NNLO phase-space integrals for semi-inclusive deep-inelastic scattering

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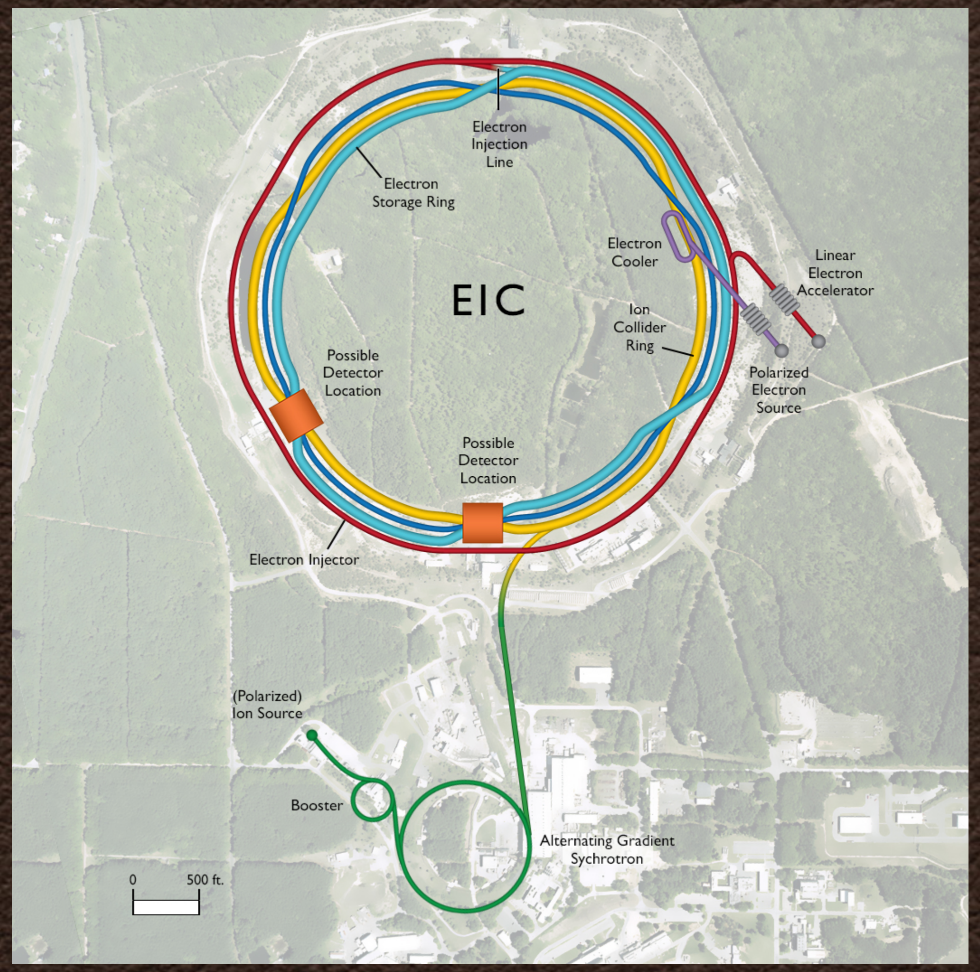

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Roman Lee (Novosibirsk, IYF)
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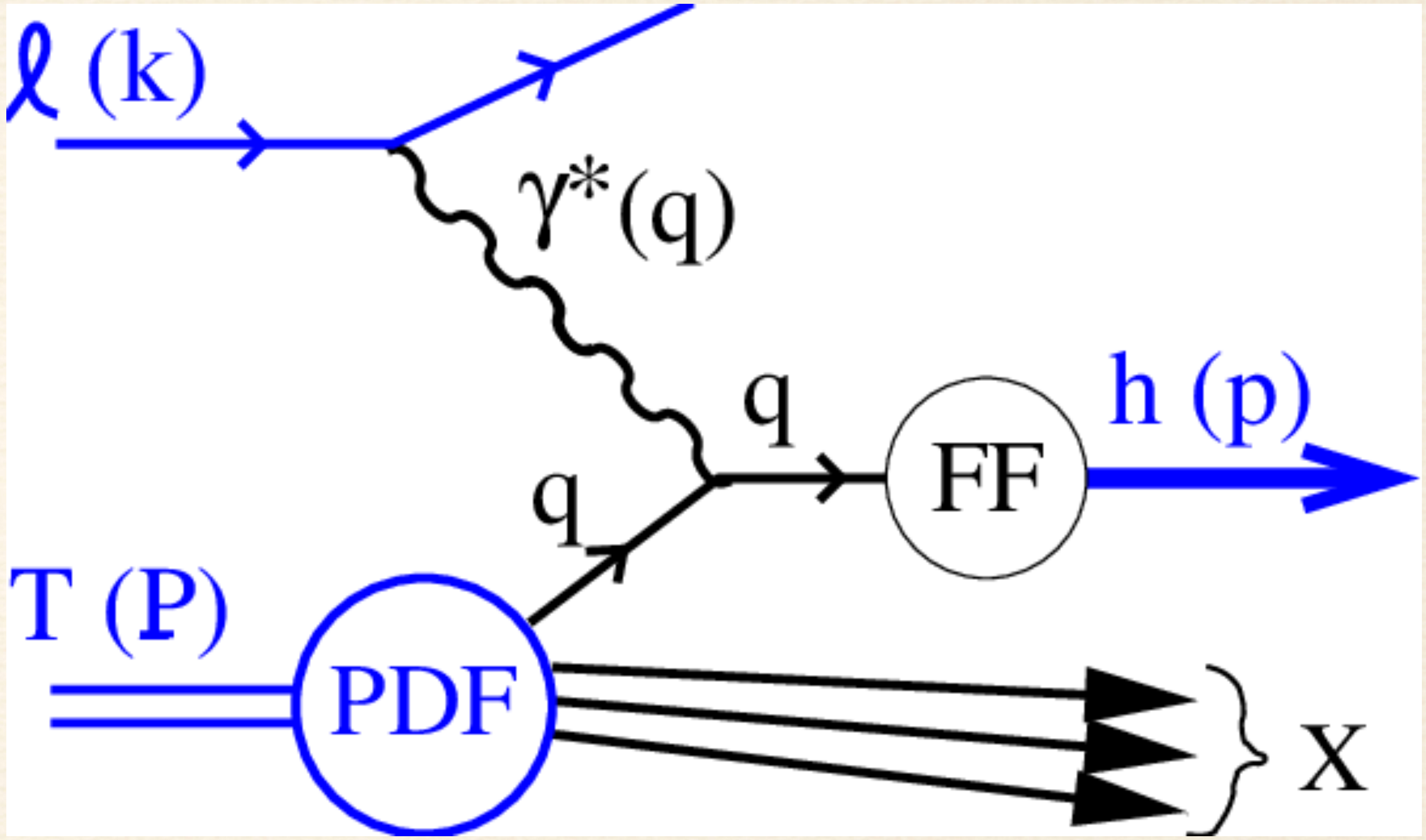
arXiv: 2412.16509v1

DFG Deutsche Forschungsgemeinschaft

Matter To The Deepest
Katowice 2025

Semi-Inclusive-Deep-Inelastic-Scattering process

- *DIS + one final state hadron observed*
- *Important tool for extracting PDF's, TMD's*
- *3D tomography of the proton*
- *EIC*



State of the art

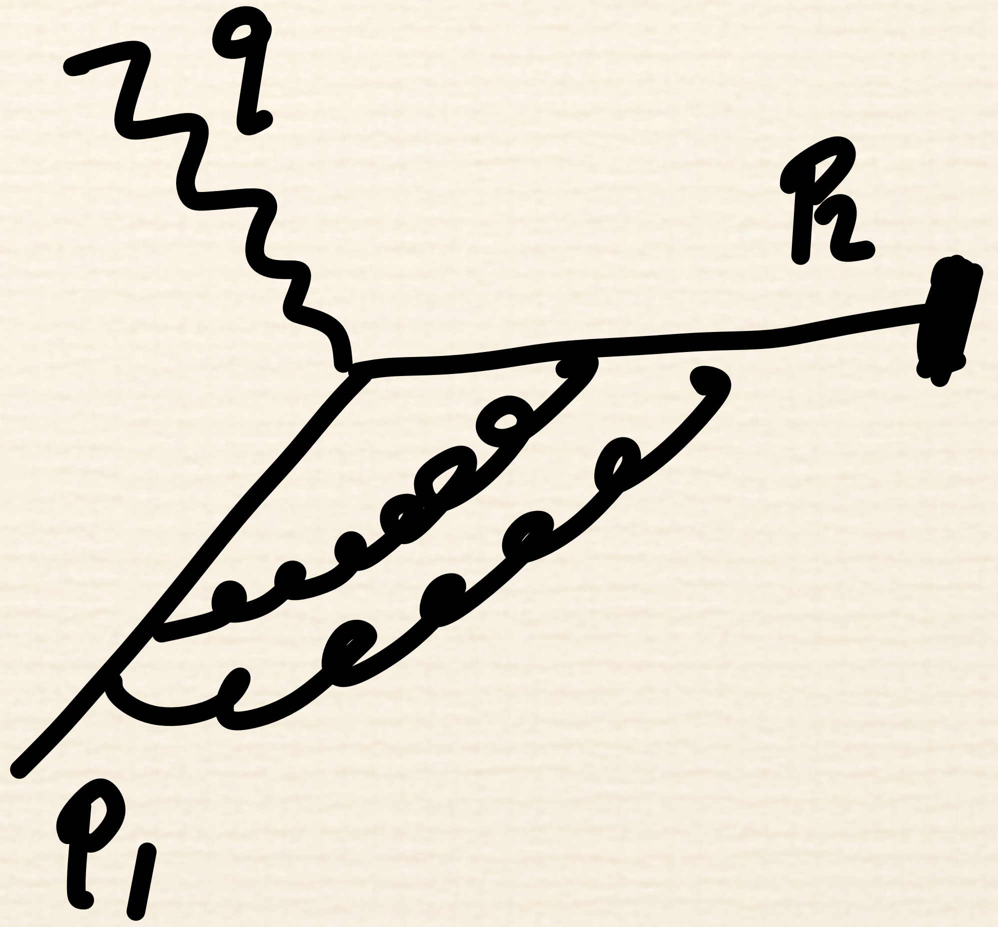
Recent NNLO calculation of SIDIS cross-section:

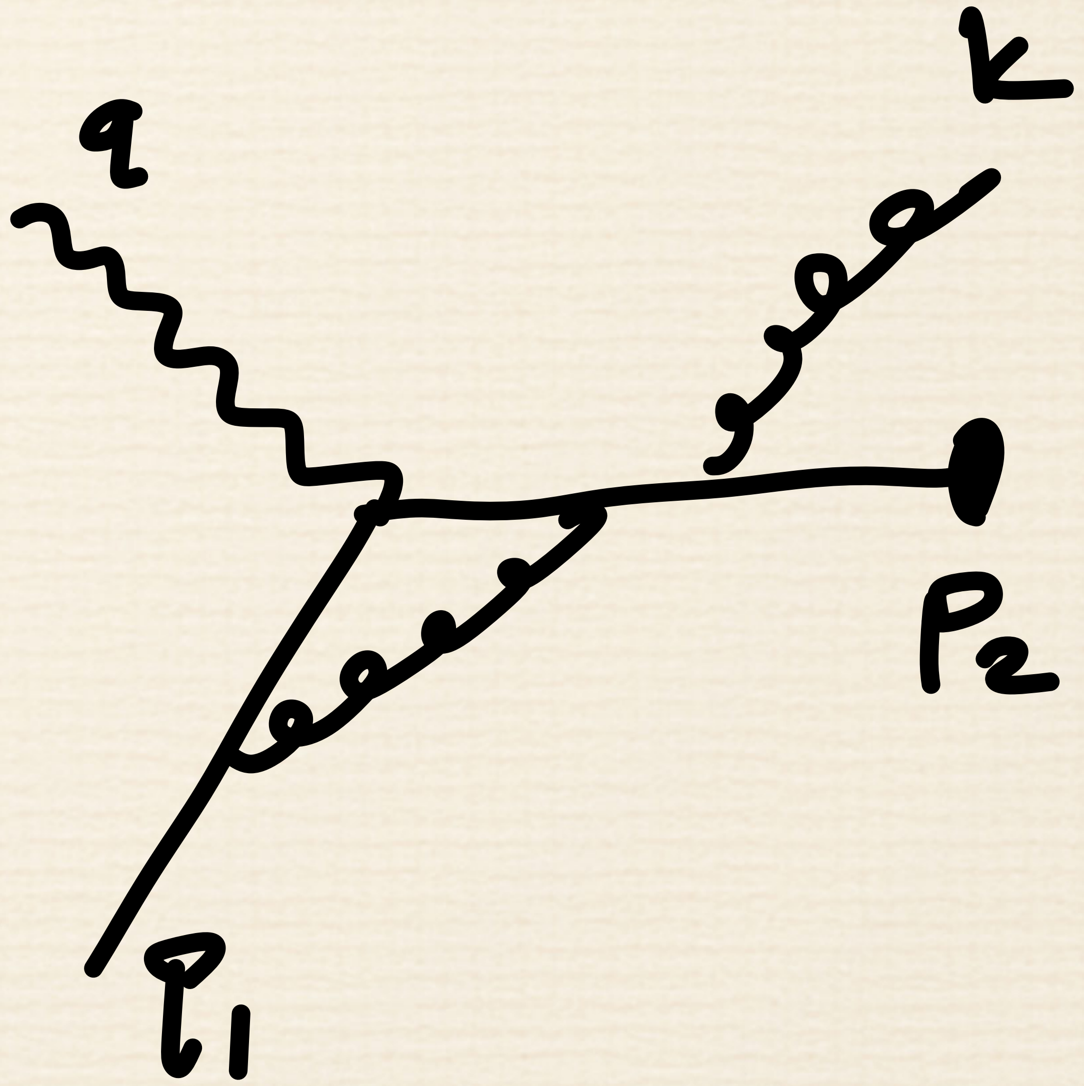
Ravindran, Sven-Olaf Moch, Roman Lee, Saurav Goyal, Vaibhav Pathak, Narayan Rana '24

Gehrmann, Bonino, Stagnitto, Schönwald, Löchner '24

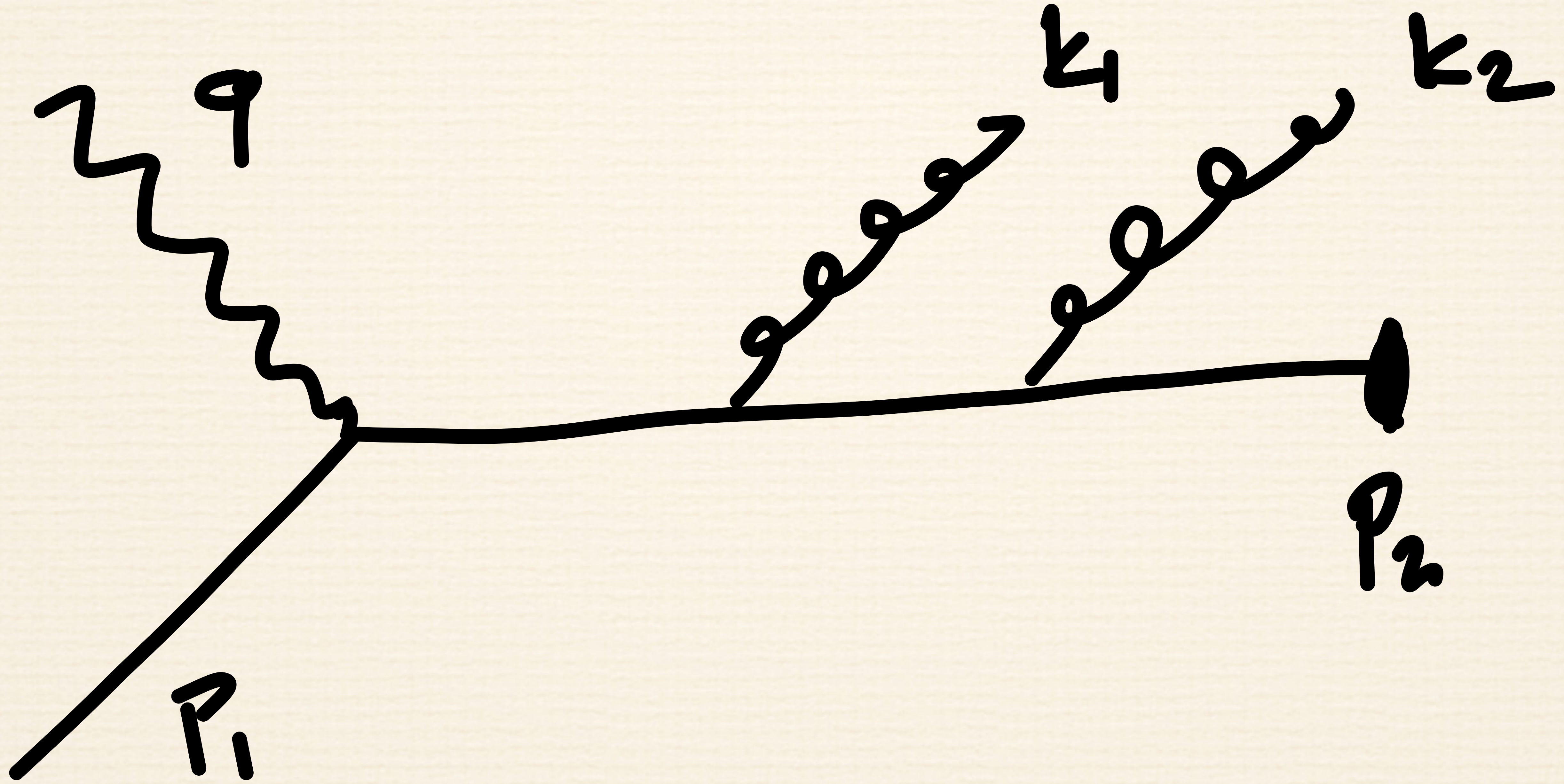
Is needed:

- 2 loop virtual ➔ Long Known
- 1 loop + 1 real
- 2 real ➔ Our Focus





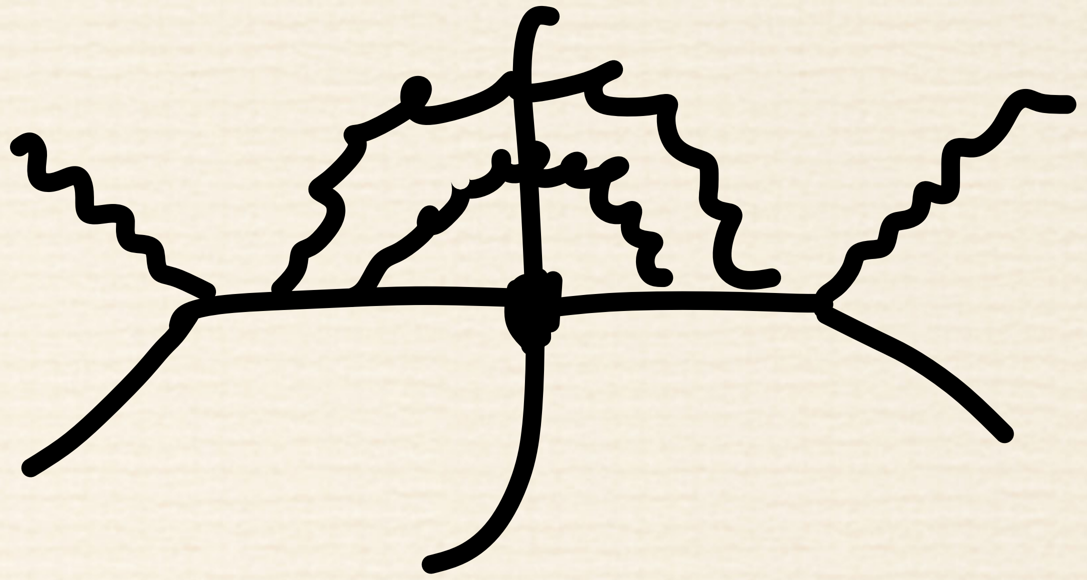
Goal: Calculate Phase space integrals



Reverse Unitarity

Anastasiou, Melnikov 2002

Real Emission Diagrams ➔ Loop Diagrams



<u>Reason</u>: Apply IBP identities

Multi-loop Feynman Integrals in dim reg are convergent

↓

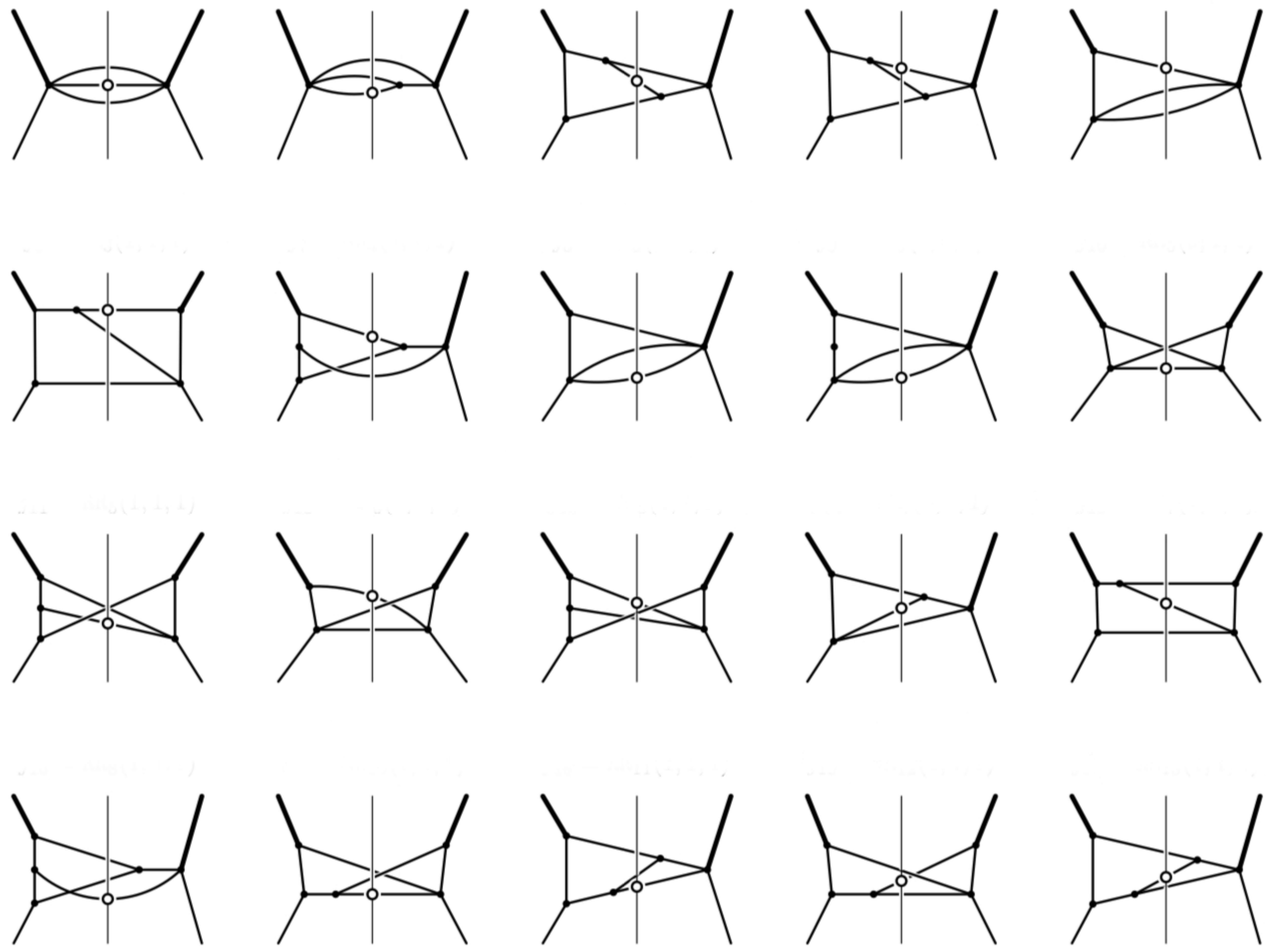
Boundary terms = 0 ➔ Relation among integrals

↓

Master Integrals

20 Master Integrals

$$\int dPS_3 \frac{1}{D_1^{n_1} D_2^{n_2} D_3^{n_3}} \delta\left(z' - \frac{p_1 . p_2}{p_1 . q}\right)$$



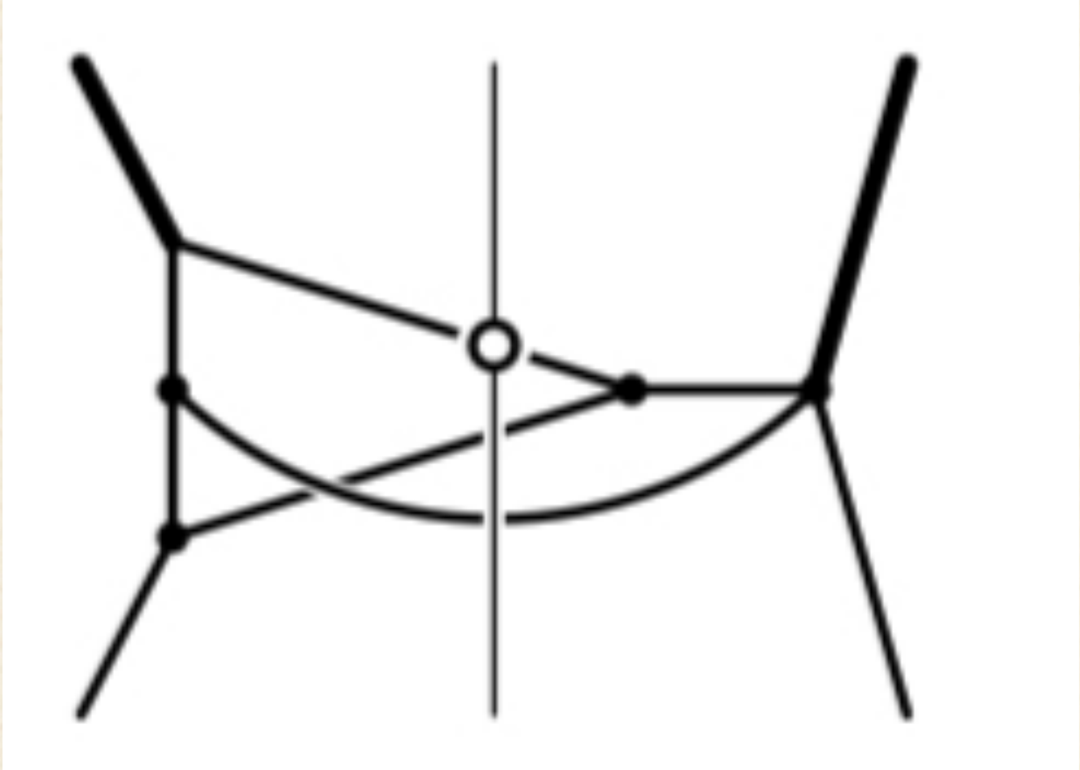
Methods

- DE Method (Canonical form, Iterative solution) Henn 2013

- Radial - Angular Decomposition (RAD) → Focus

Demonstration: Example

$$\int dPS_3 \frac{1}{(k_1 - p_1)^2 (k_2 - p_1 - q)^2 (k_1 + k_2 - p_1)^2} \delta(z' - \frac{p_1 . p_2}{p_1 . q})$$



Spherical coordinates

$$\int dPS'_{3,\mathrm{r}} \frac{1}{r} \int dPS'_{3,\mathrm{ang}} \frac{1}{(a + b\cos\theta)(A + B\cos\theta + C\sin\theta\cos\phi)}$$

Radial ← MB ← Angular

Angular Integrals

Somogyi '11

Lyubovitskij, Wunder, Zhevlakov '21

General two denominator case in Neerven parametrization:

$$\int d\Omega_{d-1} \frac{1}{(a + bcos\theta_1)^j (A + Bcos\theta_1 + Csin\theta_1 cos\theta_2)^l}$$

General two denominator case in Somogyi parametrization:

$$\frac{1}{a^j A^l} \int d\Omega_{d-1} \frac{1}{(p_1 \cdot q)^j (p_2 \cdot q)^l}$$

We categorize it

Massless: $a^2 = b^2, A^2 = B^2$

Single massive: $a^2 = b^2$ or $A^2 = B^2 + C^2$

Double massive: $a^2 \neq b^2, A^2 \neq B^2 + C^2$

$p_1 = (1, 0_{d-2}, -\frac{b}{a})$, $p_2 = (1, 0_{d-3}, -\frac{C}{A}, -\frac{B}{A})$

$q = (1, \ldots, sin\theta_1 cos\theta_2, cos\theta_1)$

Mellin-Barnes Technique

Taushif Ahmed, Syed Mehedi Hasan, **AR** , '24, arxiv: 2410.18886v1

Angular Integral

Three denominator massless

$$\int d\Omega_{d-1}(q)\frac{1}{(p_1\cdot q)^{j_1}(p_2\cdot q)^{j_2}(p_3\cdot q)^{j_3}}$$

$$p_1^\mu=(1,0,0,1)$$
$$p_2^\mu=(1,0,sinx_2^{(1)},cosx_2^{(1)})$$
$$p_3^\mu=(1,sinx_3^{(2)}sinx_3^{(1)},cosx_3^{(2)}sin(x_3^{(1)}),cosx_3^{(1)})$$
$$q^\mu=(1,cos\theta_3 sin\theta_2 sin\theta_1,cos\theta_2 sin\theta_1,cos\theta_1)$$

Haug, Wunder '24

Mellin-Barnes representation

$$2^{2-j-k-l-2\epsilon}\pi^{1-\epsilon}\frac{1}{\Gamma(j)\Gamma(k)\Gamma(l)\Gamma(2-j-k-l-2\epsilon)}$$
$$\times\int_{-i\infty}^{+i\infty}\frac{dz_{12}dz_{13}dz_{23}}{(2\pi i)^3}\Gamma(-z_{12})\Gamma(-z_{13})\Gamma(-z_{23})\Gamma(j+z_{12}+z_{13})\Gamma(k+z_{12}+z_{23})$$
$$\times\Gamma(l+z_{13}+z_{23})\Gamma(1-j-k-l-\epsilon-z_{12}-z_{13}-z_{23})(u_{12})^{z_{12}}(u_{13})^{z_{13}}(u_{23})^{z_{23}}$$

Expand in ε + solving the coefficients

Mellin-Barnes Technique

Four denominator case:

Taushif Ahmed, Syed Mehedi Hasan, **AR ,** ‘25, arxiv:2508.15952v1

Haug, A.Smirnov, Wunder ‘25

Mellin-Barnes Technique

Gamma functions $\to$ Beta functions

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Real integral representations:

$$B(a,b) = \int_0^1 dx x^{a-1}(1-x)^{b-1}, \quad \mathcal{R}(a), \mathcal{R}(b) > 0$$

$$B(a,b) = \int_0^\infty dx x^{a-1}(1+x)^{-a-b}, \quad \mathcal{R}(a), \mathcal{R}(b) > 0$$

Mellin-Barnes Technique

we get:

$$\int_0^1 \left(\prod_{k=1}^{K} dx_k \right) R_0(x,u) \int_{-i\infty}^{+i\infty} \prod_{l=1}^{L} \frac{dz_l}{2\pi i} \left[R_l(x,u) \right]^{z_l}$$

Mellin-Barnes Technique

Use the formula:

$$\int_{-i\infty+z_0}^{+i\infty+z_0} \frac{dz}{2\pi i} A^z = \delta(1-A), \quad A > 0$$

Then:

$$\int_0^1 \left(\prod_{k=1}^{K} dx_k\right) R_0(x,u) \prod_{l=1}^{L} \delta\left[1 - R_l(x,u)\right]$$

Mellin-Barnes Technique

Real Integrations over rational functions $\rightarrow$ Multiple Polylogarithms (MPLs):

$$G(a_1, ..., a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, ..., a_n; t)$$

Goal: Combining Radial + Angular

Result in terms of GPLs or MPLs

↓

Angular part GPLs manipulation → Pole Subtraction Method

↓

Singularity Resolution

Angular Part: Singularity Resolution

Why can we expand the angular part?

- Angular part → Collinear Singularities
- Parametric part → Soft Singularities

Conjecture:

Angular part soft singularity → All order Resummation + Factorization

Conjecture Verification → NNLO SIDIS

Example

Angular Part :

$$\Omega'_z = \frac{2\pi}{\varepsilon z}\left(z' + z - zz'\right) + \frac{\pi}{z}\left(z' + z - zz'\right)\ln\left(\frac{z}{z' + z - zz'}\right)$$
$$+ \frac{\pi\varepsilon}{4z}\left(z' + z - zz'\right)\left\{2\,\mathrm{Li}_2\left(\frac{z}{zz' - z' - z} + 1\right) + \ln^2\left(\frac{z}{-zz' + z' + z}\right)\right\}$$
$$+ \frac{\varepsilon^2}{24z}\pi\left(z' + z - zz'\right)\left\{-6\,\mathrm{Li}_3\left(\frac{z}{-z'z + z + z'}\right) - 6\,\mathrm{Li}_3\left(\frac{z}{zz' - z' - z} + 1\right)\right.$$
$$+ \ln^3\left(\frac{z}{-zz' + z' + z}\right) - 3\ln\left(\frac{z}{zz' - z' - z} + 1\right)\ln^2\left(\frac{z}{-zz' + z' + z}\right)$$
$$\left. + \pi^2\ln\left(\frac{z}{-zz' + z' + z}\right) + 6\zeta(3)\right\} + \frac{\pi\varepsilon^3}{8z}\left(zz' - z' - z\right)\mathrm{Li}_4\left(\frac{zz' - z'}{z}\right) + O\left(\varepsilon^4\right)$$

Extract

$$z^{-1+\frac{\epsilon}{2}}$$

Push to Radial Part

Radial Part: Singularity Resolution

Consider $g(z)$ singular at $z = b$. Then separate the singular part $S(z)$ and the finite part $F(z)$ as:

$$g(z) = S(z)F(z)$$

Next we define:

$$\widetilde{F} \equiv \lim_{z \to b}(F(z))$$

So we can separate the singularity as:

$$\int_a^b g(z)dz = \int_a^b [S(z)F(z) - \tilde{F}S(z)]dz + \int_a^b \tilde{F}S(z)dz$$

The integrand $S(z)F(z) - \tilde{F}S(z)$ in non singular and $\int_a^b \tilde{F}S(z)dz$ can be integrated easily.

Results

- Fully analytic form in GPLs
- One-fold numerical integration over GPLs

Verification through DE method → Perfect numerical agreement

Up to ε^2 → Useful for N^3LO SIDIS

Advantages of the RAD

- Angular Part → Universal Structure → Process Independent
- Transparent Singularity Structure
- It can handle Massive cases

Conclusion & Outlook

- Solution of NNLO 2 real SIDIS MI
- DE and RAD methods
- MB Technique for the Angular Part
- Transparent singularity structure in RAD
- RAD can handle Massive Cases

Thank you for listening!