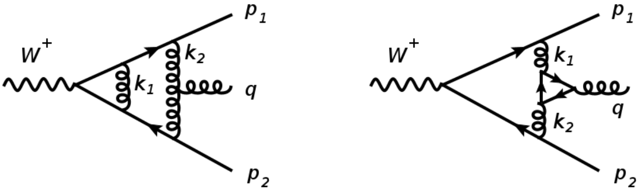
$$
\begin{aligned}
s_{1q} &= (p_1 + q)^2 \\
s_{2q} &= (p_2 + q)^2 \\
s_{12} &= (p_1 + p_2)^2 \\
m_t^2 &= p_1^2 \\
m_b^2 &= p_2^2
\end{aligned}
$$

Details of the calculation

- Generate two-loop diagrams (196 in total) for the process:

$$W^+ \to t + \bar{b} + g$$

in Feynman gauge, with FeynArts



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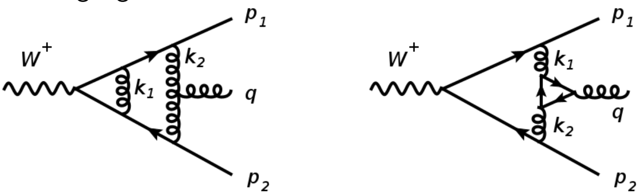

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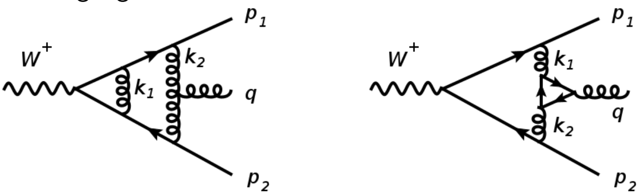
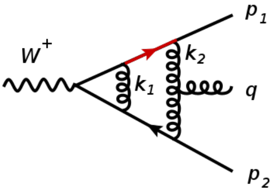


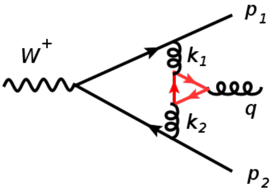
- Generate corresponding amplitude $\mathcal{A}^{(2)}_{W^+\to t\bar{b}g}$ with FEYNCALC
- Parameterize the gluon momenta

$$k_1 \to \lambda k_1\,,$$
$$k_2 \to \lambda k_2\,,$$
$$q \to \lambda k_1\,,$$

expand the amplitude in λ and take the leading (most singular) term.
This is the soft limit of $\mathcal{A}^{(2)}_{W^+\to t\bar{b}g}$

What happened to the propagators (and vertices)?





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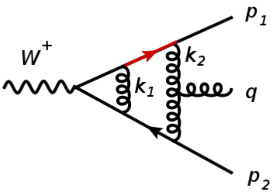

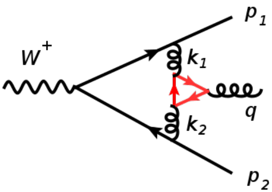

- External quarks:

$$\frac{\not{p}_1 - \lambda \not{k}_2}{(p_1 - \lambda k_2)^2 - m_t^2} = \simeq \frac{-\not{p}_1}{\lambda\, p_1 \cdot k_2} \quad \text{(eikonal)}$$

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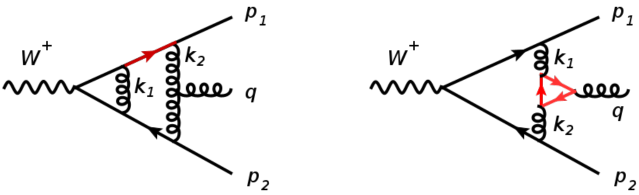

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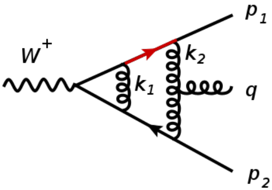

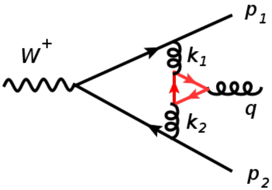

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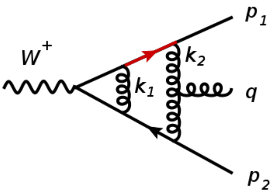

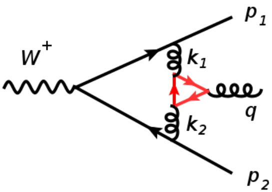

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- Tipple-gluon vertex: exact

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- These integrals are build out of subsets of 22 propagators:

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\begin{aligned}
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&(k_1+k_2+q)^2 + i\epsilon \\
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- We can significantly reduce the number of integrals by employing IBP identities

Integration by parts (IBPs)

In dimensional regularization, the integral over total derivative is zero

$$\int d^d k_1 \, \ldots d^d k_L \frac{\partial}{\partial k_i^\mu} \left(\frac{q^\mu}{P_1^{a_1} \cdots P_N^{a_N}} \right) = 0 \, ,$$

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Example:

$$\mathrm{top}_1(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9) =$$

$$\int \frac{d^d k_1 d^d k_2}{k_1^{2a_1} k_2^{2a_2} (k_1+k_2)^{2a_3} (k_2+q)^{2a_4} (k_1+k_2+q)^{2a_5} (2k_2 p_1)^{a_6} (2p_2(k_1+q))^{a_7} (-2k_1 p_1)^{a_8} (-2k_2 p_2)^{a_9}}$$

Variables

As mentioned earlier, the process is characterized by five invariants, or, equivalently, by five scalar products: $p_1\cdot q,\ p_2\cdot q,\ p_1\cdot p_2,\ p_1^2,\ p_2^2$

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Hence, our integrals will also be invariant under rescaling of the heavy quark momenta, p_1 and p_2. This can be achieved only by the three ratios

$$\underset{\sim\, m^{-2}}{\frac{(p_1 \cdot p_2)}{(p_1 \cdot 1)(p_2 \cdot q)}}\,, \qquad \underset{\sim\, 1}{\frac{(p_1 \cdot p_1)\,(p_2 \cdot q)}{(p_1 \cdot p_2)(p_1 \cdot q)}}\,, \qquad \underset{\sim\, 1}{\frac{(p_2 \cdot p_2)\,(p_1 \cdot q)}{(p_1 \cdot p_2)(p_2 \cdot q)}}$$

Variables

As mentioned earlier, the process is characterized by five invariants, or, equivalently, by five scalar products: $p_1 \cdot q,\ p_2 \cdot q,\ p_1 \cdot p_2,\ p_1^2,\ p_2^2$

However, notice that the eikonal propagators have the following scaling property

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Loop integrals:

$$\int \prod dk_i^4 \to \text{ dimensionless}$$

$$\int \prod dk_i^{4-2\epsilon} \to\ m^{d-4} \text{ per loop}$$

Variables

Hence, our integrals will evaluate to the following functions:

$$I_i(p_1\cdot q, p_2\cdot q, p_1\cdot p_2, p_1^2, p_2^2) = q_\epsilon^{-2\epsilon} M_i(\alpha_1, \alpha_2)$$

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$$q_\epsilon = \frac{(p_1 \cdot p_2)}{(p_1 \cdot 1)(p_2 \cdot q)}\,,$$

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And we can write differential equations our masters in terms of dimensionless functions M_i of dimensionless variables α_1, α_2:

$$\frac{\partial}{\partial\alpha_1}\vec{M}(\alpha_1,\alpha_2) = \boldsymbol{a}_1(\epsilon,\alpha_1,\alpha_2)\,\vec{M}(\alpha_1,\alpha_2)$$
$$\frac{\partial}{\partial\alpha_2}\vec{M}(\alpha_1,\alpha_2) = \boldsymbol{b}_2(\epsilon,\alpha_1,\alpha_2)\,\vec{M}(\alpha_1,\alpha_2)$$

System of differential equations[1]

$$\frac{\partial}{\partial \alpha_1}\begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{65} \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,65} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,65} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,65} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{65,1} & a_{65,2} & a_{65,3} & \cdots & a_{65,65} \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{65} \end{bmatrix}$$

$$\frac{\partial}{\partial \alpha_2}\begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{65} \end{bmatrix} = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \cdots & b_{1,65} \\ b_{2,1} & b_{2,2} & b_{2,3} & \cdots & b_{2,65} \\ b_{3,1} & b_{3,2} & b_{3,3} & \cdots & b_{3,65} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{65,1} & b_{65,2} & b_{65,3} & \cdots & b_{65,65} \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{65} \end{bmatrix}$$

[1]Slide powered by ChatGPT...

Closed subsystems

No.	Size homogeneous	Size inhomogeneous
1-14	1	1
15-26	2	1
27	2	2
28	2	2
29	3	1
30	3	1
31	4	2
32	4	2
33	5	1
34	5	2
35	5	1
36	5	2
37	5	1
38	6	2
39	6	2
40	8	4
41	10	1
42	10	1
43	12	2
44	13	1
45	13	1
46	16	2
47	29	3
48	29	3

Canonical form

All our differential systems, $s \in \{1, \ldots, 48\}$, have the form

$$\frac{\partial}{\partial \alpha_i} \vec{M}_s = \boldsymbol{A}_{si}(\alpha_i, \epsilon)\, \vec{M}_s$$

where $\vec{M}_s = \{M_1, \ldots, M_n\} \subset \{M_1, \ldots, M_{65}\}$.

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- But the set of masters $\{M_1, \ldots, M_{65}\}$ corresponds just to a particular choice of basis in the space of integrals.

As observed in [Henn '13], by a proper change of basis of masters

$$\vec{M}_s = \boldsymbol{T}_s \vec{J}_s\,,$$

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- The dependence on ϵ factorizes! This is the so-called canonical form.

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- Lee algorithm [Lee '15], LIBRA [Lee '21]
- EPSILON [Prausa '17]
- CANONICA [Meyer '18]
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is a rational function, *i.e.*

$$T_s^{jk}(x_i) = \frac{P(x_i)}{Q(x_i)} \,,$$

where P and Q are polynomials.

Letters and alphabet

Let's have a look at the canonical form again

$$\partial_i \vec{J} = \epsilon\, \boldsymbol{S}_i(\alpha_i)\, \vec{J}$$

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$$S_{i,mn} = \sum_j c_j^{mn}(\epsilon) \frac{1}{L_j(\alpha_i)}$$

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$$L_j \in \{\alpha_1,\, \alpha_2,\, 1 - \alpha_1 - \alpha_2,\, 1 - 4\alpha_1\alpha_2\}$$

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When the matrix $\boldsymbol{S}_i$ is being integrated iteratively, the integrals evaluate to multiple polylogarithms:

$$G(0;x) = \log(x),\;\; G(a;x) = \log\left(1-\tfrac{x}{a}\right),\;\; G(0,1;x) = -\mathrm{Li}_2(x)$$

Square roots

A subset of our 48 systems can be put into canonical form directly by finding a rational transformation $\boldsymbol{T}$.

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$$t_1 = 2\alpha_2, \quad t_2 = \sqrt{1-4\alpha_1\alpha_2}\,,$$

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However, the following change of kinematic variables

$$t_1 = 2\alpha_2, \quad t_2 = \sqrt{1-4\alpha_1\alpha_2}\,,$$

leading to a new alphabet

$$\{t_1,\ t_2,\ 1-t_1,\ 1-t_2,\ 1-t_1-t_2,\ 1-t_1+t_2\}\ ,$$

allowed us to find the canonical form for those cases.

So let's see what we have got

No.	Size homogeneous	Size inhomogeneous	Canonical form
1-42	1-10	1-4	✓
43	12	2	✗
44	13	1	✓
45	13	1	✓
46	16	2	✗
47	29	3	✓
48	29	3	✓

Evaluating integrals

Let's put the systems 43 and 46 aside for a moment, and proceed with the remaining ones, which we have in the canonical form.

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These systems are fully solved according to:

$$J^{(0)}(x_1, x_2) = B^{(0)}$$

$$J^{(i)}(x_1, x_2) = \int\limits_{(a_1,a_2)}^{(x_1,x_2)} (\boldsymbol{S}_1 dx_1' + \boldsymbol{S}_2 dx_2') J^{(i-1)}(x_1', x_2') + B^{(i)}$$

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  $\hookrightarrow$ AMFlow [Liu, Ma, Wang '18-'23]

AMFlow can also be used to numerically compute $J^{(i)}(x_1, x_2)$ outside of the boundary and this can serve an ultimate validation of our solutions!

So let's see what we have got

No.	Size homogeneous	Size inhomogeneous	Canonical form	Solved and validated with AMFlow
1-42	1-10	1-4	✓	✓
43	12	2	✗	✗
44	13	1	✓	✓
45	13	1	✓	✓
46	16	2	✗	✗
47	29	3	✓	✗
48	29	3	✓	✗

System 43

$$\frac{\partial}{\partial t_1}\begin{bmatrix} \boldsymbol{m} \\ M_{44} \\ M_{61} \end{bmatrix} = \begin{bmatrix} \epsilon \boldsymbol{S}_a & 0 & 0 \\ \boldsymbol{R}_{1,1} & a_{1,1} & a_{1,2} \\ \boldsymbol{R}_{1,2} & a_{2,1} & a_{2,2} \end{bmatrix} \begin{bmatrix} \boldsymbol{m} \\ M_{44} \\ M_{61} \end{bmatrix}$$

$$\frac{\partial}{\partial t_2}\begin{bmatrix} \boldsymbol{m} \\ M_{44} \\ M_{61} \end{bmatrix} = \begin{bmatrix} \epsilon \boldsymbol{S}_b & 0 & 0 \\ \boldsymbol{R}_{2,1} & b_{1,1} & b_{1,2} \\ \boldsymbol{R}_{2,2} & b_{2,1} & b_{2,2} \end{bmatrix} \begin{bmatrix} \boldsymbol{m} \\ M_{44} \\ M_{61} \end{bmatrix}$$

where $\boldsymbol{S}_a$, $\boldsymbol{S}_b$ are canonical submatrices for 10 out of 12 masters

$$\boldsymbol{m} = [M_1, M_2, M_{15}, M_{18}, M_{20}, M_{26}, M_{32}, M_{53}, M_{54}, M_{55}]$$

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The question is: can we find a canonical of form the matrices:

$$\begin{bmatrix}a_{1,1} & a_{1,2}\\ a_{2,1} & a_{2,2}\end{bmatrix} \qquad \begin{bmatrix}b_{1,1} & b_{1,2}\\ b_{2,1} & b_{2,2}\end{bmatrix}$$

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- Standard algorithms of CANONICA and LIBRA do not find a rational transformation. No surprise.

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- Standard algorithms of CANONICA and LIBRA do not find a rational transformation. No surprise.
- We could however try to find it manually!

System 43

Because there is still one thing I didn't tell you...

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By definition, canonical form is achieved though the following transformation

$$\epsilon\, \boldsymbol{S} = \boldsymbol{T}^{-1} \boldsymbol{A} \boldsymbol{T} - \boldsymbol{T}^{-1} d\, \boldsymbol{T} \qquad (❀)$$

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where $\boldsymbol{A}$ is our original matrix and $\boldsymbol{T}$ is the transformation matrix we are looking for

$$\boldsymbol{T} = \begin{bmatrix} v_{11}(t_1, t_2) & v_{12}(t_1, t_2) \\ v_{21}(t_1, t_2) & v_{22}(t_1, t_2) \end{bmatrix}$$

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Eq. (❀) can be used to generate four conditions for the entries of $\boldsymbol{T}$

$$\left(\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} - \boldsymbol{T}^{-1}d\boldsymbol{T}\right)\Big|_{\epsilon=0} = 0$$

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$$\epsilon\,\boldsymbol{S} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} - \boldsymbol{T}^{-1}d\,\boldsymbol{T} \qquad (❀)$$

where $\boldsymbol{A}$ is our original matrix and $\boldsymbol{T}$ is the transformation matrix we are looking for

$$\boldsymbol{T} = \begin{bmatrix} v_{11}(t_1,t_2) & v_{12}(t_1,t_2) \\ v_{21}(t_1,t_2) & v_{22}(t_1,t_2) \end{bmatrix}$$

Eq. (❀) can be used to generate four conditions for the entries of $\boldsymbol{T}$

$$\left(\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} - \boldsymbol{T}^{-1}d\,\boldsymbol{T}\right)\Big|_{\epsilon=0} = 0$$

This leads to four partial differential equations for $v_{ij}(t_1,t_2)$

System 43

$$\boldsymbol{T} = \frac{1}{\sqrt{t_1 t_2}} \begin{bmatrix} Q_{-\frac{1}{2}}(x) & P_{-\frac{1}{2}}(x) \\ \frac{1}{t_2} Q_{\frac{1}{2}}(x) & \frac{1}{t_2} P_{\frac{1}{2}}(x) \end{bmatrix},$$

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$$P_{\frac{1}{2}}(x) = \frac{2}{\pi}\left[2E\left(\frac{1-x}{2}\right) - K\left(\frac{1-x}{2}\right)\right]$$

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- Hence, we found transformation to canonical form! We checked that it's invertible and it works.
- The transformation is not rational and it involves elliptic integrals.

Solutions for systems 43 and 46

Just like for other systems

$$J^{(i)}(x_1, x_2) = \int_{(a_1,a_2)}^{(x_1,x_2)} (\boldsymbol{S}_1 dt_1 + \boldsymbol{S}_2 dt_2) J^{(i-1)}(t_1, t_2) + B^{(i)}$$

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And those, and only those terms need to be integrated numerically.

State of the art

No.	Size homogeneous	Size inhomogeneous	Canonical form	Solved and validated with AMFlow
1-42	1-10	1-4	✓	✓
43	12	2	✗	
44	13	1	✓	✓
45	13	1	✓	✓
46	16	2	✗	
47	29	3	✓	
48	29	3	✓	

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- This is not surprising for a two-loop calculation with massive particles.