

Automated analytic evaluation of Mellin-Barnes integrals having polygamma functions in their integrand

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Mellin-Barnes integrals

General facts

One-fold case:

$$I_1 = \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} x^z \frac{\prod_{i=1}^k \Gamma^{a_i}(e_i z + g_i)}{\prod_{j=1}^l \Gamma^{b_j}(f_j z + h_j)}$$

N -fold case:

$$I_N = \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \cdots \int_{-i\infty}^{+i\infty} \frac{dz_N}{2\pi i} x_1^{z_1} \cdots x_N^{z_N} \frac{\prod_{i=1}^k \Gamma^{a_i}(\vec{e}_i \cdot \vec{z} + g_i)}{\prod_{j=1}^l \Gamma^{b_j}(\vec{f}_j \cdot \vec{z} + h_j)}$$

where a_i, b_j, k, l and N are positive integers and e_i, f_j are real numbers and g_i, h_j can be complex ($\vec{e}_i, \vec{f}_j$ are N -dimensional coefficient vectors and $\vec{z} = (z_1, \dots, z_N)$).

Mellin-Barnes integrals

General facts

The class of integrals we consider in this work

$$\int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \cdots \int_{-i\infty}^{+i\infty} \frac{dz_N}{2\pi i} x_1^{z_1} \cdots x_N^{z_N} \frac{\prod_{i=1}^k \Gamma^{a_i}(\vec{e}_i \cdot \vec{z} + g_i) \prod_{k=1}^n \psi^{c_k}(m_k, \vec{i}_k \cdot \vec{z} + j_k)}{\prod_{j=1}^l \Gamma^{b_j}(\vec{f}_j \cdot \vec{z} + h_j)}$$

Mellin-Barnes integrals

General facts

MB integrals historically come from [hypergeometric function theory](#).

In high energy physics, they can be used to compute [Feynman integrals](#).

[AMBRE](#) [Gluza et al. 2007] automatizes the derivation of their MB representations.

[MBConicHulls](#) allows the automatic analytic computation of multiple MB integrals.

[arXiv:2309.00409 \(PRD\)](#)

[arXiv:2212.1139 \(PRD\)](#)

[arXiv:2012.15108 \(PRL\)](#)



[MTTD 2023](#)

[Acta. Phys. Polon. Supp. 17 \(2024\)](#)

Mellin-Barnes integrals

The `MBConicHulls` package

What do we mean by analytic computation?

MB integral $\Rightarrow$ `MBConicHulls` $\Rightarrow$ series representations.

General form of the MB $\Rightarrow$ These series converge or not
(asymptotic)

If converging (in the degenerate case):

- Regions of convergence can/must be found.
- They provide transformation formulas, analytic continuations.

`MBConicHulls` also works with straight contours of integration.

Mellin-Barnes integrals

The MBConicHulls package

A typical MB integral that the package can compute:

$$\begin{aligned}
& \frac{1}{(2\pi i)^9} \int_{-i\infty}^{+i\infty} \dots \int_{-i\infty}^{+i\infty} \prod_{i=1}^9 (dz_i w_i^{z_i} \Gamma(-z_i)) \\
& \times \Gamma(\alpha_1 + z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 + z_8 + z_9) \\
& \times \Gamma(\alpha_2 + z_1 + z_5 + z_8 + z_9) \Gamma(\alpha_3 + z_2 + z_6 + z_7 + z_8) \\
& \times \Gamma(\alpha_4 + z_3 + z_4 + z_5 + z_6) \Gamma(\alpha_5 - z_4 - z_5 - z_6 - z_7 - z_8 - z_9) \\
& \times \Gamma(\alpha_6 - z_1 - z_2 - z_3 - z_5 - z_6 - z_8)
\end{aligned}$$

Mellin-Barnes integrals

Feynman integrals

Now, what happens with Feynman integrals?

- The MB representations are degenerate
- Often the number of different scales is smaller than the number of integration variables
- The MB integrals are regularized (ϵ appears in the gamma functions)
- The usual strategy is to first solve the ϵ singularities
- One can then perform the ϵ expansion
 $\Rightarrow$ This is where polygamma functions can appear

Mellin-Barnes integrals

Feynman integrals

A simple example: the **J011**(1,2,2) sunset [Davydychev and Kalmykov 2001]

Sunsets (or sunrises) are two-loop Feynman integrals of the type

$$H_{\sigma, \nu_1, \nu_2}(m_1, m_2, m_3; p^2) \equiv \int \frac{d^d k_1 d^d k_2}{[(k_1 - p)^2 - m_1^2 + i0]^{\nu_1} [k_2^2 - m_2^2 + i0]^{\sigma} [(k_1 - k_2)^2 - m_3^2 + i0]^{\nu_2}}$$

where $d = 4 - 2\epsilon$.

In our case,

$$\mathbf{J011}(1, 2, 2) = (i\pi^{2-\epsilon}\Gamma(1+\epsilon))^{-2} H_{1,2,2}(m, 0, m; m^2)|_{m \rightarrow 1}.$$

Mellin-Barnes integrals

Feynman integrals

A simple example: the **J011**(1,2,2) sunset

Davydychev and Kalmykov have shown that

$$\mathbf{J011}(1, 2, 2) = \frac{1}{3\epsilon^2} \left(\frac{\Gamma(1+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(1+\epsilon)} - 1 \right)$$

whose ϵ expansion reads

$$\mathbf{J011}(1, 2, 2) = \frac{\pi^2}{9} + \frac{1}{3}\psi(2, 1)\epsilon + \frac{\pi^4}{30}\epsilon^2 + \mathcal{O}(\epsilon^3)$$

which can also be written as

$$\mathbf{J011}(1, 2, 2) = \frac{2}{3}\zeta_2 - \frac{2}{3}\zeta_3\epsilon + 3\zeta_4\epsilon^2 + \mathcal{O}(\epsilon^3)$$

Mellin-Barnes integrals

Feynman integrals

A simple example: the **J011**(1,2,2) sunset

Mellin-Barnes approach

Find the MB representation of **J011**(1,2,2) using **AMBRE** [Gluza et al. 2007]

$$\begin{aligned} \mathbf{J011}(1, 2, 2) &= \frac{\sqrt{\pi}}{2^{1+2\epsilon}} \frac{\Gamma(1-\epsilon)}{\Gamma(1+\epsilon)^2} \\ &\times \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} \left(-\frac{1}{4}\right)^z \frac{\Gamma(-z)\Gamma(1+z)\Gamma(1+\epsilon+z)\Gamma(1+2\epsilon+z)}{\Gamma\left(\frac{3}{2}+\epsilon+z\right)\Gamma(2-\epsilon+z)} \end{aligned}$$

Mellin-Barnes integrals

Feynman integrals

A simple example: the **J011**(1,2,2) sunset

Performing the ϵ expansion at the integral level, we get, for the first three terms:

$$\begin{aligned}
 \text{J011}(1, 2, 2) &= \frac{\sqrt{\pi}}{2} \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} \left(-\frac{1}{4}\right)^z \frac{\Gamma(-z)\Gamma(1+z)^3}{\Gamma\left(\frac{3}{2}+z\right)\Gamma(2+z)} \\
 &\quad \times \left[1 + \epsilon \left(-2 + 4\gamma_E + \psi\left(0, \frac{3}{2}\right) + 3\psi(0, 1+z) - \psi\left(0, \frac{3}{2}+z\right) + \psi(0, 2+z) \right) + \right. \\
 &\quad + \epsilon^2 \left(2 - 8\gamma_E + 8\gamma_E^2 - \frac{\pi^2}{12} - (2 - 4\gamma_E)\psi\left(0, \frac{3}{2}\right) + \frac{1}{2}\psi\left(0, \frac{3}{2}\right)^2 + \left[-2 + 4\gamma_E + \psi\left(0, \frac{3}{2}\right) \right] \right. \\
 &\quad \times \left[3\psi(0, 1+z) - \psi\left(0, \frac{3}{2}+z\right) + \psi(0, 2+z) \right] + \frac{9}{2}\psi(0, 1+z)^2 - 3\psi(0, 1+z)\psi\left(0, \frac{3}{2}+z\right) \\
 &\quad + \frac{1}{2}\psi\left(0, \frac{3}{2}+z\right)^2 + 3\psi(0, 1+z)\psi(0, 2+z) - \psi\left(0, \frac{3}{2}+z\right)\psi(0, 2+z) + \frac{1}{2}\psi(0, 2+z)^2 \\
 &\quad \left. \left. + \frac{5}{2}\psi(1, 1+z) - \frac{1}{2}\psi\left(1, \frac{3}{2}+z\right) - \frac{1}{2}\psi(1, 2+z) \right) + \mathcal{O}(\epsilon^3) \right]
 \end{aligned}$$

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Polygamma functions

Definition and basic properties

Polygamma functions of order m are defined as

$$\psi(m, z) = \frac{d^m}{dz^m} \psi(0, z)$$

where $\psi(0, z) = \frac{d}{dz} \ln \Gamma(z)$ is the digamma function.

At all non positive integers, polygamma functions of order m have poles of order $m + 1$.

Polygamma functions

Definition and basic properties

In a similar way as the relation

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$

can be used iteratively to obtain

$$\Gamma(z-n) = \frac{\Gamma(z+1)\Gamma(1-z)(-1)^n}{z \Gamma(n+1-z)}$$

one can get, from the relation

$$\psi(m, z) = \psi(m, z+1) + \frac{(-1)^{m+1} m!}{z^{m+1}}$$

the following useful result

$$\psi(m, z-n) = \frac{(-1)^{m+1} m!}{z^{m+1}} + \psi(m, z+1) + (-1)^{m+1} m! \sum_{k=0}^{n-1} \frac{1}{(z-n+k)^{m+1}}$$

Polygamma functions

How to handle these with MBConicHulls

How to handle these with MBConicHulls?

Let us focus on the following simple one-fold integral

$$3 \frac{\sqrt{\pi}}{2} \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} \left(-\frac{1}{4}\right)^z \frac{\Gamma(-z)\Gamma(1+z)^3}{\Gamma\left(\frac{3}{2}+z\right)\Gamma(2+z)} \psi(0, 1+z)$$

This integral, in the MBConicHulls syntax, reads

```
MBRepOut = MBRep[3  $\frac{\sqrt{\pi}}{2}$ , {z}, {-1/4}, {{-z, 1+z, 1+z, 1+z, {0, 1+z}}}, {3/2+z, 2+z}]];
```

Non-Straight Contours.

Time Taken 0.029413 seconds

Polygamma functions

How to handle these with MBConicHulls

Solving this integral is next performed as usual with `ResolveMB[]`:

```
ResolveMBRepOut = ResolveMB[MBRepOut];
```

Degenerate case with 5 conic hulls

Found 2 series solutions.

Cardinality 1:: Solution found 1.

Cardinality 4:: Solution found 1.

Series Solution 1 :: Cardinality 1. Intersecting Conic Hulls

$\{C_1\}$. Set of poles :: $\{\{n_1\}\}$ with master series characteristic list and variables $\{n_1, \{-\frac{1}{4}\}\}$.

Series Solution 2 :: Cardinality 4. Intersecting Conic Hulls $\{C_2, C_3, C_4, C_5\}$. Set of

poles :: $\{\{-1-n_1\}, \{-1-n_1\}, \{-1-n_1\}, \{-1-n_1\}\}$ with master series characteristic list and variables $\{n_1, \{-4\}\}$.

Time Taken 0.113935 seconds

Polygamma functions

How to handle these with MBConicHulls

Focusing on the first series, we get

```
SeriesNumber = 1;
```

```
EvaluateSeriesOut1 = EvaluateSeries[ResolveMBRepOut, {}, SeriesNumber] // Quiet;
```

The series solution is a sum of the following 1 series.

Series Number 1 ::
$$\frac{3 (-1)^{2n_1} 2^{-1-2n_1} \sqrt{\pi} \Gamma[1+n_1]^2 \text{PolyGamma}[0, 1+n_1]}{\Gamma\left[\frac{3}{2} + n_1\right] \Gamma[2+n_1]} \text{ valid for } n_1 \geq 0$$

Time Taken 0.130477 seconds

which can be summed as

$$\text{Sum}\left[\frac{3 (-1)^{2n_1} 2^{-1-2n_1} \sqrt{\pi} \Gamma[1+n_1]^2 \text{PolyGamma}[0, 1+n_1]}{\Gamma\left[\frac{3}{2} + n_1\right] \Gamma[2+n_1]}, \{n_1, 0, \infty\}\right]$$

$$\frac{1}{18} \left(-6 \text{EulerGamma} \pi^2 + \pi \left(- \left((i + \sqrt{3}) \text{PolyGamma}\left[1, \frac{1}{6}\right] \right) + (3i + \sqrt{3}) \text{PolyGamma}\left[1, \frac{1}{3}\right] + \right. \right. \\ \left. \left. 3i \text{PolyGamma}\left[1, \frac{2}{3}\right] - \sqrt{3} \text{PolyGamma}\left[1, \frac{2}{3}\right] - i \text{PolyGamma}\left[1, \frac{5}{6}\right] + \sqrt{3} \text{PolyGamma}\left[1, \frac{5}{6}\right] \right) + 132 \text{Zeta}[3] \right)$$

Polygamma functions

How to handle these with MBConicHulls

Doing the same exercise with all the other integrals

$$\begin{aligned}
 J_{011}(1, 2, 2) &= \frac{\sqrt{\pi}}{2} \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} \left(-\frac{1}{4}\right)^z \frac{\Gamma(-z)\Gamma(1+z)^3}{\Gamma\left(\frac{3}{2}+z\right)\Gamma(2+z)} \\
 &\quad \times \left[1 + \epsilon \left(-2 + 4\gamma_E + \psi\left(0, \frac{3}{2}\right) + 3\psi(0, 1+z) - \psi\left(0, \frac{3}{2}+z\right) + \psi(0, 2+z)\right) + \right. \\
 &\quad + \epsilon^2 \left(2 - 8\gamma_E + 8\gamma_E^2 - \frac{\pi^2}{12} - (2 - 4\gamma_E)\psi\left(0, \frac{3}{2}\right) + \frac{1}{2}\psi\left(0, \frac{3}{2}\right)^2 + \left[-2 + 4\gamma_E + \psi\left(0, \frac{3}{2}\right)\right] \right. \\
 &\quad \times \left[3\psi(0, 1+z) - \psi\left(0, \frac{3}{2}+z\right) + \psi(0, 2+z)\right] + \frac{9}{2}\psi(0, 1+z)^2 - 3\psi(0, 1+z)\psi\left(0, \frac{3}{2}+z\right) \\
 &\quad + \frac{1}{2}\psi\left(0, \frac{3}{2}+z\right)^2 + 3\psi(0, 1+z)\psi(0, 2+z) - \psi\left(0, \frac{3}{2}+z\right)\psi(0, 2+z) + \frac{1}{2}\psi(0, 2+z)^2 \\
 &\quad \left. \left. + \frac{5}{2}\psi(1, 1+z) - \frac{1}{2}\psi\left(1, \frac{3}{2}+z\right) - \frac{1}{2}\psi(1, 2+z)\right) + \mathcal{O}(\epsilon^3)\right]
 \end{aligned}$$

we confirm (numerically) Davydychev and Kalmykov's result.

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Summary

- The *Mathematica* package `MBConicHulls.wl` has been improved to include the computation of MB integrals with polygamma functions in their integrand.
- Non-trivial checks have been performed to test the validity of the code.
- The present version of the package provides a full automatic analysis only for non-straight contours of integration.

Conclusion

Future Work

There is still a lot of work to be done:

- Automatize the straight contours case.
- Study asymptotic cases involving polygamma functions

More generally for `MBConicHulls.wl`:

- Improve the method for Feynman integrals having less scales than integration variables.
- Better understand the properties of the master series.
- Develop a systematic way to reach the "white zones".
- Study non-degenerate MB integrals deeper (links with asymptotics in several variables, resurgence, etc.).