Kira 3 – Whats Next?

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In higher-order calculations, IBP reductions are the vital oil that ensures the complex engine of computation runs without friction.

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Kira versions

- Kira 1 [Maierhöfer,Usovitsch,Uwer,arXiv:1705.05610] made possible many 2->2 (2L) and 1->2 (3L) non-planar (massive) processes.
  - uses finite field (FF) methods to get linearly independent system
  - uses Fermat for coefficient simplification.
- Kira 2 [Klappert,Lange,Maierhöfer,Usovitsch,arXiv:2008.06494] compatibility with FireFly to reconstruct the coefficients provided with Kira's FF methods.
  - Introduction of user defined systems
  - More main memory efficient than Kira 1
- Kira 3 [Lange, Usovitsch, Wu, arXiv:2505.20197] solves 4-loop 9 scalar product integral reductions in 5th post-Minkowskien (PM) GW calculation [Driesse, Jakobsen, Klemm, Mogull, Nega, Plefka, Sauer, Usovitsch, arXiv:2403.07781, arXiv:2411.11846].
  - Nature publication + established a link to 5PM GW and the Calabi-Yau
  - $1 \to 2$, $2 \to 2$ forward scattering (4L), $2 \to 2$ (3L), and $2 \to 3$ (2L), $2 \to 4$ (2L)
  - *small* sized system of equations

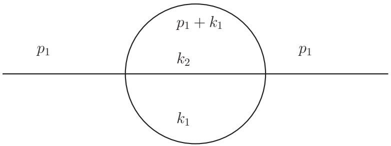
Feynman integral reduction methods

- The number of master integrals is finite [A. V. Smirnov, A.V. Petukhov, arXiv:1004.4199], however the proof is non-constructive
- Syzygy approach, [J. Gluza, K. Kajda, D. A. Kosower arXiv:1009.0472, Z. Wu, J. Boehm, R. Ma, H. Xu, Y. Zhang, arXiv:2305.08783]
- Linear relations from syzygies [B. Agarwal, S. P. Jones, A. von Manteuffel, arXiv:2011.15113]
- Reconstruct the rational coefficients on-the-fly, directly in a form which is decomposed in partial fractions [S. Badger, C. Brønnum-Hansen, D. Chicherin, T. Gehrmann, H. B. Hartanto, J. Henn, M. Marcoli, R. Moodie, T. Peraro, S. Zoia, arXiv:2106.08664]
- Block triangular form [X. Liu, Y.-Q. Ma, arXiv:1801.10523]
- Reduction to master integrals via intersection numbers [S. Mizera, arXiv:1711.00469, G. Fontana, T. Peraro, arXiv:2304.14336]
- Modern Gröbner bases [Barakat,Brüser,Fieker,Huber,Piclum,arXiv:2210.05347]

Reduction programs

- Laporta algorithm implementations available:
  - FIRE 6.5 [A. V. Smirnov, M. Zeng, arXiv:2311.02370]
  - FiniteFlow [T. Peraro, arXiv:1905.08019]
  - Reduze 2 [A. von Manteuffel, C. Studerus, arXiv:1201.4330]
  - First mover Laporta integral implementation: AIR [C. Anastasiou, A. Lazopoulos, arXiv:hep-ph/0404258]
  - IBP tool box: `LiteRed` [R. N. Lee, arXiv:1212.2685]
  - Feynman parameter IBP reduction Ampred: [W. Chen, arXiv:2408.06426]
  - `Blade` [Guan,Liu,Yan-Qing ,Wen-Hao,arXiv:2405.14621]
  - `neatIBP` [,Wu,Boehm, Ma,Xu,Zhang,arXiv:2305.08783,Wu,Böhm,Ma,Usovitsch,Xu,Zhang,arXiv:2502.20778]

Integral family



$$\rightarrow \quad \frac{\int d^D k_1 d^D k_2}{\underbrace{[k_1{}^2 - m_1^2]^{a_1}}_{P_1^{a_1}} \underbrace{[(p_1 + k_1)^2]^{a_2}}_{P_2^{a_2}} \underbrace{[k_2{}^2]^{a_3}}_{P_3^{a_3}} \underbrace{[(p_1 + k_2)^2]^{a_4}}_{P_4^{a_4}} \underbrace{[(k_2 - k_1)^2]^{a_5}}_{P_5^{a_5}}}$$

$$= I(\vec{s}, D|a_1, \ldots, a_5)$$

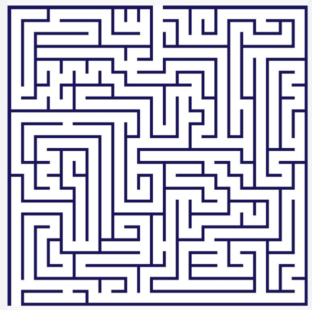
- $q_j = k_1, \ldots, k_L, p_1, \ldots, p_E$
- $s_{ij} = q_i\, q_j, \quad i = 1, \ldots, L, \quad j = i, \ldots, L + E$
- $\vec{s} = (\{s_i\}, \{m_i^2\})$, dimensional regularization parameter $D = 4 - 2\epsilon$
- The integral family definition is complete, if all P_i are linearly independent in the s_{ij}
- $s_{11} = m_1^2 + P_1, \quad s_{12} = \frac{1}{2}(m_1^2 + P_1 + P_3 - P_5), \quad s_{22} = P_3, \quad s_{13} = \frac{1}{2}(-m_1^2 - p_1^2 - P_1 + P_2),$
  $s_{23} = \frac{1}{2}(-p_1^2 - P_3 + P_4)$

Reduction specifications

- Each integral gives $L\,E + L^2$ IBP equations [K. G. Chetyrkin, F. V. Tkachov, Nucl.Phys.B 192 (1981) 159-204] and $E(E-1)/2$ Lorentz-Invariance equations [T. Gehrmann,E. Remiddi,arXiv:hep-ph/9912329]
- $c_1(\{a_f\}, \vec{s}, D) I(a_1, \ldots, a_N \mathbf{-1}) + \cdots + c_m(\{a_f\}, \vec{s}, D) I(a_1 \mathbf{+1}, \ldots, a_N) = 0$
- Reduction: express all integrals with different exponents a_f as a linear combination of some basis integrals (master integrals)
- Generate a **large** set of equations: Laporta algorithm [S. Laporta, arXiv:hep-ph/0102033]
- N is number of propagators, $t = \sum_{f=1}^{N} \theta(a_f - \frac{1}{2})$,

$$r = \sum_{f=1}^{N} a_f \theta(a_f - \tfrac{1}{2}), \qquad s = -\sum_{f=1}^{N} a_f \theta(\tfrac{1}{2} - a_f), \qquad d = \sum_{f=1}^{N} (a_f - 1)\theta(a_f - \tfrac{1}{2})$$

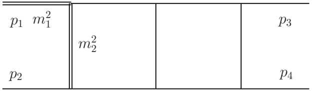
The maze problem



- A reduction is a multidimensional maze with up to 20 dimensions at 4-loops.
- There are multiple possible paths to the goal.
- Identifying the fastest path is highly non-trivial.
- Designing code to follow the fastest path is far from straightforward.
- `Kira` 3 is a C++ program.

The Don't-Lose-the-Knot Problem

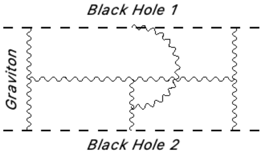
- The knot represents the initial system of equations
- Every laze represents a chain of equations
- Pulling the correct lazes is non trivial
- Pulling the correct chain of euations representing on chain is even more non-trivial
- **Math problem: which laze to pull without destroying the knot ↔ which equations to drop without destorying the system of equations**



	2.3, $s_{\max} = 5$	2.3, $s_{\max} = 6$	3.0, $s_{\max} = 5$	3.0, $s_{\max} = 6$
# of generated eqs.	3 317 357	6 009 193	4 053 617	7 511 785
# of selected eqs.	62 514	85 119	30984	30984

- `Kira 3` generates IBP for all sectors even if they are symmetric unlike in `Kira 2.3`.
- Generating IBPs for all sectors gives Kira more freedom to choose more efficient equations.
- New: Increasing $s_{\max}$ does not change the number of selected integrals. `Kira 3` selects 50% less equations than `Kira 2.3`.
- I am still suprised to see improvements with topo7 (figure above), a topology I study since my master thesis.

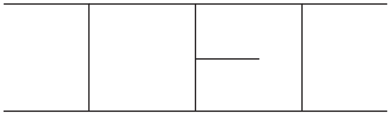
Main feature: Small sized system of equations



- Real example: SF2[1, 1, 1, 1, -2, -1, 0, -5, 1, 0, 1, 1, 1, 1, 1, -1, 1, 0, 0, 1, 0, 1], s=9, r=13, d=0
- `Kira 2.3` would seed integrals like: SF2[1, 1, 1, 1, -2, -1, 0, -5, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0], s=9, r=4, d=0
- Laporta algorithm with s=9 leads to *4,686,825* necessary seeds, very bad
- `Kira 3` only seeds integrals like: SF2[1, 1, 1, 1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], s=1, r=4, d=0
- This time only *19* seeds are needed, very good
- New option: `truncate_sp: [1: 5]`
  forces `Kira` to generate seeds with $s \leq \max(1,\ t-l+1,\ s_{\text{max,sector}})$

"That's great progress, but it's also a bit frustrating. Over almost twenty-five years, no one had guessed this one simple change?"
– Matthew von Hippel, 4Gravitons blog

Comparison of methods

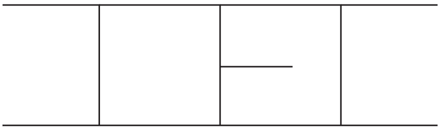


(r.s.d) 8.5.0	Kira 3	syzygies	syzygies (spc.)	block triangular
T_{gen}	8.8 s	$\sim$ hours	$\sim$ hours	$\sim$ hours
# eq.	41 998	26 106	23 834	3 497
# terms	734 833	1 498 728	544 355	388 973
disk [MiB]	3.7	46	8.6	23
$T_{\texttt{pyRed}}$ [s]	1.3	3.3	0.42	0.63
$T_{\texttt{Ratracer}}$ [s]	0.25	1.31	0.068	0.5

- Kira 3 generates the system in under 10 seconds and uses significantly less disk space. In contrast, system generation with NeatIBP [arXiv:2305.08783, arXiv:2502.20778] and blade [arXiv:2405.14621] takes hours and requires more storage.
- Number of equations is the biggest with Kira 3 but the coefficients are as simple as they can get.
- Run time of Kira 3 is either comparable or better, especially in combination with Ratracer.
- Current implementation of spanning cuts (spc.) in NeatIBP is not generic.
- Implementing spc. in Kira 3 is possible.

Kira effect universal

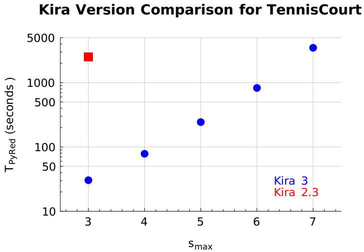
problem	r.s.d	8.5.0 (doublePentagon) (l=4)				
	version	Kira 2.3	Kira 3			
	topology	massless (108)		one-mass (142)	two-mass (185)	three-mass (172)
generation	# generated	16 872 564	73 902	76045	77286	78480
	# independent	4 842 650	53 648	55596	57034	57918
	# selected	1 157 381	41 998	43827	45359	46231
	RAM [GiB]	25.04	0.34	0.37	0.46	0.43
	time to generate	33:46.0	0:08.8	0:09.9	0:12.8	0:11.6
FF	[Kira/Ratracer]	33/-	1.3/0.25	1.6/0.38	2.5/0.45	2.1/0.43

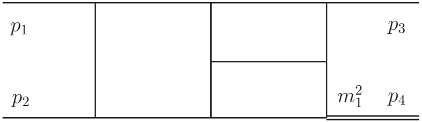


- 230 times less equations compared to version Kira 2.3
- 25 faster numerical evaluation of the system of equations on a prime finite field
- High performance sustained with next to no difference in number of scales ("difficulties")

3-loop Example

problem	r.s.d	10.3.0		10.4.0	10.5.0	10.6.0	10.7.0
	version	Kira 2.3	Kira 3				
	truncate_sp	-	l=7	l=6	l=5	l=4	l=3
generation	# gen.	8 272 762	460 257	1 054 294	3 719 838	12 011 203	34 303 672
	# indep.	4 890 067	333 097	719 512	2 211 669	6 300 414	16 169 898
	# sel.	2 768 144	140 098	389 248	1 187 692	3 363 240	8 319 835
	RAM [GiB]	35.44	0.64	1.87	5.97	18.23	54.07
	T_{gen} [min:s]	41:36.3	0:30.4	1:18.4	4:03.6	13:47.0	58:05.9
FF	T_{pyRed} [s]	146	1.0	4.4	15.6	53	206





- Generic algorithm works with tennis-court topology and beyond
- Kira 3 generates 18 times less, selects 20 times less, uses 56 times less main memory, and 83 times faster in generating the system than Kira 2.3
- Effect of algorithm improvements is illustrated in the plot

Numerical IBP

- IBP reductons can be saved with finite field coefficients
- Instead of using `firefly: true` we use the new option `numerical_points: numerics`
- Use case: Generate with `Kira` fast and small sized system of equations.
- The file `numerics` contains user input,

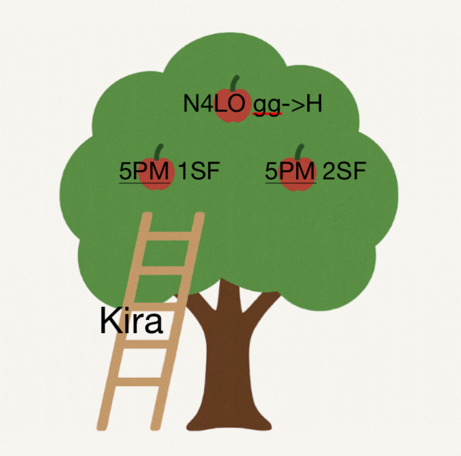
```
prime 9223372036854771977
d s12 s23 s34 s45 s56 s16 s123 s234 s345
11111 -1 -38 -1039/6 -2712776 -50409 -1662120 -95 -19926 -2752175
11112 -1 -38 -1039/6 -2712776 -50409 -1662120 -95 -19926 -2752175
```

- Used in the study of $2 \to 4$ planar differential equations [Abreu, Monni, Page, Usovitsch, 2412.19884], similar to [Henn, Matijašić, Miczajka, Peraro, Xu, Zhang, arXiv:2501.01847]

New directions

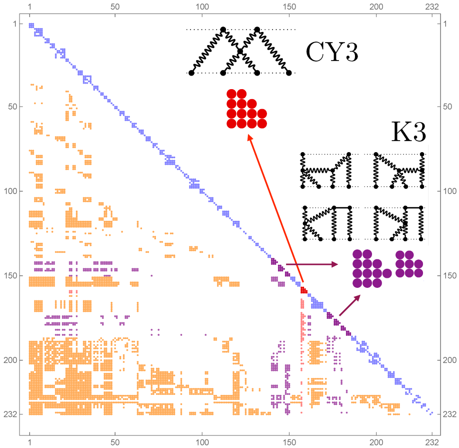
- Will AI give the next hint towards a new reduction algorithm?
- Will AI be able to find the fastest path in the maze?
- Will AI be able to minimize the Don't-Lose-the-Knot Problem?
- Link to other software: calculation of integrals (improving Ampred, AMFlow), generation of diagrams (Diagen in collaboration With Michał Czakon and Marco Niggetiedt)

Conclusions



- Kira hopefully is the tool to make the new generation of perturbative calculations happen.
- Is floating point Gaussian elimination possible?
- `Kira` is universal tool, all algorithms also work with user defined systems
- Tested up to 4-loops
- *To some, integrals are numbers; to others, they are art.*

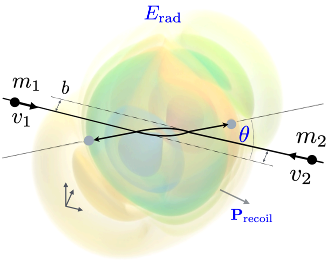
Differential Equation Matrix



- $(d + A(\epsilon, \vec{s}))\vec{I} = 0$, where $D = 4 - 2\epsilon$
- $(d + \epsilon A(\vec{s}))\vec{I} = 0$ [arXiv:1304.1806]
- CY3: $\left[\left(x\frac{d}{dx} - 1\right)^4 - x^4\left(x\frac{d}{dx} + 1\right)^4\right]\varpi(x) = 0$

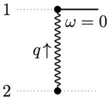
  [arXiv:2108.05310, arXiv:2503.20655]

Gravitational Waves



6.1 Leading order (1PM)

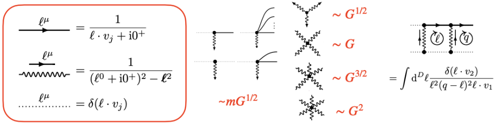
At leading order Δp_1^μ is described by a single diagram:



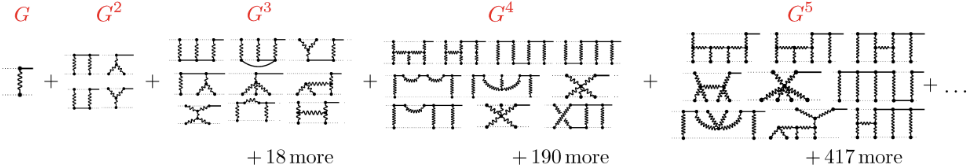
$$= i\frac{m_1 m_2}{4m_{\text{Pl}}^2}\int_q e^{iq\cdot b}\delta\!\!\!{}^-(q\cdot v_1)\delta\!\!\!{}^-(q\cdot v_2)(-v_1^\nu v_1^\rho q^\mu)\frac{P_{\nu\rho;\sigma\lambda}}{q^2}v_2^\sigma v_2^\lambda\,, \qquad (6.4)$$

- Classical physics, scattering of black holes, following Worldline QFT in [arXiv:2010.02865]
- Post-Minkowskian (PM) expanding the metric tensor in a power series of the gravitational constant G
- Third generation telescopes, Einstein Telescope, Cosmic Explorer, and LISA wish list: [arXiv: 2203.08228]
- Detection of black hole scattering events with LISA, launching in 2035
- Analytic calculations for speed / used next to the numerical relativity calculations

Post-Minkowskian (PM) Expansion



$\Delta p_1 =$



- The 5PM integrand is generated using the Berends-Giele type recursion relation, with collected self-force (SF) sectors according to their scaling with the masses m_1 and m_2:

$$\begin{aligned}\Delta p_1^{(5)\mu} =& m_1 m_2 \Big(m_2^4 \Delta p_{\mathsf{0SF}}^{(5)\mu} + m_1 m_2^3 \Delta p_{\mathsf{1SF}}^{(5)\mu} + m_1^2 m_2^2 \Delta p_{\mathsf{2SF}}^{(5)\mu} \\ &+ m_1^3 m_2 \Delta p_{\mathsf{1SF}}^{(5)\mu} + m_1^4 \Delta p_{\mathsf{0SF}}^{(5)\mu} \Big)\end{aligned}$$