Recent progress on three-loop Higgs plus jet amplitudes

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Matter To The deepest 2025

Based on [2410.19088] and ongoing work with Cesare Carlo Mella and Petr Jakubčík

MAX-PLANCK-INSTITUT FÜR PHYSIK

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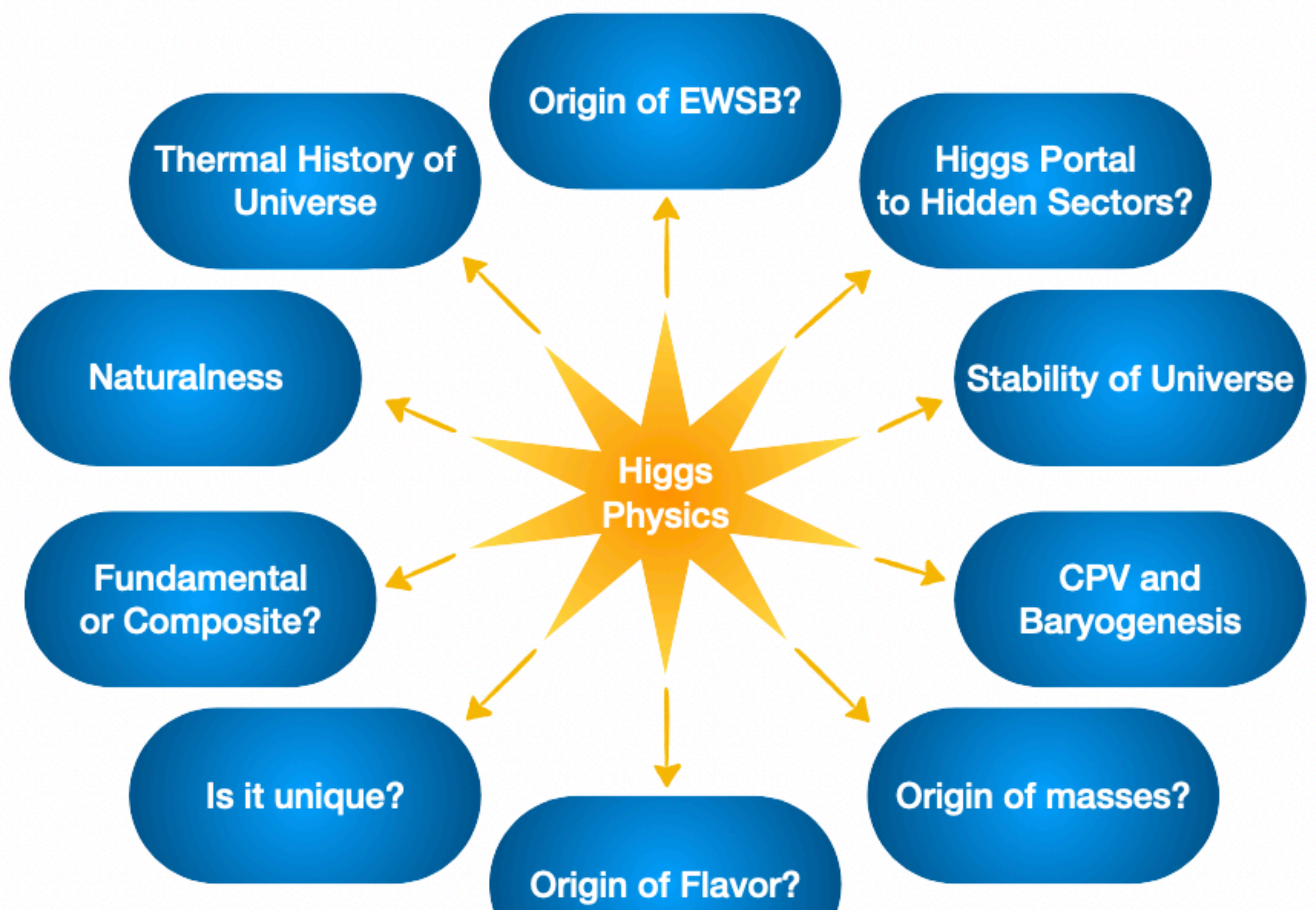
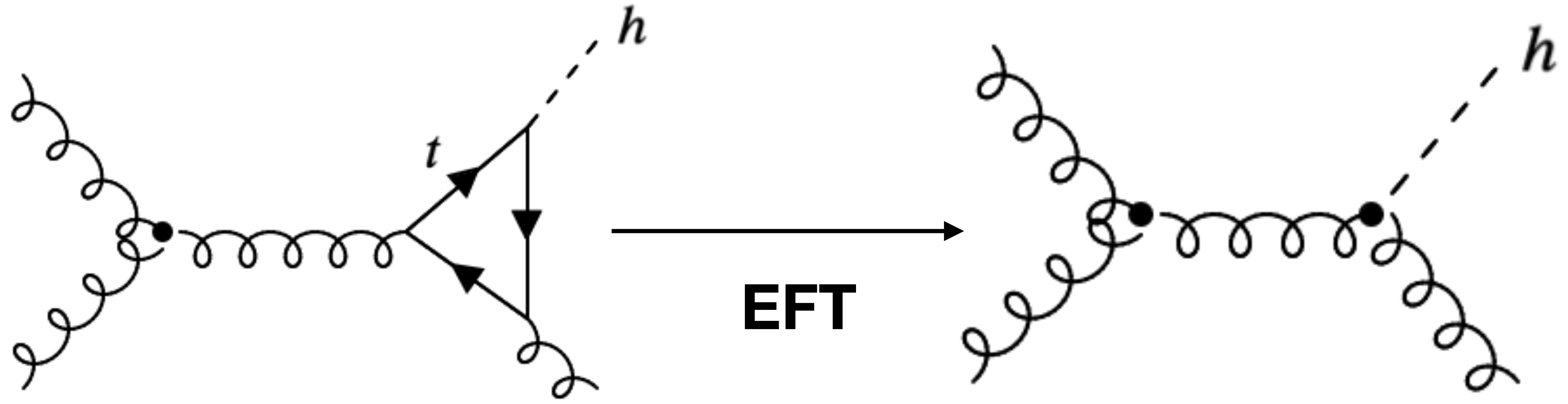
Motivation

(1) Phenomenology (Higgs production)

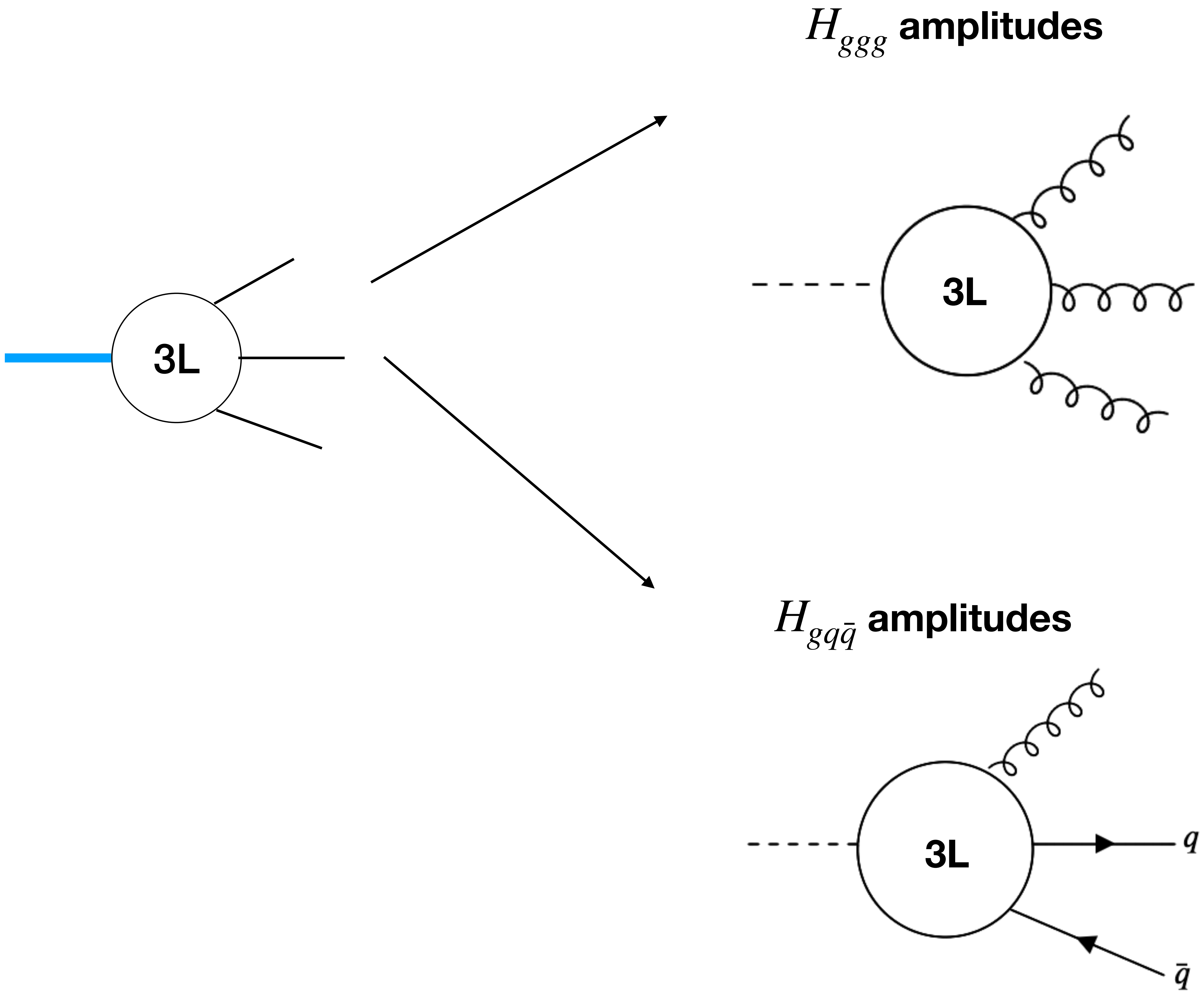


[Snowmass 2022]



mediated by top quark

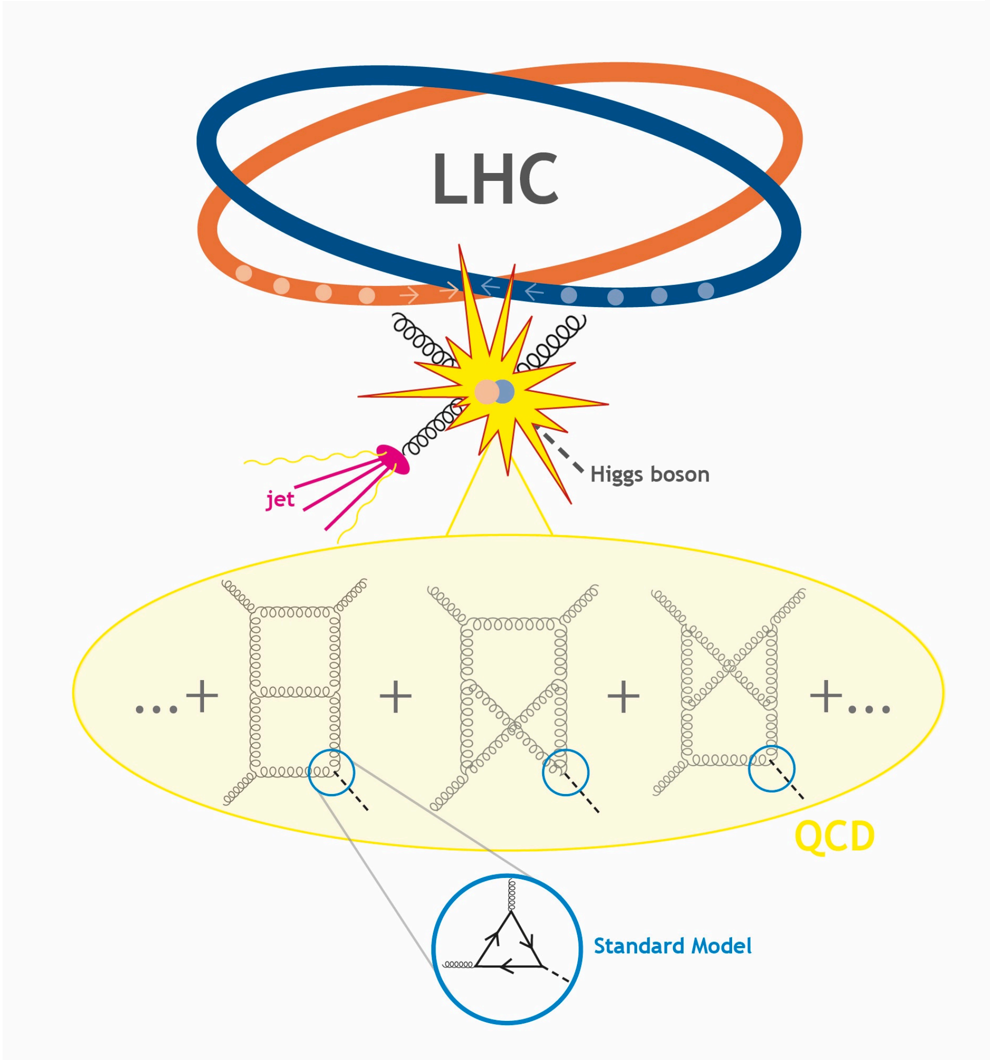
For percentage level precision, we need <u>N3LO</u>



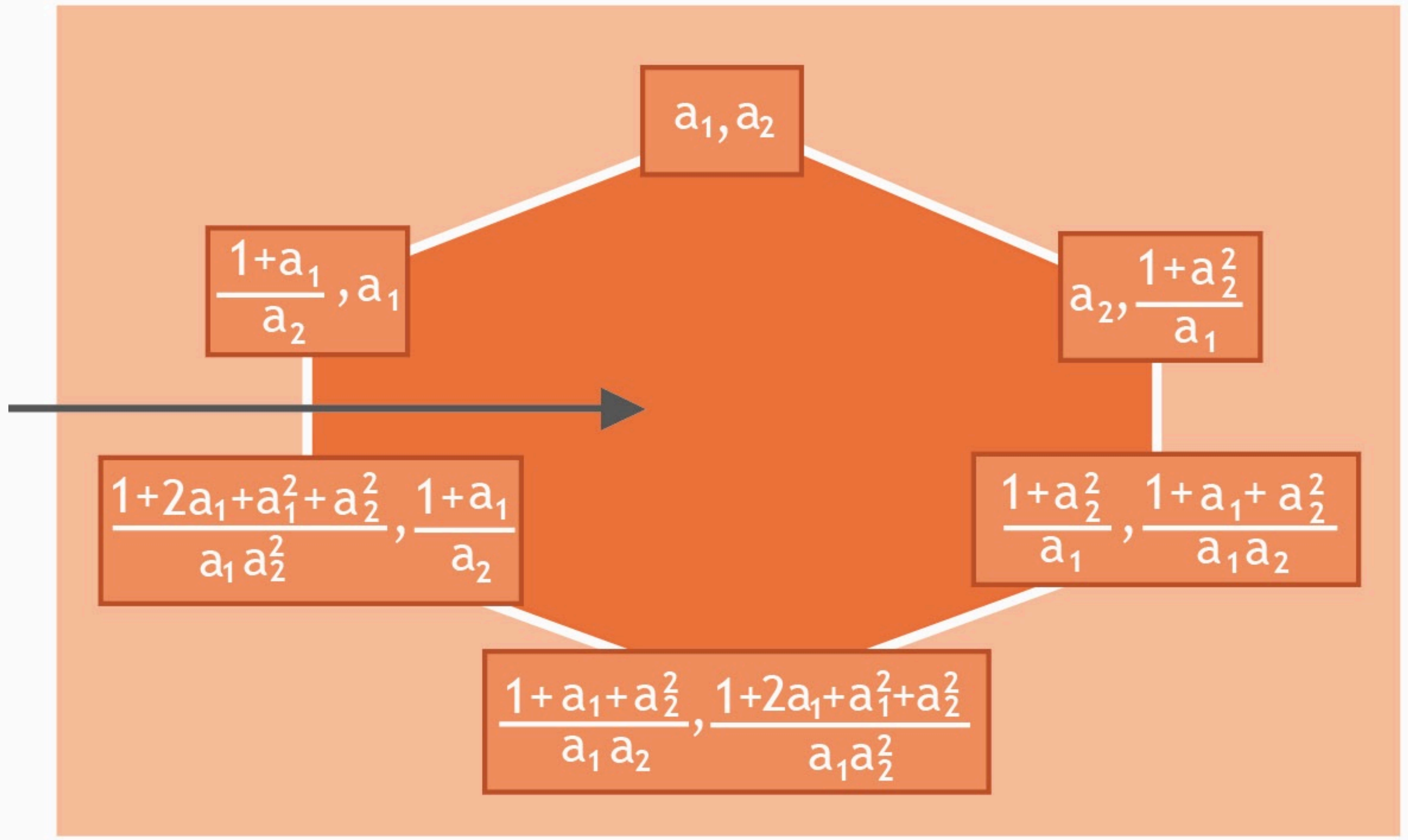
Motivation

(2) Hidden structure

- C_2 cluster algebra, extended-Steinmann-like condition [Chicherin, Henn, Papathanasiou 2020] [Dixon, McLeod, Wilhelm 2020]



www.scattering-amplitudes.com

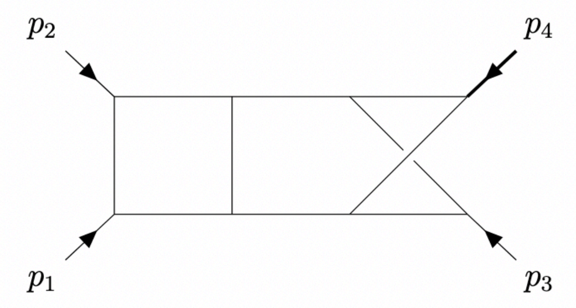


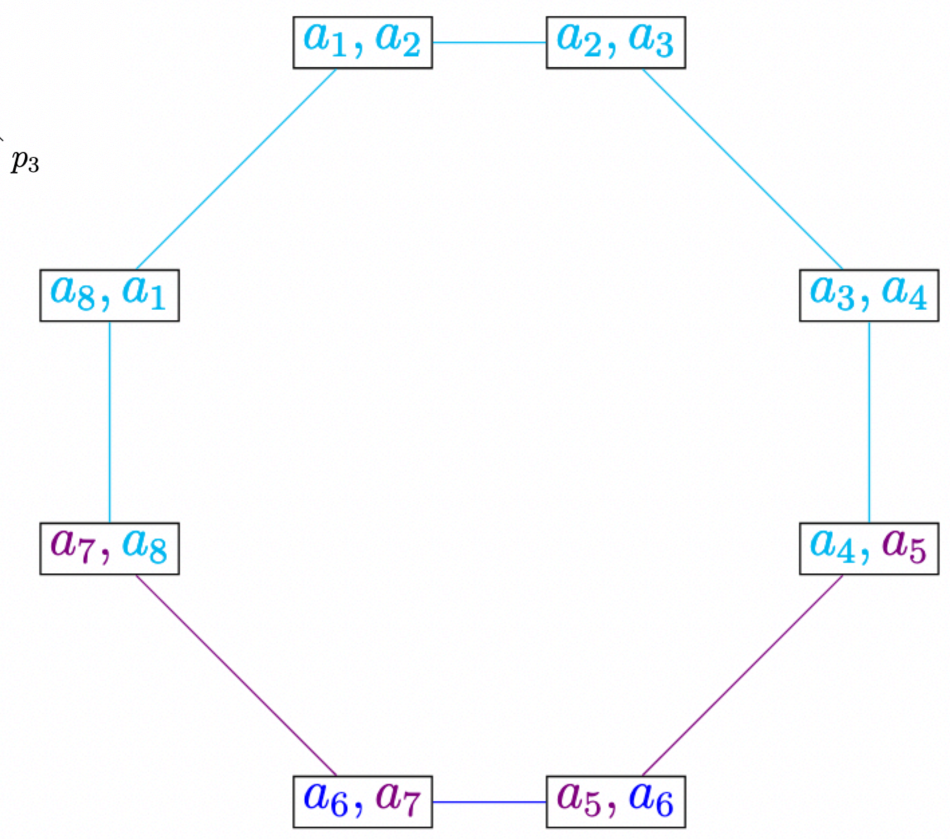
$$\alpha = \{z_1, z_2, 1 - z_1 - z_2, 1 - z_1, 1 - z_2, z_1 + z_2\}$$

$$\tilde{A}_i \cdot \tilde{A}_j = 0 \Longrightarrow \ldots \otimes \cancel{\alpha_i \otimes \alpha_j} \otimes \ldots$$ for $i, j \in \{4,5,6\}$ with $i \neq j$

- G_2 cluster algebra

[Aliaj, Papathanasiou 2024]





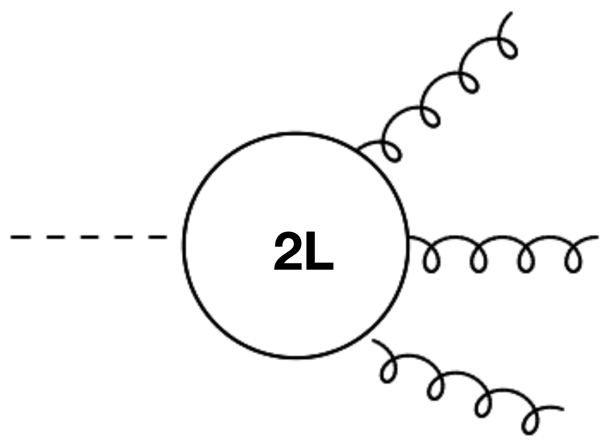
$$\alpha = \{z_1, z_2, 1 - z_1 - z_2, 1 - z_1, 1 - z_2, z_1 + z_2, 1 - 2z_1 + z_1^2 - z_2, z_1 - z_1^2 - z_2\}$$

Motivation

(2) Hidden structure

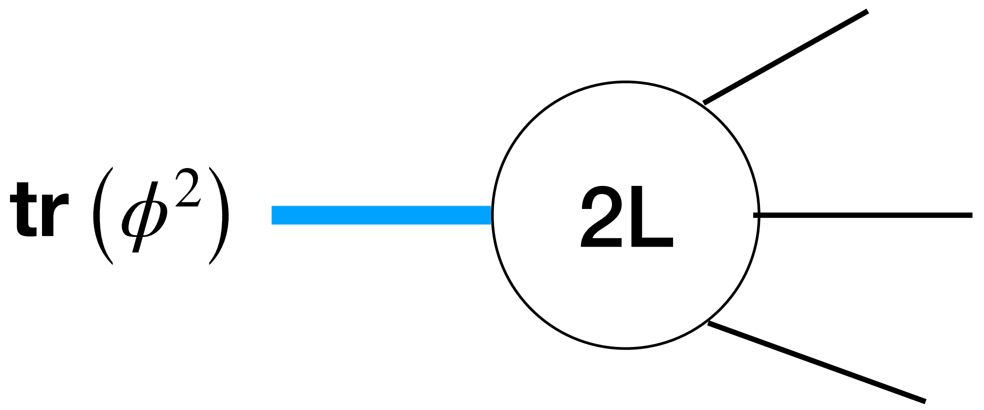
- Maximal transcendentality conjecture

H_{ggg} amplitudes



Maximally transcendental part

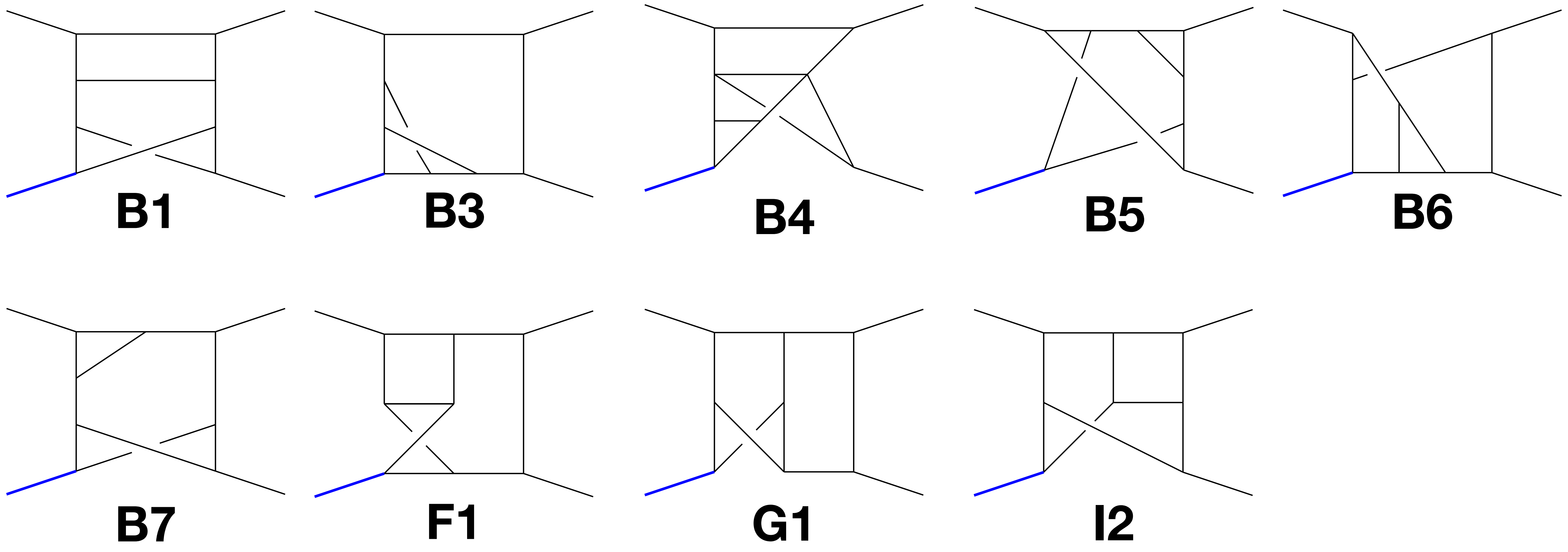
3-point tr $\left(\phi^2\right)$ form factor in N=4 SYM



[Brandhuber, Travaglini, Yang 2012]

[Gehrmann, Henn, Jakubčík, JL, Mella, Syrrakos, Tancredi, Torres Bobadilla 2024]

Integral Families @leading color



+ Planar families

	# MI
B1	150
B3	90
<u>B4</u>	143
<u>B5</u>	70
<u>B6</u>	150
<u>B7</u>	89
F1	214
G1	254
I2	305

(<u>Family</u> : reducible top sector)

Differential equation method

Building canonical differential equations

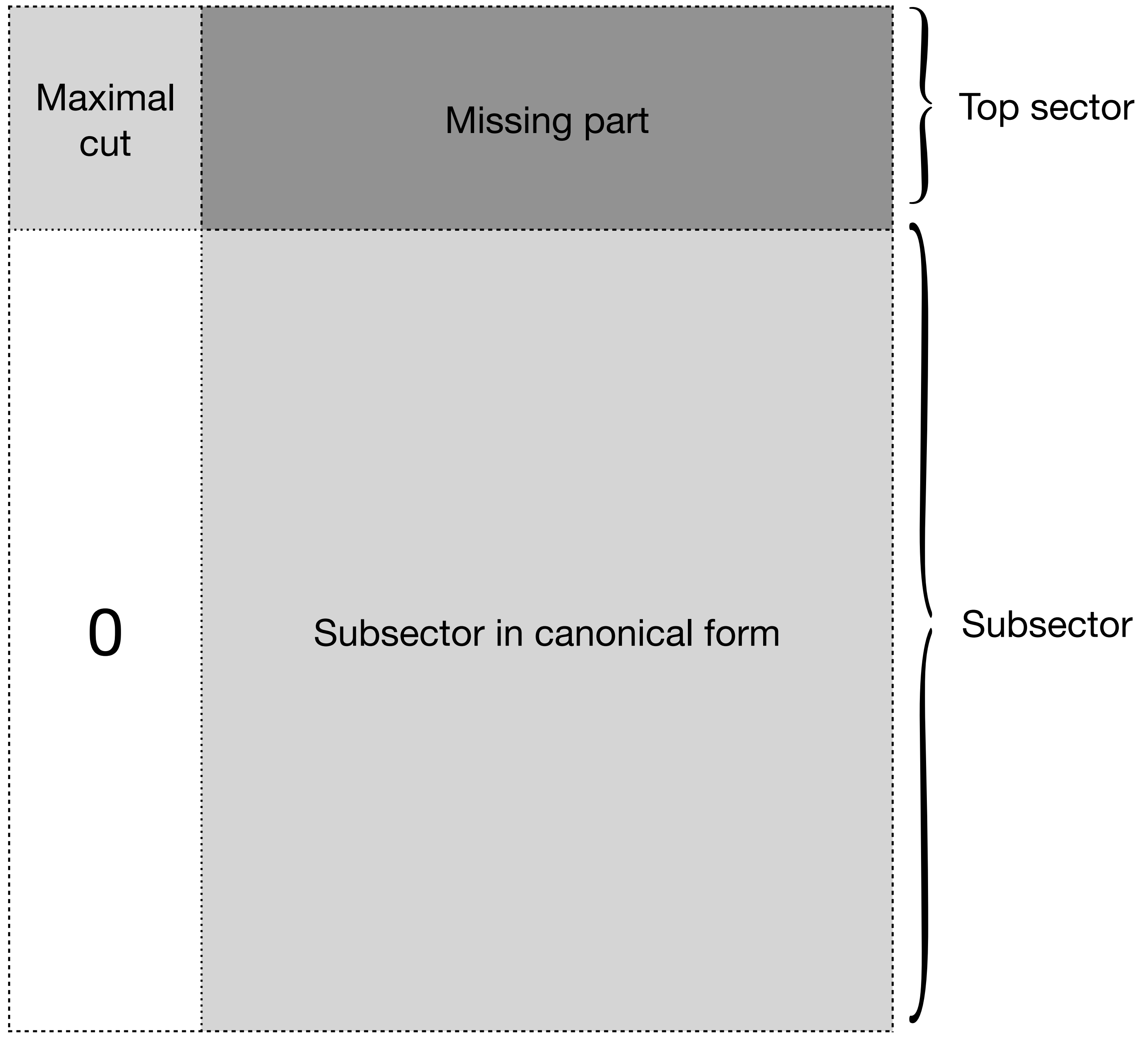
[Henn 2013]

Good choice of basis for Feynman integrals can significantly simplify the computation of differential equation.

$$d\vec{f}\left(\vec{x},\epsilon\right) = \epsilon\left(d\tilde{A}\right)\vec{f}\left(\vec{x};\epsilon\right), \text{ with } \tilde{A} = \left[\sum_k A_k \log \alpha_k(x)\right]$$

- Subsector : DlogBasis, Mapping from other families, loop-by-loop approach
[Wasser 2022]

- Topsector : Matrix rotation, loop-by-loop approach



By FiniteFlow, Kira
[Peraro, 2019] [Klappert, Lange, Maierhöfer, Usovitsch, 2020]

Iterated integrals

$$\vec{f}\left(\overrightarrow{x},\epsilon\right) = \mathbb{P}\exp\left[\epsilon\int_\gamma d\tilde{A}\right]\vec{f}_0(\epsilon)$$ where $\vec{f}_0(\epsilon)$ is a boundary vector

$\gamma*\left(\omega_i\right) = k_i(t)\,dt$ function k_i are defined by pulling back the 1-form ω_i to the interval [0,1]

An ordinary line integral is given by $\int_\gamma \omega_1 = \int_{[0,1]}\gamma*\left(\omega_1\right) = \int_0^1 k_1\left(t_1\right)dt_1$

Iterated integral of $\omega_1 \ldots \omega_n$ along γ is defined by $\int_\gamma \omega_1\ldots\omega_n = \int_{0\le t_1\le\ldots\le t_n\le 1} k_1\left(t_1\right)dt_1\ldots k_n\left(t_n\right)dt_n$

[Chen 1977]

If the alphabet is rational functions , one can write the answer in terms of Goncharov polylogarithms

$$G(\overrightarrow{a}_n;z) \equiv G(\overrightarrow{a}_1,\overrightarrow{a}_{n-1};z) \equiv \int_0^z \frac{dt}{t-a_1}G(\overrightarrow{a}_{n-1};t)$$

with $G\left(a_1;z\right) = \int_0^z \frac{dt}{t-a_1}$ and $G(\overrightarrow{0}_n;z) \equiv \frac{1}{n!}\log^n(z)$

Function space of 3loop integrals @leading colors

$$\vec{\alpha} = \{p_4^2, s, t, p_4^2 - s - t, p_4^2 - s, p_4^2 - t, s + t, \frac{(p_4^2 - s - t)s - R}{(p_4^2 - s - t)s + R}, \frac{st - R}{st + R}, p_4^4 - t\left(p_4^2 + s\right), p_4^4 - s\left(p_4^2 + t\right),\ t^2 + p_4^2(s - t), s^2 - p_4^2(s - t), -p_4^2 t + \left(p_4^2 - s\right)^2\}$$

with $R = \sqrt{-p_4^2 s(p_4^2 - s - t)t}$

20 letters (including kinematic crossings)

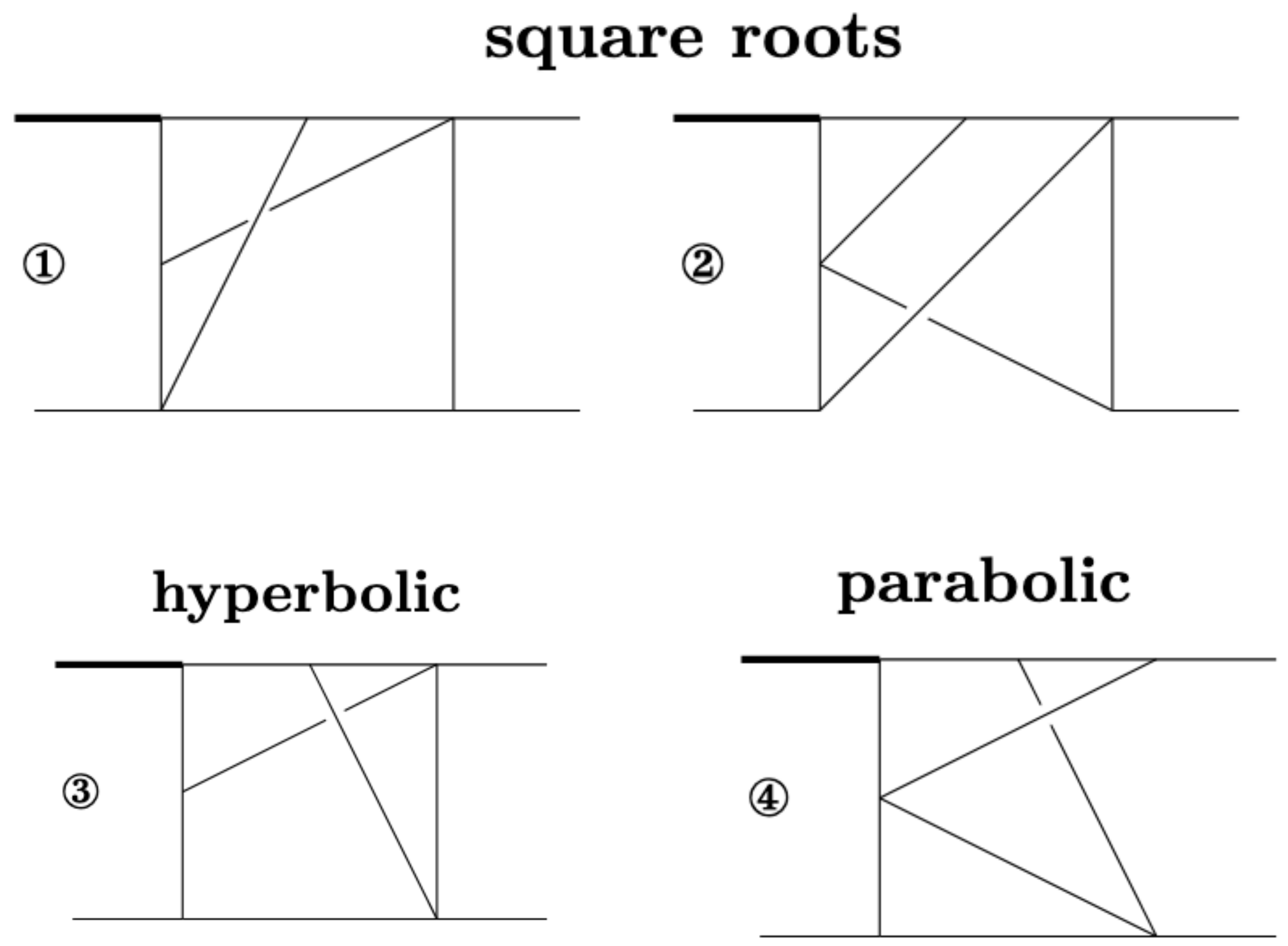
Simple planar letters

$$\{x, y, z, 1 - x, 1 - y, 1 - z\} \longrightarrow l_{1-6}$$

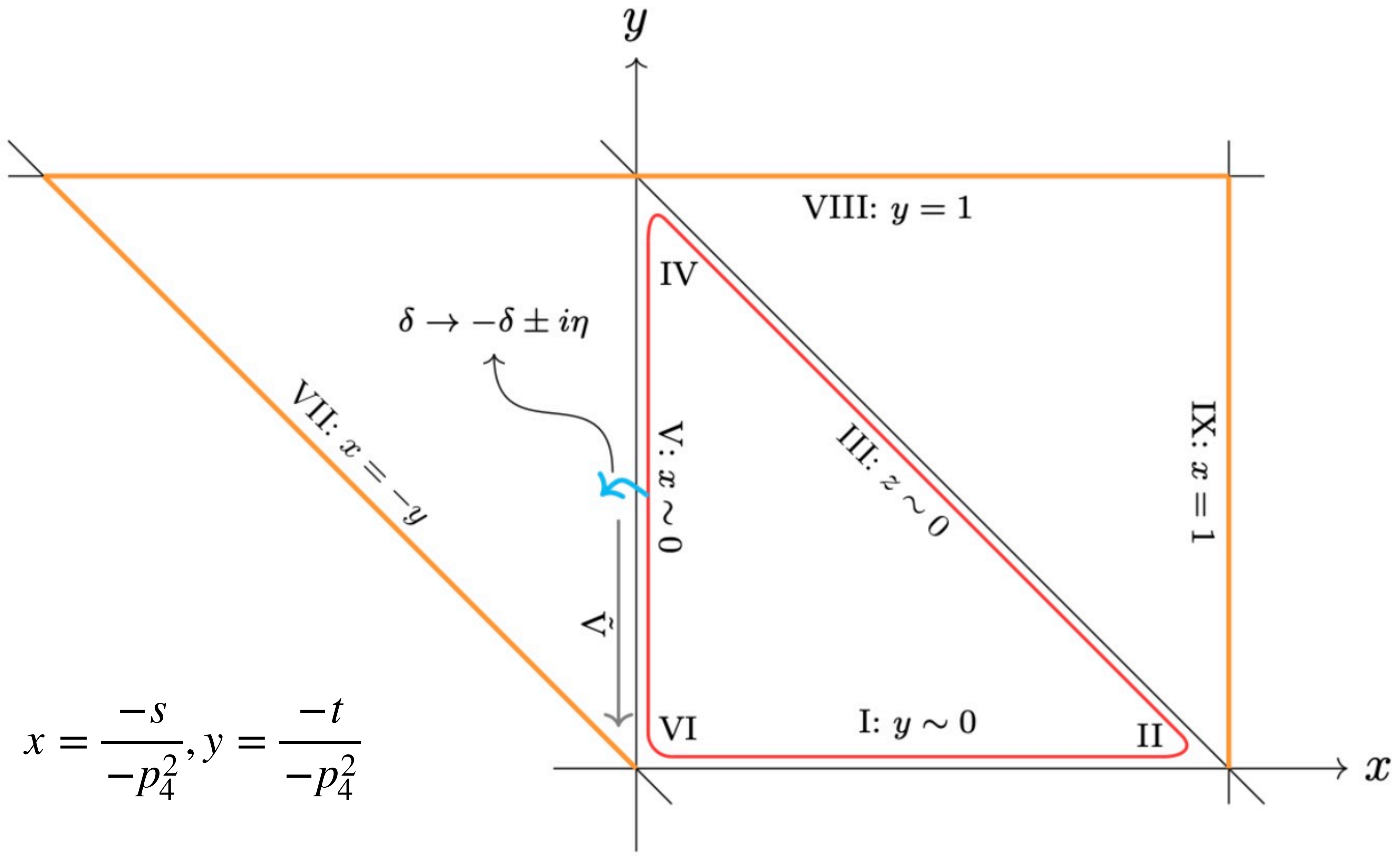
$$x = \frac{-s}{-p_4^2}, y = \frac{-t}{-p_4^2}, z = \frac{-p_4^2 + s + t}{-p_4^2}$$

New letter types

- Sqaure root letter : $\dfrac{xy - \sqrt{xyz}}{xy + \sqrt{xyz}}$ $\quad l_{7-8}$
- Hyperbolic letter : $x^2 + xy + y$ $\quad l_{9-10}, l_{17-20}$
- Parabolic letter : $x^2 - x + y$ $\quad l_{11-16}$



Fixing boundary constants



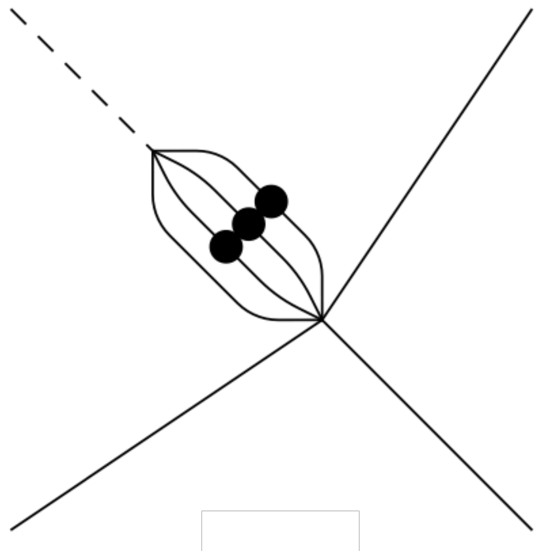
segment	start-/end-point	$x(\delta,t)$	$y(\delta,t)$
I	$P_1 \to P_2$	t	δ
II	$P_2 \to P_3$	$1-\delta\left[(1-t)^2+t^2\right]$	δt^2
III	$P_3 \to P_4$	$1-t-\delta$	t
IV	$P_4 \to P_5$	$\delta(1-t)^2$	$1-\delta\left[(1-t)^2+t^2\right]$
V	$P_5 \to P_6$	δ	$1-t$
VI	$P_6 \to P_7$	δt^2	$\delta(1-t)^2$
VII	$P_7 \to P_8$	$-t(1-\delta)$	t
VIII	$P_8 \to P_9$	$(-1+t)\cup t$	$1-\delta$
IX	$P_9 \to P_{10}$	$1-\delta$	$1-t$

At each segment, your "effective" alphabet is only rational. $\vec{\alpha}_t = \{t, t-\frac{1}{2}, t-1, t-\frac{e^{i\pi/4}}{\sqrt{2}}, t-\frac{e^{-i\pi/4}}{\sqrt{2}}, t \pm e^{i\pi/3}, t \pm e^{2i\pi/3}\}$

At each segment, one can impose the constraint with the information of singularities.

By matching, one can relate the boundary vector in one segment to another.

$\longrightarrow$ Fix all the boundary constants up to one integral.



Non-adjacency conditions

G_2 cluster algebra conjecture the relation between simple planar letters and parabolic letters

[Aliaj, Papathanasiou 2024]

Namely, in terms of the differential equation matrices

$$d\vec{f}\left(\overrightarrow{x};\epsilon\right) = \epsilon\left[\sum_i A_i d\log l_i\left(\overrightarrow{x}\right)\right]\vec{f}\left(\overrightarrow{x};\epsilon\right)$$

$\vec{l}_{1-6} = \{x, y, 1-x-y, 1-x, 1-y, x+y\}$, and $l_{12} = x^2 - x + y, l_{13} = (1-x)^2 - y$

$$A_i \cdot A_j = 0, \qquad (i,j) = (1,12), (1,13), (3,12), (3,13), (12,13)$$

and
$$A_{12} \cdot \left(A_1 + A_2 + A_6 + A_{12}\right) = 0,$$
$$\left(A_1 + A_3 + A_5 + A_{13}\right) \cdot A_{13} = 0$$

These conditions and their kinematic crossing are satisfied for all integral families

Minimal space and grading the functions

Our function space is the vector space of transcendental function

We solve differential equations in canonical form and the solution can be expressed by Chen iterated integrals

$$I\left(\omega_1,\ldots,\omega_n;\vec{x}\right)=\int_\gamma \omega_1\omega_2\cdots\omega_n, \qquad I\left(;\vec{x}\right)=1,$$

where $\omega_i=\omega_i\left(\vec{x}\right)$ are differential forms in the kinematic invariants and
$\gamma=\gamma\left(\vec{1}_0,\vec{x}\right)$ is a curve connecting the base point $\vec{1}_0$ to a generic kinematic point $\vec{x}$.

$\longrightarrow$ **Length n iterated integral has transcendental weight n**

Same definition can be extended to transcendental number $\xi_n=\pi^2,\zeta_n,\ldots,$ which correspond to special values of the iterated integrals, and we also assign weight −1 to ϵ.

Ex) $st^2 J_{E1;1,1,1,1,1,1,1,1,1,1,-1,0,0,0,0}$

$$=\frac{2}{9}+\epsilon\left(-\frac{2}{3}I\left(\omega_1\right)-\frac{2}{3}I\left(\omega_2\right)\right)+\epsilon^2\left(\frac{8\pi^2}{27}-\frac{2}{3}I\left(\omega_1,\omega_4\right)+2I\left(\omega_1,\omega_1\right)+2I\left(\omega_1,\omega_2\right)+2I\left(\omega_2,\omega_1\right)-\frac{2}{3}I\left(\omega_2,\omega_5\right)+2I\left(\omega_2,\omega_2\right)\right)+\mathcal{O}\left(\epsilon^3\right)$$

Minimal space and grading the functions

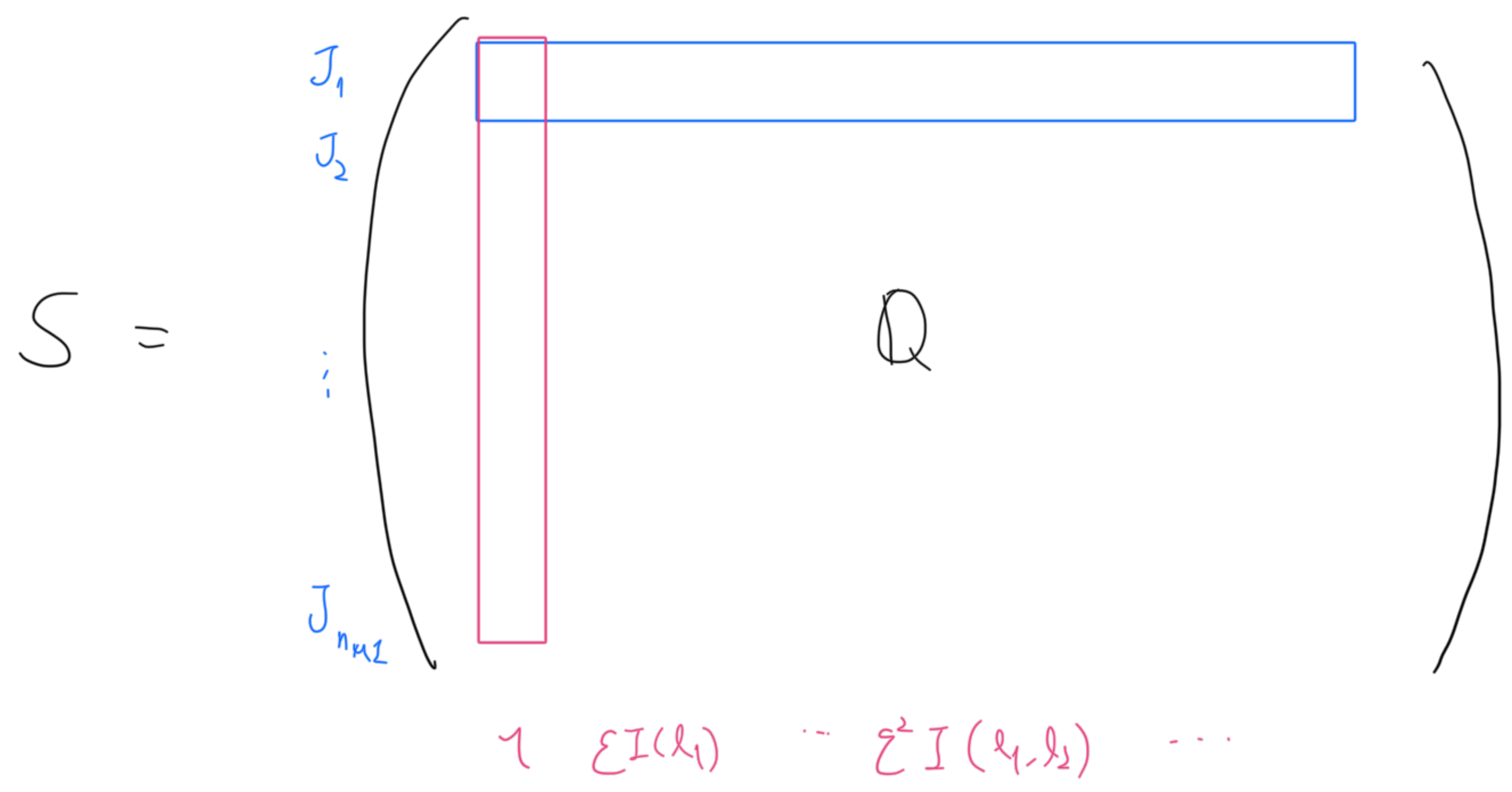
Master integrals are not the minimal basis

The basis of our space is $b_{\mathcal{T}_\omega} = \{\epsilon^{-a}\xi_n^b I\left(\omega_1,\ldots,\omega_c;\vec{x}\right)\}$, with weight $w = a + nb + c$

Scattering amplitudes truncated at an ϵ order containing functions of weight at most w are combinations of some set of transcendental functions with algebraic functions as prefactors,

$$\boxed{\vec{f} = \sum a\left(\vec{x}\right)\cdot b_{\mathcal{T}_\omega}.}$$

$$S_{ij} = \left[f_i\right] b_{\mathcal{T}_{w,j}}, \qquad i = 1,\ldots,n_{MI}, \qquad j = 1,\ldots,\dim\left(\mathcal{T}_w\right).$$



Most of the cases in scattering amplitudes, the number of master integrals is significantly lower than the number of elemental transcendental functions I.

⟶ The number of independent combinations of the functions are bounded by the number of master integrals

Further, not all the master integrals are independent Rank $(S) \leq n_{\text{MI}} \leq$ dim $\left(\mathcal{T}_\omega\right)$

[Jakubčík @HP2 2024]

We can write the full transformation T' as an invertible matrix over numbers,

$$\vec{J}_\omega = T'\cdot\begin{pmatrix} b_{\mathcal{M}_\omega} \\ \vec{0}_\omega \end{pmatrix}$$

$b_{\mathcal{M}_\omega}$: Rank (S)-vector

⟶ **Our minimal basis!**

Minimal space and grading the functions

Grading using physical properties

$$\vec{J}_\omega = T' \cdot \begin{pmatrix} b_{\mathscr{M}_\omega} \\ \vec{0}_\omega \end{pmatrix}$$

$b_{\mathscr{M}_\omega}$: Rank (S) vector

$$\mathscr{M}_\omega = A_{\parallel,\omega} \oplus A_{\perp,\omega}$$

Physical Unphysical

Grading : Organizing the functions in $\mathscr{M}_\omega$ so that $A_{\parallel,\omega}$ and $A_{\perp,\omega}$ does not mixed

All the components of $b_{\mathscr{M}_\omega}$ are independent. $\longrightarrow$ No further cancellation between the functions.

We can remove the component that violates the constraints from physics.

$$\begin{pmatrix} \vec{\psi}_w \\ \vec{0}_w \end{pmatrix} = \begin{pmatrix} T'' & \\ & \mathbb{I} \end{pmatrix} \cdot (T')^{-1} \cdot \vec{J}_w$$
$$\equiv T \cdot \vec{J}_w \,,$$

Minimal space and grading the functions

Higgs plus jet amplitudes in leading color

13812 seemingly different canonical combination

1282 unique combination of functions

	ψ_1 $-\psi_3$	$\boldsymbol{\psi_4}$ $\boldsymbol{-\psi_9}$	ψ_{10} $-\psi_{18}$	ψ_{19} $-\psi_{24}$	$\boldsymbol{\psi_{25}}$ $\boldsymbol{-\psi_{36}}$	ψ_{37} $-\psi_{51}$	ψ_{52} $-\psi_{63}$	$\boldsymbol{\psi_{64}}$ $\boldsymbol{-\psi_{93}}$	ψ_{94} $-\psi_{111}$	ψ_{112} $-\psi_{1281}$	$\boldsymbol{\psi_{1282}}$
equals FF in sYM											✓
violates (3.13)			*	*		*	*		✓		
satisfies (3.14, 3.16)	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
l_{7-20} appears	✓	✓	✓	✓	✓	✓	✓	✓			
• from $\mathcal{O}(\epsilon^4)$	✓	✓									
↪ only parabolic		✓									
↪ also roots	✓										
• from $\mathcal{O}(\epsilon^5)$			✓	✓	✓						
↪ only parabolic					✓						
↪ also hyperbolic			✓	✓							
↪ also roots			✓								
• from $\mathcal{O}(\epsilon^6)$						✓	✓	✓			
↪ only parabolic								✓			
↪ also hyperbolic						✓	✓				
↪ also roots						✓					

Minimal space and grading the functions

Analytic structure of Higgs plus jet amplitudes in leading color

Numerical reduction is enough to see appearance of the functions and hence study the analytic structures

1) The square root letters and hyperbolic letters drop out of the finite reminders in spite of their presence in master integrals

2) For the $Hggg$ amplitudes, we found the hint of maximal transcendentality such as there is no new letters appear at weight 6

$$
\begin{aligned}
\alpha^{(3)}|_{N^3} = \epsilon^{-6} \Big[& \underbrace{+ \frac{1746}{48841}\epsilon^2\,\psi_4 + \frac{22}{289}\epsilon^2\,\psi_6 + \frac{10}{169}\epsilon^2\,\psi_7}_{\epsilon^2 \times \left(\text{parabolic letter from } \mathcal{O}\left(\epsilon^4\right)\right)} \\
& \underbrace{+ \frac{11}{18}\Big(\frac{37}{3}\epsilon\,\psi_{25} + 5\epsilon\,\psi_{26} + 6\epsilon\,\psi_{27} + \frac{11}{2}\epsilon\,\psi_{28} + 8\epsilon\,\psi_{29} - 7\epsilon\,\psi_{30} - 5\epsilon\,\psi_{31} - \frac{5}{2}\epsilon\,\psi_{32} + 2\epsilon\,\psi_{33} - \frac{22}{3}\epsilon\,\psi_{34} + 2\epsilon\,\psi_{35}\Big)}_{\epsilon \times \left(\text{parabolic letter from } \mathcal{O}\left(\epsilon^5\right)\right)}\Big] \\
& + \text{terms with letters } l_{1-6} \text{ only} \\
& + \mathcal{O}(\epsilon)\,.
\end{aligned}
$$

Recently, analytic reduction of the Higgs plus jet amplitudes at the leading color was done and it has been shown that indeed it is satisfied

$$
\alpha^{(3)}\Big|_{N^3} = \mathcal{G}_2^{(3)} + \sum_{i=1}^{1281} c_i \psi_i \text{ with } c_i = \mathcal{O}(\epsilon)
$$

, where $\mathcal{G}_2^{(3)}$ is three loop tr $\left(\phi^2\right)$ form factor in N=4 sYM

[Chen. Guan, Mistlberger 2025]

Minimal space and grading the functions

Analytic structure of Higgs plus jet amplitudes in leading color

3) At weight 4 and 5, all six parabolic letters l_{11-16} are still present in the finite remainder

4) For $Hq\bar{q}g$ amplitudes, only 4 parabolic letters appear

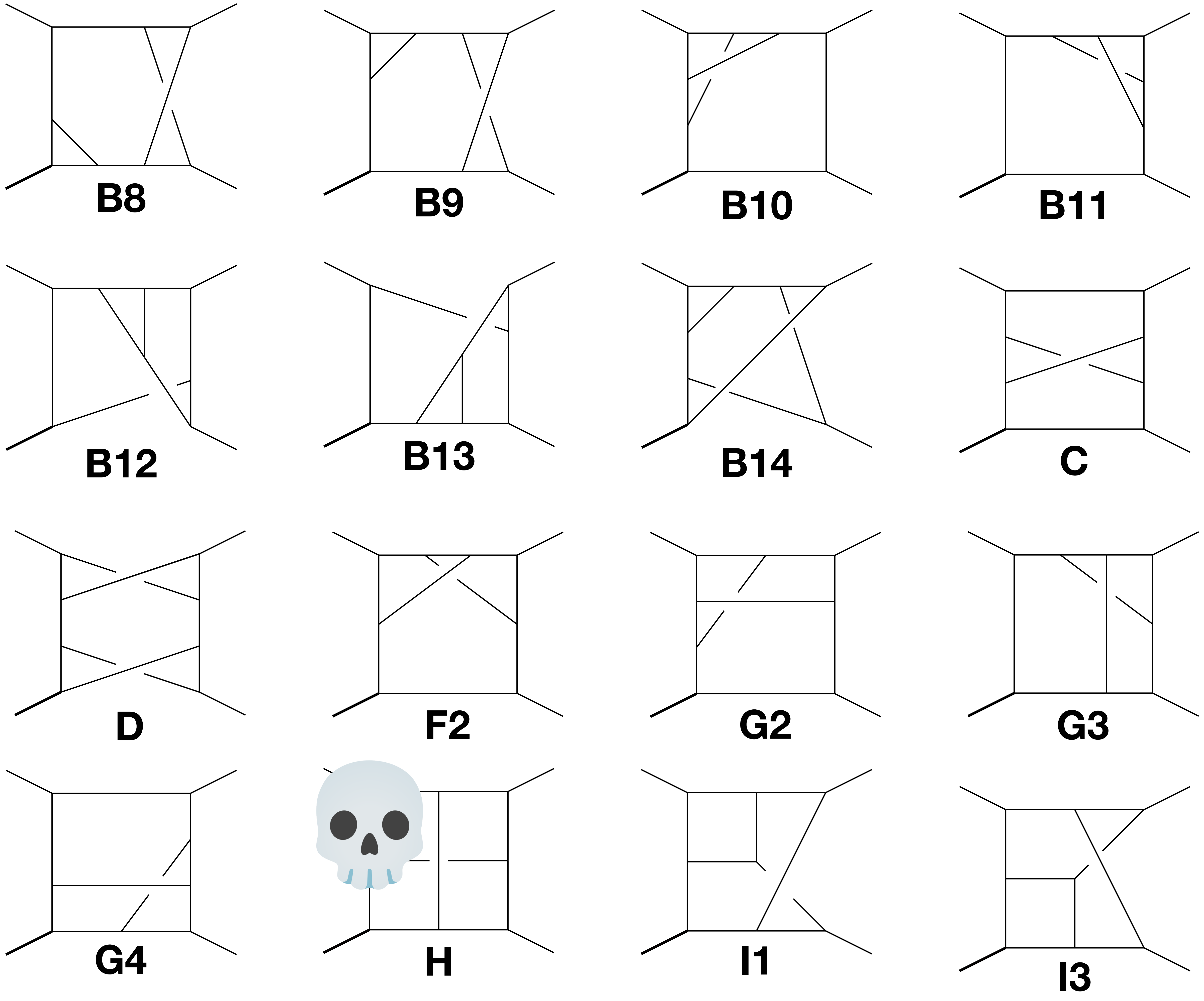
5) The new quadratic letters appear only as the fourth integration kernel. All iterated integrals of weight 5 or 6 with the parabolic letters as fifth or sixth integration kernel drop from the amplitudes. This observation can be extended to the functions with transcendental constants.

Ex) $\pi^2 I\left(l_i, l_{11}, l_j, l_k\right), \zeta_3 I\left(l_{11}, l_j, l_k\right)$ with $i, j, k \in \{1,\ldots,6\}$

and terms with π^4 or ζ_5 and a parabolic letter are completely excluded.

$$\vec{\alpha}_{H+\text{jet}}\Big|_{\text{L.C}} = \left\{x, y, 1-x-y, 1-x, 1-y, x+y, x^2-x+y, y^2-y+x, \left(x+y\right)^2-x, (1-x)^2-y, \left(x+y\right)^2-y, \left(1-y\right)^2-x\right\}$$

New integral families @sub-leading color



+ planar families

Family	# MI(#topsector)	Status
B8	131(0)	Canonical Deq
B9	58(0)	Canonical Deq
B10	32(0)	Canonical Deq
B11	36(0)	Canonical Deq
B12	114(0)	Canonical Deq
B13	112(0)	Canonical Deq
B14	52(0)	Canonical Deq
C	173(4)	Canonical Deq
D	121(4)	Canonical Deq
F2	136(4)	Canonical Deq
G2	174(4)	Canonical Deq
G3	176(4)	Canonical Deq
G4	212(5)	Canonical Deq
H	371(19)	Canonical subsector
I1	277(8)	Canonical Deq
I3	333(12)	Canonical Deq

(Family : reducible top sector)

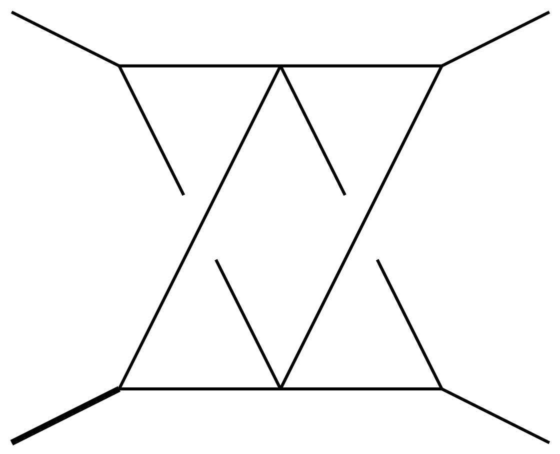
Function space of integrals @ subleading color

- **We are conjecture to have the same alphabet as leading color**

$$\vec{\alpha} = \{p_4^2, s, t, p_4^2 - s - t, p_4^2 - s, p_4^2 - t, s + t, \frac{(p_4^2 - s - t)s - R}{(p_4^2 - s - t)s + R}, \frac{st - R}{st + R}, p_4^4 - t\left(p_4^2 + s\right), p_4^4 - s\left(p_4^2 + t\right),\ t^2 + p_4^2(s - t), s^2 - p_4^2(s - t), -p_4^2 t + \left(p_4^2 - s\right)^2\}$$

with $R = \sqrt{-p_4^2 s(p_4^2 - s - t)t}$

- **New contribution for the square root letter**



$\longrightarrow$ Square root letters $l_{7,8}$ from $\mathcal{O}\left(\epsilon^3\right)$

This is also the source of hidden region $I \sim \lambda^{-\frac{1}{2}-3\epsilon}$

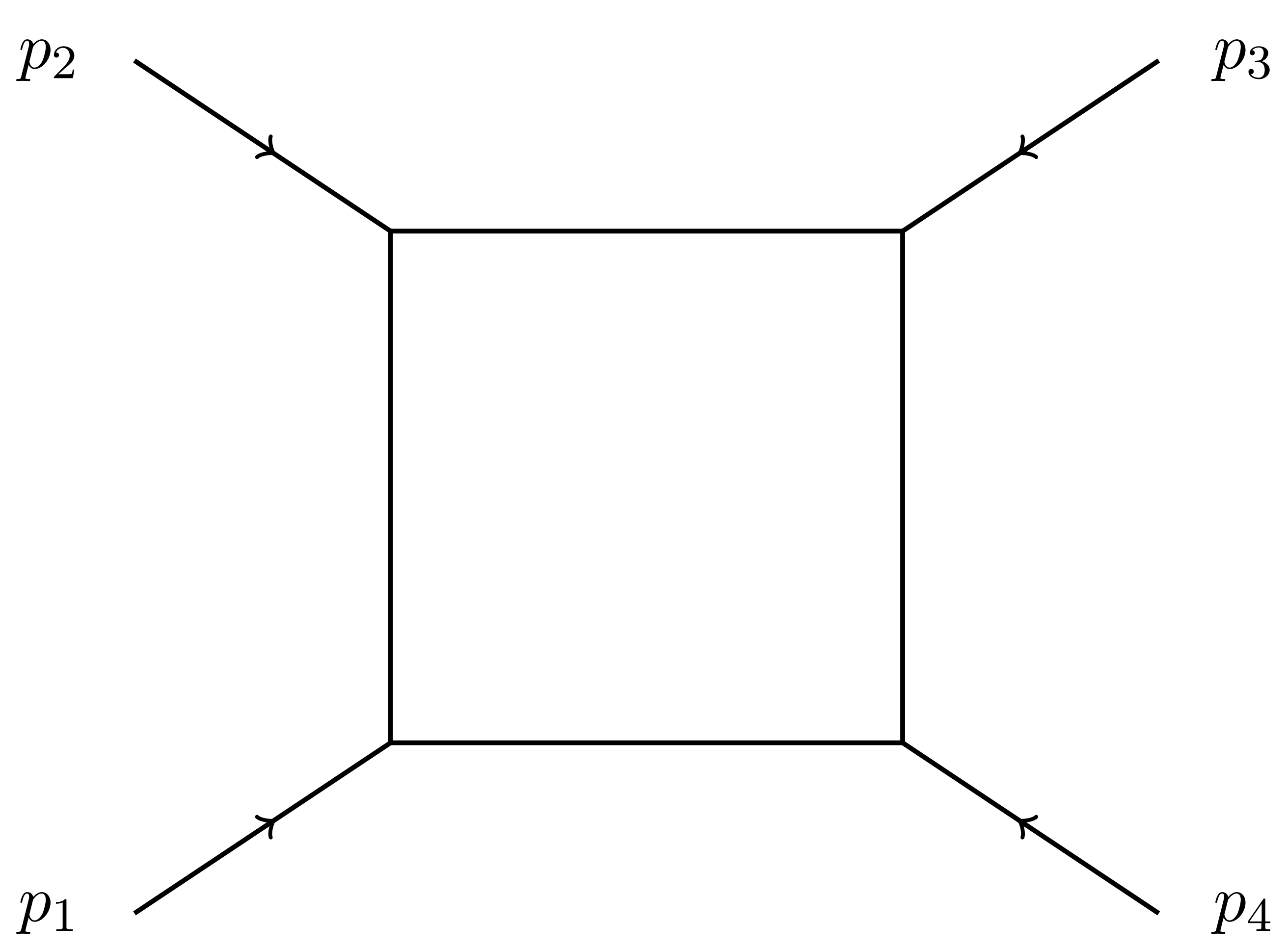
[Gardi, Herzog, Jones, Ma 2024]

This can be seen from the differential equations : $\partial_{p_4^2} I = \left(-\frac{1}{2} - 3\epsilon\right) I$

Finding canonical basis of family H

Improved Baikov representations [Henn, Korchemsky, Mistlberger, 2019]

Idea : Using the fact that the box integral has dlogform as



$$dI = \frac{st d^4 k}{k^2 \left(k+p+1\right)^2 \left(k+p_1+p_2\right)^2 \left(k+p_1+p_2+p_3\right)^2} = \pm \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4},$$

where

$$\alpha_1 = \frac{k^2}{\left(k-k_\pm^*\right)^2}, \quad \alpha_2 = \frac{\left(k+p_1\right)^2}{\left(k-k_\pm^*\right)^2}, \quad \alpha_3 = \frac{\left(k+p_1+p_2\right)^2}{\left(k-k_\pm^*\right)^2}, \quad \alpha_4 = \frac{\left(k+p_1+p_2+p_3\right)^2}{\left(k-k_\pm^*\right)^2},$$

and $k_\pm^*$ are solutions of $k^2 = \left(k+p_1\right)^2 = \left(k+p_1+p_2\right)^2 = \left(k+p_1+p_2+p_3\right)^2 = 0.$

$\longrightarrow$ In this way, one can obtain the expression similar to Baikov representation but with simpler terms.

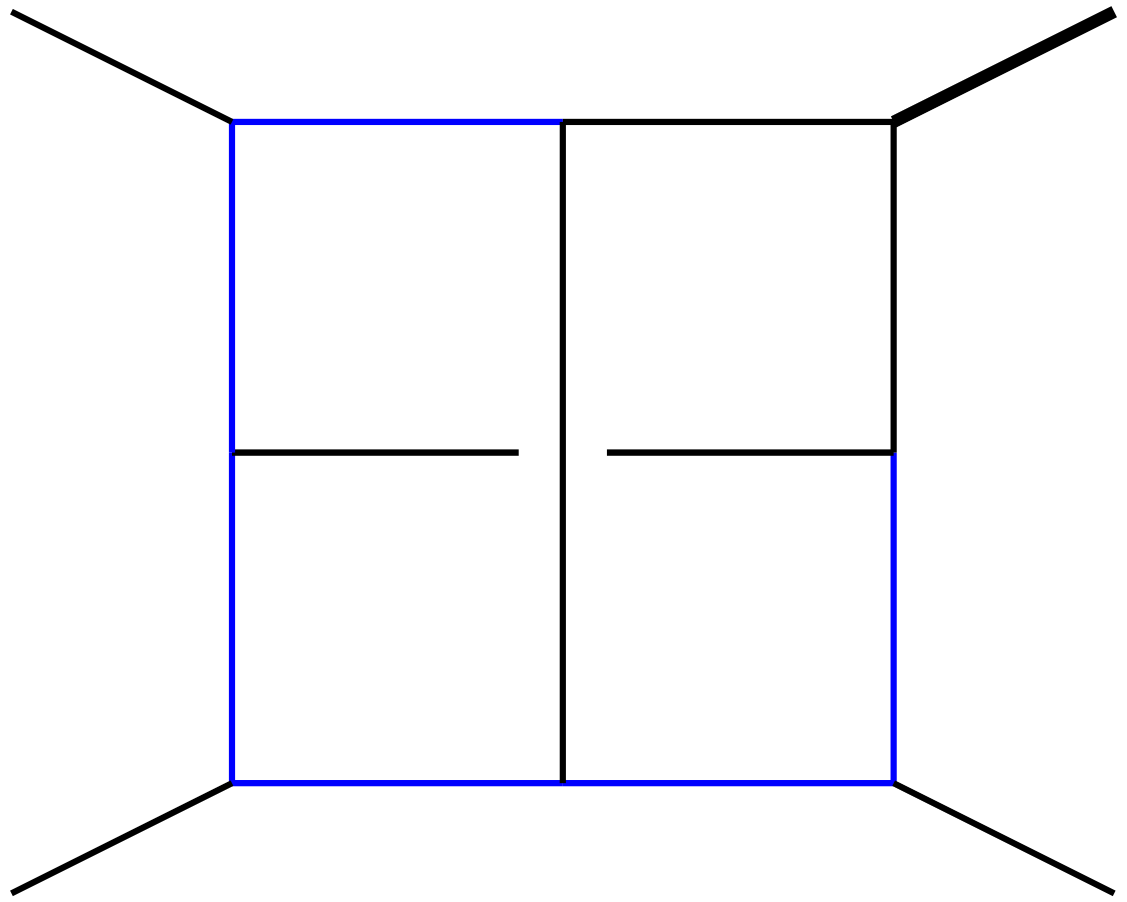
Namely, one can relate each propagator to the new variables α's up to the factor $\left(k-k_\pm^*\right)^2$ and cutting propagator is done by setting corresponding $\alpha_i \to 0$

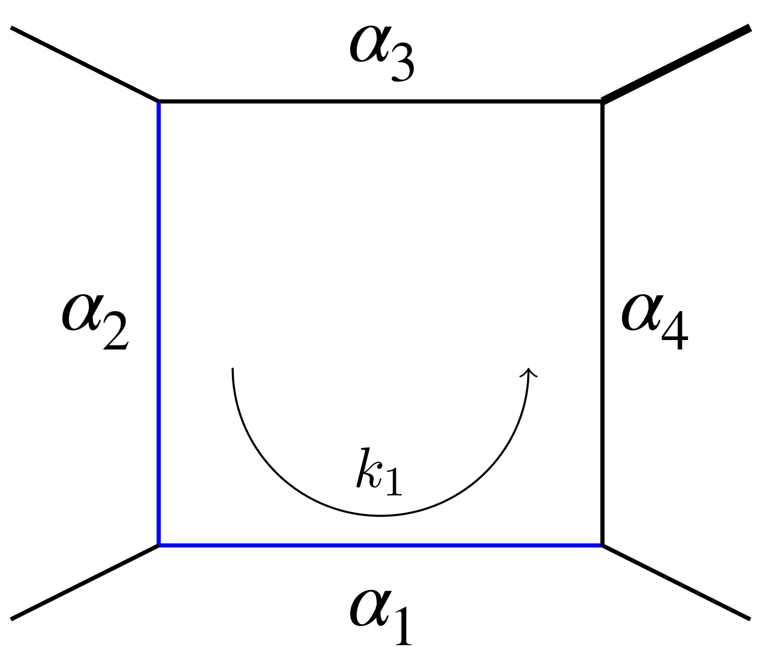
Finding canonical basis of family H

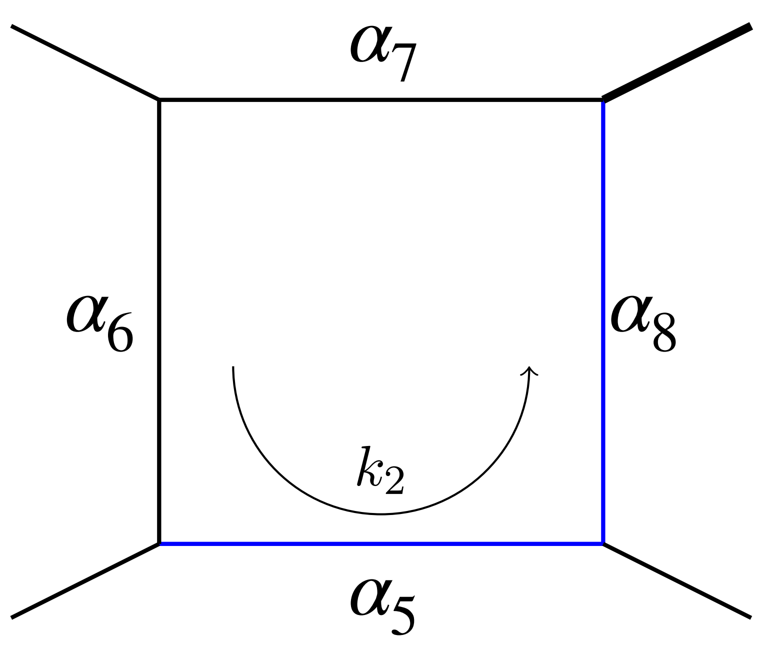
Improved Baikov representations

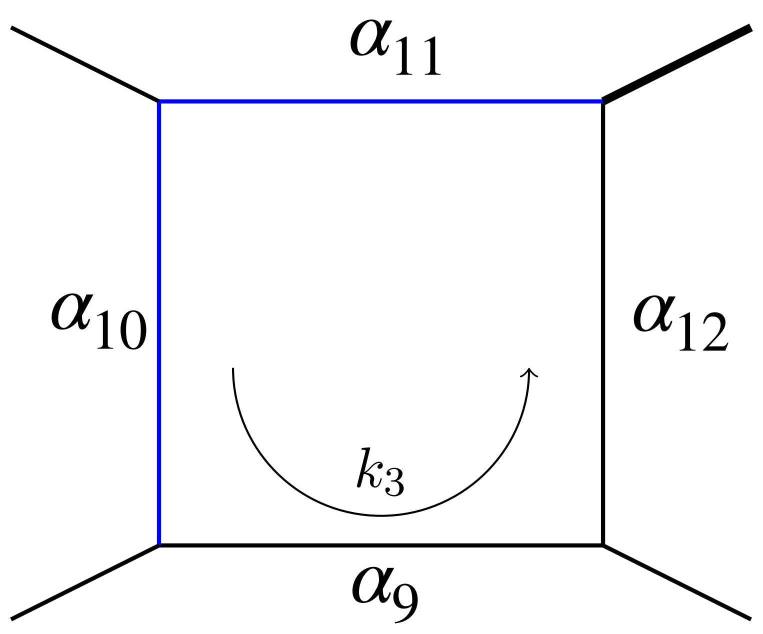
[Henn, Korchemsky, Mistlberger, 2019]

For H,









$$dI_{i\text{–th box}} = \frac{st d^4 k_i}{k_i^2 \left(k_i + p + 1\right)^2 \left(k_i + p_1 + p_2\right)^2 \left(k_i - p_3\right)^2} = \pm \frac{d\alpha_{4(i-1)+1}}{\alpha_{4(i-1)+1}} \frac{d\alpha_{4(i-1)+2}}{\alpha_{4(i-1)+2}} \frac{d\alpha_{4(i-1)+3}}{\alpha_{4(i-1)+3}} \frac{d\alpha_{4(i-1)+4}}{\alpha_{4(i-1)+4}},$$

where $\alpha_{4(i-1)+1} = \dfrac{k_i^2}{\left(k_i - k_{i,\pm}^*\right)^2}, \quad \alpha_{4(i-1)+2} = \dfrac{\left(k_i + p_1\right)^2}{\left(k_i - k_{i,\pm}^*\right)^2}, \quad \alpha_{4(i-1)+3} = \dfrac{\left(k_i + p_1 + p_2\right)^2}{\left(k_i - k_{i,\pm}^*\right)^2}, \quad \alpha_{4(i-1)+4} = \dfrac{\left(k_i - p_3\right)^2}{\left(k_i - k_{i,\pm}^*\right)^2},$

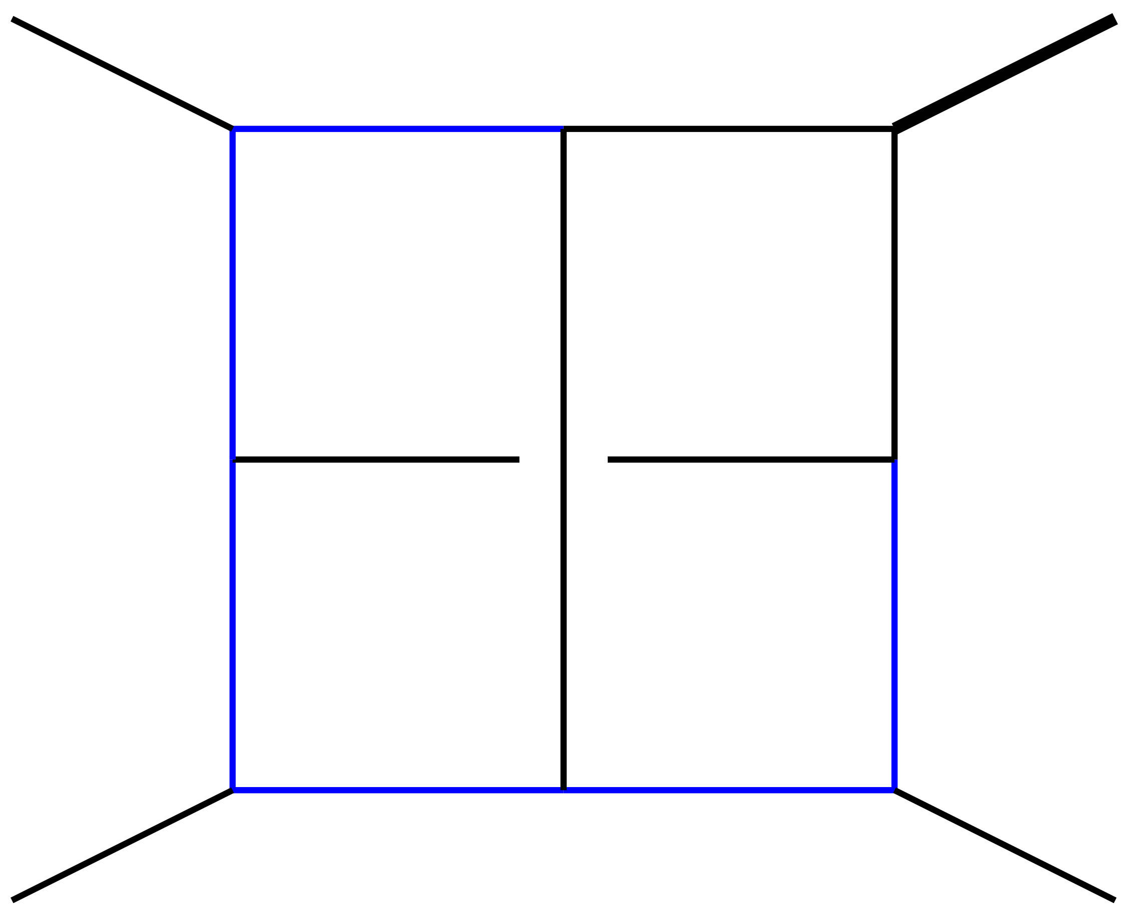
and $k_{i,\pm}^*$ are solutions of $k_i^2 = \left(k_i + p_1\right)^2 = \left(k_i + p_1 + p_2\right)^2 = \left(k_i - p_3\right)^2 = 0.$

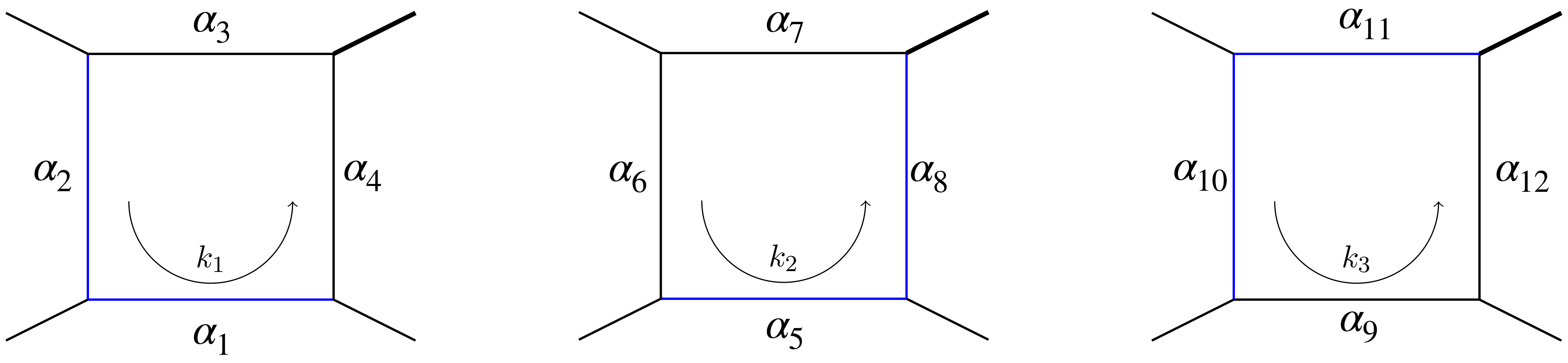
Finding canonical basis of family H

Improved Baikov representations

[Henn, Korchemsky, Mistlberger, 2019]

For H,





$$dI_H = \frac{d\alpha_1 \ldots d\alpha_{12}}{\alpha_1\alpha_2\alpha_5\alpha_8\alpha_{10}\alpha_{11}} \times \left(\text{rational function of } \alpha_1, \ldots, \alpha_{12}\right)$$

- The overall expression becomes simpler
- Taking cut of the propagators represented by α's is trivial

$\longrightarrow$ We manage find canonical top sector at specific three propagator cut

But it is still not enough, when we reconstruct the differential equations, there are terms with $\sim \mathcal{O}\left(\epsilon^{10}\right)$

Summary & Outlook

- We computed all master integrals contributing leading color three-loop Higgs plus jet amplitudes
- By grading the functions and numerical reduction, we could study the analytic properties of the amplitudes
- Recently, analytic reduction was computed for leading color Higgs plus jet amplitudes and the maximal transcendentality was checked. [Chen. Guan, Mistlberger 2025]
- For the subleading color amplitudes, there are new contributions of integral families. We found the canonical differential equations for all the families except one integral family called H
- One needs to check if the maximal transcendentality holds for subleading color by computing subleading part of tr $\left(\phi^2\right)$

Thank You!