Annihilators and D-modules for Feynman integrals

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arXiv:2506.10456, 2508.04309

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Matter To The Deepest 2025

September 15

Motivation

- Feynman integrals are complicated functions whose computation is difficult.
- We would like to understand and utilize underlying mathematical structures.
- $\mathcal{D}$-module approach to Feynman integrals [1-3].
- Can we find an algorithm to construct differential operators annihilating Feynman integrals?
- Can we build an annihilating ideal?
- Does there exist a generalization of the GKZ [4,5] construction for Feynman integrals?

Feynman integrals

Feynman parametrization:

$$\pi(s) = \int_\Gamma \frac{\mathrm{U}(\alpha)^\kappa}{\mathrm{F}(\alpha, s)^\eta} \prod_{i=1}^{n} \alpha_i^{\nu_i - 1} \mu \equiv \int_\Gamma \Omega \,,$$

- $\alpha = (\alpha_1, \ldots, \alpha_n)$: integration variables,
- $s = (s_1, \ldots, s_m)$: external variables,
- $\mu \equiv \sum_{i=1}^{n} (-1)^{i-1} \alpha_i \, d\alpha_1 \wedge \cdots \wedge \widehat{d\alpha_i} \wedge \cdots \wedge d\alpha_n \,,$
- $\Gamma = \{(\alpha_1, \ldots, \alpha_n) | \sum_{i=1}^{n} \alpha_i = 1 \,, \alpha_i \geq 0\} \,,$
- U: homogeneous polynomial of degree ℓ,
- F: homogeneous polynomial of $\ell + 1$ and also homogeneous in s of degree 1,
- $\kappa = \sum_{i=1}^{n} \nu_i - (\ell + 1)d/2$ and $\eta = \sum_{i=1}^{n} \nu_i - \ell d/2$,
- d: space-time dimension.

assumption: $\nu_i = 1$ for $i = 1, \ldots, n$

$\mathcal{D}$-modules

- m-th rational Weyl algebra:

$$\mathcal{R}_m = \mathbb{C}(x_1, \ldots, x_m)\langle \partial_1, \ldots, \partial_m \rangle \text{ with } [\partial_i, c(s)] = \frac{\partial c(s)}{\partial s_i} \,,$$

- Any $L \in \mathcal{R}_m$ can be written in a normally ordered form as

$$L = \sum c_q(s)\, \partial^q \,,$$

  multi-index notation, $\partial^q \equiv \partial_{s_1}^{q_1} \ldots \partial_{s_m}^{q_m}$, $q = \{q_1, \ldots, q_m\}$.
- Holonomic rank

$$r(\mathcal{I}) = \dim_{\mathbb{C}(x)} \mathbb{C}(x)[\xi] / \left(\mathbb{C}(x)[\xi] \cdot \mathrm{char}(\mathcal{I})\right) = \dim_{\mathbb{C}(x)} \left(\mathcal{R}/\mathcal{R}\mathcal{I}\right) \,.$$

Macaulay matrix

- Macaulay matrix

$$M = (\partial^I \mathbf{A}_p) = 0\,, \quad |I| = 0, \ldots, l\,,$$

  where l is a chosen maximal degree of the derivatives, *i.e.* $\partial^I = \partial_{s_1}^{i_1} \cdots \partial_{s_m}^{i_m}$ with $i_1 + \ldots i_m = |I|$.

- Standard monomials

$$\mathrm{Std} = \{\partial^k\}\,,$$

- Holonomic rank

$$r(\mathcal{I}) = |\mathrm{Std}|\,.$$

Differential operators

- Goal:

$$\mathcal{I} = \{\mathbf{A}_p \mid \mathbf{A}_p \pi(s) = 0\}\,,$$

- Shape:

$$\mathbf{A}_p \equiv \mathbf{D}_p + \mathbf{D}_{p-1}\,,$$

- Ansatz w.r.t. external variables:

$$\mathbf{D}_p = \sum_{I, |I| = p} c_I\, \partial_s^I \Longrightarrow \mathbf{D}_p\, \pi(s) = \int_\Gamma \frac{\mathrm{N}_p}{\mathrm{F}(\alpha, s)^p} \Omega\,,$$

  where c_I are coefficients we want to fix.

- Action on $\pi(s)$:

$$\mathbf{A}_p\, \pi(s) = (\mathbf{D}_p + \mathbf{D}_{p-1})\, \pi(s) = \int_\Gamma d\gamma = \int_{\partial\Gamma} \gamma = 0\,.$$

Griffiths-Dwork reduction

- Projective differential form [6-13]:

$$\gamma \equiv \frac{1}{\mathrm{F}^{k-1}} \sum_{i<j} (-1)^{i+j} \left(\alpha_i \lambda_j - \alpha_j \lambda_i\right) \frac{\mathrm{U}(\alpha)^{\kappa}}{\mathrm{F}(\alpha, s)^{\eta}}$$
$$\times\, d\alpha_1 \wedge \ldots \widehat{d\alpha_i} \wedge \ldots \wedge \widehat{d\alpha_j} \ldots \wedge d\alpha_n \,.$$

- Total derivative

$$d\gamma = (\eta + k - 1) \frac{\sum_{i=1}^{n} \lambda_i \, \partial_i \mathrm{F}}{\mathrm{F}^k} \Omega - \frac{\sum_{i=1}^{n} \nabla_i \, \lambda_i}{\mathrm{F}^{k-1}} \Omega \,,$$

with the *twisted covariant derivative* ∇_i, defined as,

$$\nabla_i \equiv \partial_i + \omega_i \wedge \,, \quad \text{with} \quad \omega_i \equiv \partial_i \log(\mathrm{U}^{\kappa}) \,,$$

Necessary conditions

A) Boundary condition:

$$\lambda_i\Big|_{\alpha_i=0} = 0\,,$$

B) Jacobian ideal membership condition:

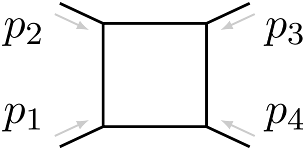
$$\mathrm{N}_p = \sum_{i=1}^{n} \lambda_i\, \partial_i \mathrm{F}\,,$$

implying that $\mathrm{N}_p \in \langle \partial_1 \mathrm{F}, \ldots, \partial_n \mathrm{F} \rangle$

C) Twisted covariant derivative condition:

$$\mathrm{N}_{p-1} = \sum_{i=1}^{n} \nabla_i\, \lambda_i\,.$$

Example: Massless Box



Symanzik polynomials:

$$\mathrm{U} = \sum_{i=1}^{4} \alpha_i \,, \quad \text{and} \quad \mathrm{F} = \alpha_1 \alpha_3 s + \alpha_2 \alpha_4 t \,,$$

with exponents $\kappa = 4 - d$ and $\eta = 4 - \frac{d}{2}$.

Example: Massless Box - First order operator

$$\mathbf{D}_1\pi(s,t) = (c_1\partial_s + c_2\partial_t)\,\pi(s,t) = \int_\Gamma \frac{\mathrm{N}_1}{\mathrm{F}}\Omega\,,$$

$$\mathbf{D}_0\pi(s,t) = \pi(s,t) = \int_\Gamma \Omega \Longrightarrow \mathrm{N}_0 = 1\,,$$

where

$$\mathrm{N}_1 = \alpha_1\alpha_3 c_1\left(\frac{d}{2}-4\right) + \alpha_2\alpha_4 c_2\left(\frac{d}{2}-4\right)\,.$$

- Conditions A and B $\rightarrow$ Syzygy equation:

$$\sum_{i=1}^{4} q_i\,\alpha_i\partial_i\mathrm{F} = q_0\,\mathrm{N}_1\,,$$

  There are four generators of Syzygies $g_1,\ldots,g_4$ and we choose the following combination:

  $q \equiv (q_1,\ldots,q_0) = r_1 g_1 + r_2 g_2 + g_3\,,$ where r_i are parameters.

Example: Massless Box - First order operator

- $\mathrm{N}_1 = \sum_{i=1}^{4} \lambda_i \partial_i \mathrm{F}$ where $\lambda_i = \frac{q_i}{q_0}\alpha_i$, explicitly:

$$\lambda_1 = \frac{\alpha_1(c_1(d-8)t + r_1)}{2st}, \quad \lambda_2 = \frac{\alpha_2(c_2(d-8)s + r_2)}{2st},$$
$$\lambda_3 = -\frac{\alpha_3 r_1}{2st}, \quad \lambda_4 = -\frac{\alpha_4 r_2}{2st}.$$

- Condition C gives:

$$\left\{ c_2 \to -\frac{4t}{(d-8)^2}, c_1 \to -\frac{4s}{(d-8)^2}, r_1 \to \frac{2st}{d-8}, r_2 \to \frac{2st}{d-8} \right\}.$$

- First order operator:

$$\mathbf{A}_1 = s\partial_s + t\partial_t + \left(4 - \frac{d}{2}\right),$$

where we took into account the prefactor $(\eta + k - 1)$ in the Griffiths-Dwork formula.

Example: Massless Box

- Generators of $\mathcal{I}$:

$$\mathbf{A}_E = s\partial_s + t\partial_t + \left(4 - \frac{d}{2}\right) ,$$

$$\mathbf{A}_3 = p_{3,0}\, \partial_s^3 + p_{2,1}\, \partial_s^2 \partial_t + p_{1,2}\, \partial_s \partial_t^2 + p_{0,2}\, \partial_t^2 .$$

- Holonomic rank:

$$r(\mathcal{I}) = 3 .$$

- singular locus:

$$\mathrm{sing}(\mathcal{I}) = st(s+t) .$$

- Picard-Fuchs operator:

$$\mathbf{A}_{\mathrm{PF}} = (\partial_s + r_1) \cdot (\partial_s + r_2) \cdot (\partial_s + r_3) .$$

Example: Massless Box

- Differential system in the basis $\vec{f} = (I_{1,1,1,1}, \partial_s I_{1,1,1,1}, \partial_s^2 I_{1,1,1,1})^T$:

$$\partial_s \vec{f} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{d-8}{2s^2(s+t)} & \frac{(d-8)^2(-t)-(64-6d)s}{4s^2(s+t)} & \frac{(d-18)s+2(d-9)t}{2s(s+t)} \end{pmatrix} \vec{f}$$

- Change to the basis $\vec{g} = (I_{1,1,1,1}, I_{1,0,1,0}, I_{0,1,0,1})$:

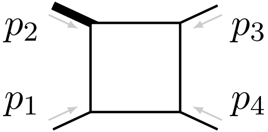
$$\vec{f} = B\vec{g}\,,$$

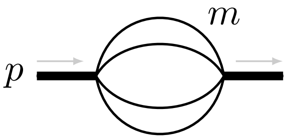
- Differential system in new basis:

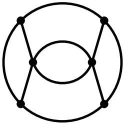
$$\partial_s \vec{g} = \begin{pmatrix} \frac{(d-6)t-2s}{2s(s+t)} & \frac{2(d-3)}{s^2(s+t)} & \frac{6-2d}{s^2t+st^2} \\ 0 & \frac{d-4}{2s} & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{g}$$

More applications

Hypergeometric ${}_2F_1$ and ${}_3F_2$







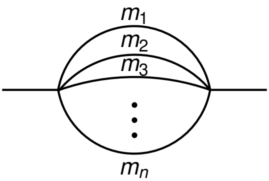
and more

Generators of the $\mathcal{D}$-ideal

Conjecture

Under genericity conditions on U, F, κ, η *and when p is sufficiently large, the holonomic rank r of the left ideal of the rational Weyl algebra* R_m *generated by* $\mathbf{D}_i + \mathbf{D}_{i-1}$, $i \leq p$ *is* $r = |\chi(V)|$ *where* $\chi(V)$ *is the Euler characteristic of the zero set V of* $\mathrm{U}(\alpha)\mathrm{F}(\alpha, s)$ *in the* α*-space* $\{\alpha_1 \cdots \alpha_n \neq 0\}$ *with generic s-parameters values.*

Banana integrals



Symanzik polynomials:

$$\mathrm{U} = \left(\prod_{k=1}^{n} \alpha_k\right) \left(\sum_{j=1}^{n} \frac{1}{\alpha_j}\right)$$

$$\mathrm{F} = \prod_{i=1}^{n} \alpha_i s_0 - \mathrm{U} \sum_{i=1}^{n} \alpha_i s_i \,,$$

with exponents $\kappa = n - \frac{\ell+1}{2} d$, and $\eta = n - \frac{\ell d}{2}$.

Banana integrals

Theorem

The following differential operators annihilate the ℓ-loop banana integral

$$1)\quad \mathcal{A}_E = \sum_{i=0}^{n} s_i \partial_i + \left(n - \frac{\ell d}{2}\right),$$

$$2)\quad \mathcal{A}_i = -s_0 \partial_0^2 + s_i \partial_i^2 - \frac{d}{2}\partial_0 + \left(2 - \frac{d}{2}\right)\partial_i\,, \quad i = 1, \ldots, n\,,$$

$$3)\quad \mathcal{A}_s = \left(\prod_{i=0}^{n} \partial_i\right)\left(\sum_{i=0}^{n} \frac{1}{\partial_i}\right).$$

Banana integrals

Theorem

The singular locus of the ideal $\mathcal{I}_B$ generated by the operators $\mathcal{A}_E$, $\mathcal{A}_s$ and $\mathcal{A}_i$, $i = 1, \dots n$, is contained in

$$\operatorname{sing}(\mathcal{I}_B) \subseteq \prod_{i=0}^{n} s_i \prod_{\{\delta_\pm\}} \left(s_0 - \left(\sum_{j=1}^{n} \delta_\pm \sqrt{s_j} \right)^2 \right), \text{ where } \delta_\pm = \pm 1\,,$$

and the second product runs over all projectively nonequivalent ways to assign $\delta_\pm$ in the sum.

Holonomic Rank:

ℓ	1	2	3	4	5	6	7	8
$r(\mathcal{I}_B)$	3	7	15	31	63	127	255	511

Banana integrals

Conjecture

The holonomic $\mathcal{D}$-ideal of rank $2^{\ell+1}-1$ for the ℓ-loop banana integral is generated by the following $\ell+3$ operators

$$\mathcal{A}_E = \sum_{i=0}^{n} x_i\partial_i + \left(n - \frac{\ell d}{2}\right) ,$$

$$\mathcal{A}_i = -x_0\partial_0^2 + x_i\partial_i^2 - \frac{d}{2}\partial_0 + \left(2 - \frac{d}{2}\right)\partial_i\,, \quad i = 1, \ldots, n\,,$$

$$\mathcal{A}_s = \left(\prod_{i=0}^{n}\partial_i\right)\left(\sum_{i=0}^{n}\frac{1}{\partial_i}\right) .$$

Summary

- We provided an algorithm based on the Griffiths-Dwork reduction together with an additional boundary condition to construct differential operators annihilating Feynman integrals.
- Constructed operators are of the form: $\mathbf{A}_p \equiv \mathbf{D}_p + \mathbf{D}_{p-1}$.
- We applied the algorithm to Feynman integrals, hypergeometric functions and Witten diagrams.
- For the examined examples the operators form a generating $\mathcal{D}$-ideal whose holonomic rank agrees with the number of master integrals.
- We found a set of simple differential operators that annihilate banana integrals.
- Singular locus of the banana operators is contained in the set of Landau singularities of the first and second type.
- We checked the holonomic rank up to $\ell = 8$ and it agrees with the formula for the number of master integrals for banana graphs.

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Extra slides

Massless box - Full expressions

$$\mathbf{A}_3 = p_{3,0}\,\partial_s^3 + p_{2,1}\,\partial_s^2\partial_t + p_{1,2}\,\partial_s\partial_t^2 + p_{0,2}\,\partial_t^2\,,$$

where

$$\begin{aligned} p_{3,0} &= -4s^2\,, \\ p_{2,1} &= s((d-8)^2 s + 2(26-3d)t)\,, \\ p_{1,2} &= -t(2(28-3d)s + (d-8)^2 t)\,, \\ p_{0,2} &= 2(d-12)t\,. \end{aligned}$$

Massless box - Full expressions

$$\mathbf{A}_{\mathrm{PF}} = (\partial_s + r_1) \cdot (\partial_s + r_2) \cdot (\partial_s + r_3) \ ,$$

where

$$\begin{aligned} r_1 =& - \frac{d-10}{2s} - \frac{1}{u}\,, \\ r_2 =& \frac{1}{s} - \frac{1}{u}\,, \\ r_3 =& - \frac{1}{2}\left(\frac{d-4}{u} + \frac{d-6}{s}\right)\ , \end{aligned}$$

and $u = -s - t$.

Massless box - Full expressions

$$B = \begin{pmatrix} 1 & 0 & 0 \\ \frac{dt-2s-6t}{2s(s+t)} & \frac{2(d-3)}{s^2(s+t)} & -\frac{2(d-3)}{st(s+t)} \\ \frac{\left(d^2-14d+48\right)t^2-8(d-6)st+8s^2}{4s^2(s+t)^2} & \frac{(d-3)((d-12)s+2(d-7)t)}{s^3(s+t)^2} & \frac{(d-3)(6s-(d-8)t)}{s^2t(s+t)^2} \end{pmatrix}$$