Monte Carlo phase space integration of multiparticle cross sections with `carlomat_4.5`

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Matter To The Deepest
Recent Developments In Physics Of Fundamental Interactions
XLVI International Conference of Theoretical Physics
Katowice, 15-19 September, 2025

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- Multichannel phase space parameterization
- Automatic generation of different phase space parameterizations and adaptation of integration weights in the multichannel probability distribution
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- Some illustrative results
- Summary

Based on a publication
K. Kołodziej, Computer Physics Communications **315** (2025) 109697, e-Print: 2504.00155 [hep-ph].

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Questions about

- the non-Abelian nature of the SM gauge symmetry group and
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can be directly addressed in processes of a few heavy particles production at a time, such as

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Reactions with multiparticle final states must also be taken into account if one wants to determine precisely hadronic contributions to the vacuum polarization which influences precision of theoretical predictions for the muon $g-2$ anomaly and plays an important role in the evolution of the fine structure constant $\alpha(Q^2)$ from the Thomson limit to high energy scales.

The hadronic contributions to the vacuum polarization can be determined through dispersion relations from the energy dependence of the ratio

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The phase space integration element is parameterized in the following way

$$\mathrm{d}^{3n_f-4}Lips = (2\pi)^4\delta^{(4)}\left(p_1 + p_2 - \sum_{i=3}^{n} p_i\right)\prod_{i=3}^{n}\frac{\mathrm{d}p_i^3}{(2\pi)^3 2E_i},$$

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Denote i-th of N phase space parameterizations generated by

$$f_i(x) = \mathrm{d}^{3n_f-4} Lips_i\,(x)\,, \qquad i = 1, \ldots, N,$$

where $x = (x_1, ..., x_{3n_f-4})$ are random arguments, $x_i \in [0, 1]$.
It must satisfy the normalization condition

$$\int_0^1 \mathrm{d}x^{3n_f-4} f_i(x) = \mathrm{vol}(Lips).$$

All the parameterizations $f_i(x)$ are then automatically combined into a single multichannel probability distribution

$$f(x) = \sum_{i=1}^{N} a_i f_i(x),$$

with non negative weights a_i, $i = 1, ..., N$, satisfying the condition

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$$a_i = \sigma_i / \sum_{j=1}^{N} \sigma_j.$$

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Generation of the particle four momenta

The actual probability density function $f_i(x)$ according to which the final state particle four momenta are generated, which are needed to calculate the corresponding matrix element or to be stored as MC events, is chosen from the set

$$\{f_j(x),\ \ j = 1, ..., N\}$$

if uniformly distributed random number $\xi \in [0, 1]$ falls into the interval

$$a_0 + ... + a_{i-1} \leqslant \xi \leqslant a_0 + ... + a_i, \text{ with } a_0 = 0.$$

In the following, the corresponding LO SM matrix elements are generated by `carlomat_4.5`, but it is also possible to use higher order matrix elements generated with other programs.

Generation of phase space parameterizations

An obvious way to follow in order to map out all the peaks is to generate one subroutine containing the phase space parameterization for each individual Feynman diagram, as it was originally done in `carlomat_1.0`.

However, for multiparticle reactions, this approach leads to a large number of subroutines and the resulting multichannel phase space routine is huge indeed and usually difficult to compile and the execution time of the MC integration may become rather long.

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To overcome these difficulties a different approach was proposed in `PSGen`, a program for generation of phase space parameterizations for the multichannel MC integration, where the phase space parameterizations of a given reaction are generated automatically according to predefined patterns which are supposed to smooth only the most relevant peaks of the matrix element.

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Some illustrative results

Let us consider the LO SM cross sections of a few physically interesting multiparticle reactions, which could potentially be measured at any future high energy e^+e^- collider.

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e^+e^- &\to \mu^+\nu_\mu\mu^-\bar{\nu}_\mu, & n_\mathrm{d} &= 8, & &19 \text{ diagrams}, & &(1)\\
e^+e^- &\to b\mu^+\nu_\mu\bar{b}\mu^-\bar{\nu}_\mu, & n_\mathrm{d} &= 14, & &452 \text{ diagrams}, & &(2)\\
e^+e^- &\to b\bar{b}b\mu^+\nu_\mu\bar{b}\mu^-\bar{\nu}_\mu, & n_\mathrm{d} &= 20, & &46890 \text{ diagrams}, & &(3)
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where dimension n_d of the corresponding phase space integral and the number of the LO SM Feynman diagrams are indicated on the right hand side of each reaction.

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To enable their identification the following cuts:

$$5^\circ < \theta(\text{l, beam}),\ \theta(\text{q, beam}) < 175^\circ, \qquad \theta(\text{l, l'}),\ \theta(\text{q, q'}),\ \theta(\text{q, l}) > 10^\circ,$$
$$E_{\text{l}},\ E_{\text{q}} > 15\,\text{GeV}, \qquad E_{T\,\text{missing}} > 15\,\text{GeV},$$

where l, l′ stand for either μ^- or μ^+ and q, q′ stand for either b or $\bar{b}$, are imposed.

$\sigma_{LO}(e^+e^- \to \mu^+\nu_\mu\mu^-\bar{\nu}_\mu)$ [fb]

$\sqrt{s}$ (GeV)	ivegas/ ipsgen	iscan=0 iwadapt=0	iscan=1 iwadapt=0	iscan=0 iwadapt=1	iscan=1 iwadapt=1
360	0/0	111.06(45)	111.48(17)	111.16(16)	111.74(15)
360	1/0	106.10(18)	111.69(5)	111.61(4)	111.62(4)
360	0/1	111.58(42)	111.71(16)	111.65(16)	111.53(15)
360	1/1	119.66(18)	112.57(5)	111.88(4)	111.81(4)
500	0/0	70.51(44)	70.85(16)	70.42(15)	70.55(15)
500	1/0	66.87(13)	70.62(4)	70.48(3)	70.50(3)
500	0/1	70.18(40)	70.66(15)	70.67(15)	70.36(14)
500	1/1	74.79(13)	70.98(4)	70.59(3)	70.58(3)
1000	0/0	21.09(24)	20.90(8)	20.93(8)	20.89(8)
1000	1/0	19.66(4)	20.88(1)	20.89(1)	20.90(1)
1000	0/1	20.75(21)	20.87(8)	20.95(8)	20.93(8)
1000	1/1	21.59(4)	20.82(1)	20.92(1)	20.92(1)

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500	0/0	70.51(44)	70.85(16)	70.42(15)	70.55(15)
500	1/0	66.87(13)	70.62(4)	70.48(3)	70.50(3)
500	0/1	70.18(40)	70.66(15)	70.67(15)	70.36(14)
500	1/1	74.79(13)	70.98(4)	70.59(3)	70.58(3)
1000	0/0	21.09(24)	20.90(8)	20.93(8)	20.89(8)
1000	1/0	19.66(4)	20.88(1)	20.89(1)	20.90(1)
1000	0/1	20.75(21)	20.87(8)	20.95(8)	20.93(8)
1000	1/1	21.59(4)	20.82(1)	20.92(1)	20.92(1)

The initial scan reduces MC error by ~ 3.

No scan $\Rightarrow$ results non reliable

$\sigma_{LO}(e^+e^- \rightarrow \mu^+\nu_\mu\mu^-\bar{\nu}_\mu)$ [fb]

$\sqrt{s}$ (GeV)	ivegas/ ipsgen	iscan=0 iwadapt=0	iscan=1 iwadapt=0	iscan=0 iwadapt=1	iscan=1
360	0/0	111.06(45)	111.48(17)	111.16(16)	111.74(15)
360	1/0	106.10(18)	111.69(5)	111.61(4)	111.62(4)
360	0/1	111.58(42)	111.71(16)	111.65(16)	111.53(15)
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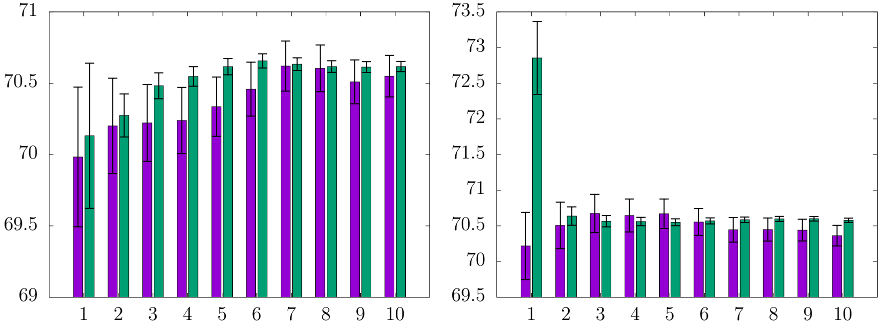
VEGAS reduces the MC error substantially. ($n_{\mathrm{d}} = 8$)

Accumulated results for the LO SM cross section at $\sqrt{s} = 500$ GeV as functions of the number of iterations, calculated with $f(x)$ of `carlomat_4.5` (left panel) and $f(x)$ of `PSGen_1.1` (right panel).



In both panels, the violet histogram has been integrated with `carlos` and the green histogram with `VEGAS`, with the initial scan (`iscan=1`) and weight adaptation (`iwadapt=1`).

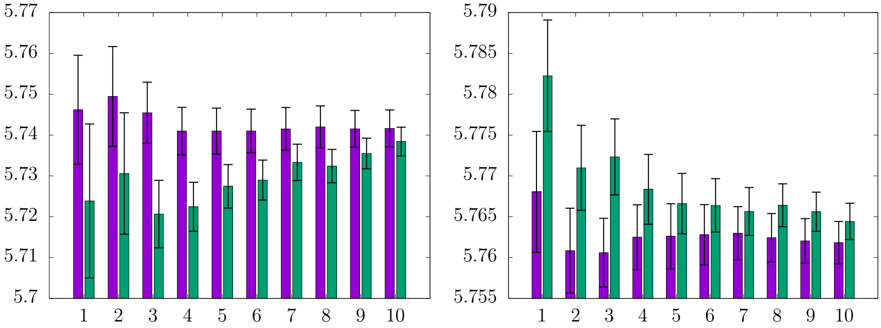
$\sigma_{LO}(e^+e^- \to b\mu^+\nu_\mu\bar{b}\mu^-\bar{\nu}_\mu)$ [fb]

$\sqrt{s}$ (GeV)	`ivegas/ipsgen`	`iscan=0 iwadapt=0`	`iscan=1 iwadapt=0`	`iscan=0 iwadapt=1`	`iscan=1 iwadapt=1`
500	0/0	5.7721(334)	5.7444(45)	5.7584(52)	5.7416(45)
500	1/0	5.3242(326)	6.2811(23)	5.7385(70)	5.7384(35)
500	0/1	5.7628(173)	5.7625(28)	5.7606(30)	5.7618(26)
500	1/1	6.0091(155)	5.8128(23)	5.7627(25)	5.7644(22)
800	0/0	2.8451(214)	2.8395(56)	2.8585(93)	2.8420(68)
800	1/0	2.4527(352)	3.2007(20)	2.8013(44)	2.7906(58)
800	0/1	2.8583(91)	2.8662(20)	2.8647(22)	2.8688(20)
800	1/1	3.0329(83)	3.0706(12)	2.8634(17)	2.8625(15)
1000	0/0	1.9306(202)	1.9433(68)	1.9363(77)	1.9230(69)
1000	1/0	2.3477(276)	2.0881(9)	0.4127(430)	1.9841(50)
1000	0/1	1.9675(65)	1.9644(18)	1.9634(18)	1.9621(17)
1000	1/1	2.0288(60)	2.0864(6)	1.8903(42)	1.8667(20)

The initial scan reduces the MC error, but the use of `VEGAS` may give non reliable results. ($n_{\mathrm{d}} = 14$)

$\sigma_{LO}(e^+e^- \to \; b\mu^+\nu_\mu\bar{b}\mu^-\bar{\nu}_\mu)$ [fb]

Accumulated results for the LO SM cross section at $\sqrt{s} = 500$ GeV as functions of the number of iterations, calculated with $f(x)$ of `carlomat_4.5` (left panel) and $f(x)$ of `PSGen_1.1` (right panel).



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The advantage of `VEGAS` is not clearly visible any more.

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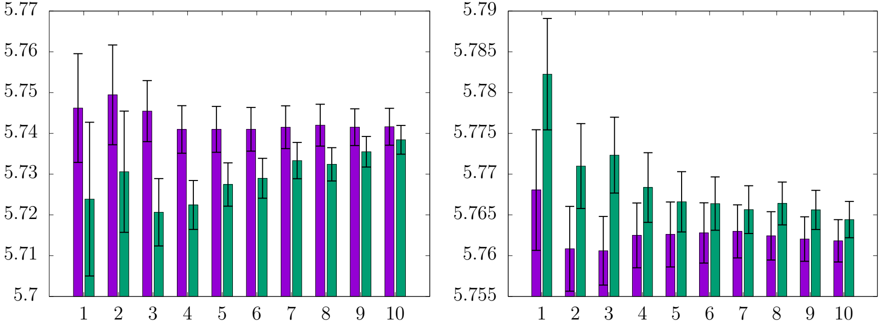

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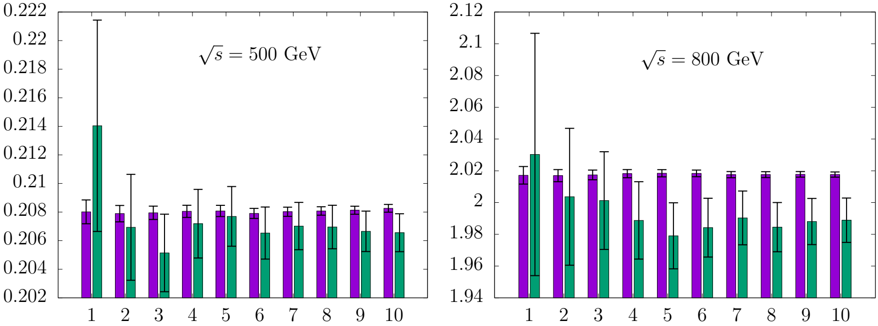
$n_d = 20 \quad \Rightarrow \quad$ VEGAS does not seem applicable any more.

Accumulated results for the LO SM cross section at $\sqrt{s} = 500$ GeV and $\sqrt{s} = 800$ GeV as functions of the number of iterations:



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$n_d = 46890$, hence very many peaks are present $\quad \Rightarrow \quad f(x)$ of `carlomat_4.5` works better.

$\sigma_{LO}(e^+e^- \to b\bar{b}b\mu^+\nu_\mu\bar{b}\mu^-\bar{\nu}_\mu)$ [fb]

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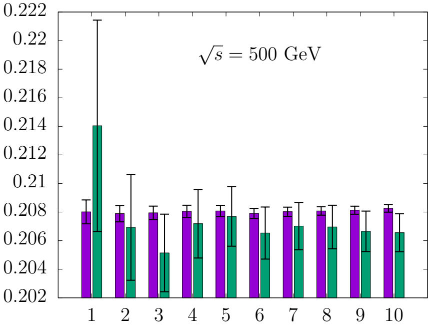

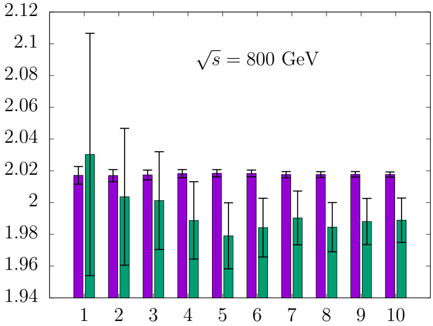

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Efficient integration of the multiparticle reaction cross sections over a multidimensional phase space is a challenge.

As the corresponding matrix elements involve many peaks, the variance of the MC integral can be reduced only if those peaks are mapped out which is achieved by the use of the multichannel MC approach, with different phase space parameterizations generated and combined in the single probability distribution in a fully automatic way.

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carlomat_4.5 and PSGen_1.1 can be useful tools to find the right solution of the problem of MC phase space integration for mutiparticle reactions.

The Fortran code with which the results shown in the present work were obtained is public. It can be downloaded from the web pages:

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