Building the cosmological models vs. different models of set theory

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MATTER TO THE DEEPEST 2015

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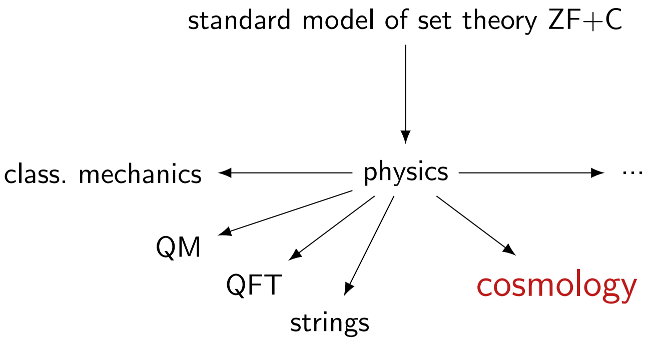


at each stage, a model of set theory is unaffected

ZFC accessible everywhere

from Planck to cosmological scales

Consequences?

- standard (boolean) logic
- standard line $\mathbb{R}$ of real numbers P. Klimasara's talk
- standard smooth structure

... everywhere

Why bother? Already pointed out:

- "logic" of QM rather "non-classical" [von Neumann,Birkhoff]
- $\mathbb{R}$ not really a best field to model QM [Isham]
- exotic smooth structures on $\mathbb{R}^4$ could act as sources of gravity [Brans,Sładkowski]

Instead, consider different (generally weaker) models — topoi [Król,Isham,Landsman,Guts,...]

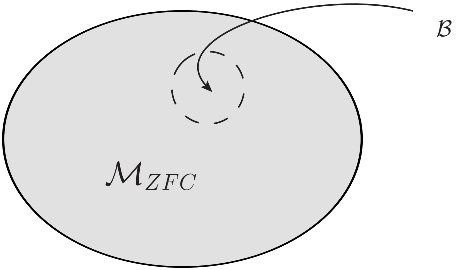
- first defined as sheaves on a topological space ...
- ... then abstracted to any “structure” containing singletons $\{*\}$, pullbacks, exponentials X^Y, analogues of “truth-value objects” $\{0,1\}$
- they are almost like universes of sets — powerful enough to perform a lot of mathematics, BUT:
- the logic is intuitionistic — NO axiom of choice, NO excluded middle (only constructive proofs!)

Guiding principle

The logic can be *locally* weakened due to local description of space-time $\mathcal{M}$ by some specific topos.

Let's focus on the specific one, called Basel topos $\mathcal{B}$ [Moerdijk-Reyes]

- sheaves on a site of C^∞-rings
- contain smooth, s-natural N and real R numbers
- s-numbers are better "adjusted", e.g. $[0, 1]$ is only s-compact, also R is archimedean only w.r.t N



Benefits?

- $\mathcal{B}$ contains both nilpotent ($d^2 = 0$) and invertible infinitesimals (non-standard analysis!) — smaller than $1/n$ for any n
- $\mathcal{B}$ contains infinite, s-natural numbers (again NSA!) — larger than any n

Note: the infinite and invertible infinitesimal numbers populate $\mathcal{B}$ as a result of model-theoretic forcing P. Klimasara's talk

Hypothesis

Does $\mathcal{B}$ somehow improve the smoothness of the underlying manifold?

Even further, we go into

distribution theory: [Moerdijk,Reyes]

All distributions in $\mathcal{B}$ are regular: even more, for any $T \in D'(R^n)$ there is smooth (non-standard) polynomial function $f : R^n \to R$ s.t.

$$T(\phi) = \int \mathrm{d}x f(x)\,\phi(x), \qquad \phi \in D(R^n)$$

Thus a multiplication of distributions is available

— a highly-demanded property!

The following applications generally obtained by the use of *Colombeau algebras*; but these methods are essentially equivalent [Todorov] (well, not exactly: one has to ensure that everything can be repeated constructively!)

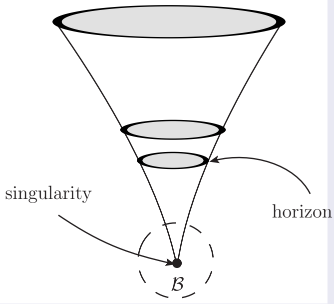
Geodesic completeness?

Let us look at the Schwarzschild metric

$$\mathrm{d}s^2 = h(r)\,\mathrm{d}t^2 - h(r)^{-1}\,\mathrm{d}r^2 + r^2\mathrm{d}\Omega, \quad h(r) = -1 + \frac{2m}{r}$$

from the $\mathcal{B}$-perspective:

- recall that classically we have geodesic incompleteness at $r = 0$: Ricci tensor $R_i^i \sim \delta$, curvature scalar $R \sim \delta$[Steinbauer],
- these are smooth solutions in $\mathcal{B}$



A Reissner–Nordström metric — point mass with charge — since [Lousto,Sanchez] apparent trouble:

(EM field tensor) $F_{ik} = 0$ while (stress-energy tensor) $T_{00} \sim F_{ik}^2 \sim \delta$

This potential contradiction is resolved when multiplication of distributions is formally taken into account [Steinbauer]:

Solution:

in fact $F_{ik} \approx 0$ (field is associated, or differs by an infinitesimal) to 0-distribution
there is no contradiction that further calculation gives

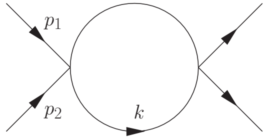
$$T_{00} \sim F_{ik}^2 \sim \delta$$

since association (resp. infinitesimal difference) is not compatible with a product!

ϕ^4 theory, UV divergence

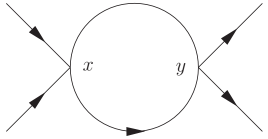
integrating over high energies

$$\int \frac{\mathrm{d}^4 k}{k^2(k+p_1+p_2)^2} \sim \int \frac{\mathrm{d}^4 k}{k^4}$$



taking products of distributions

$$\int \mathrm{d}^4 x G^2(x) \sim \int \frac{\mathrm{d}^4 x}{x^4}$$



Epstein–Glaser renormalization

restrict G^2 to $D\left(\mathbb{R}^4 \setminus \{0\}\right) \Rightarrow$ look for an extension $G^2_{\mathrm{ren}} \Rightarrow$ set the renormalization constants $\{c_i\}$ = BPHZ constants

$\mathcal{B}$-renormalization

interpret G^2 as a smooth polynomial function P in $\mathcal{B} \Rightarrow$ in fact, we obtain a whole class $\{P_1, \ldots\} \Rightarrow$ presumably, it is enumerated by $\{c_i\}$ (it has to be checked!)

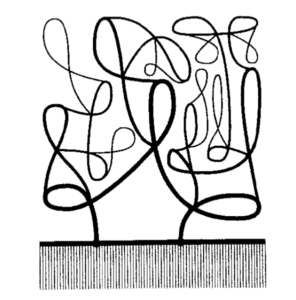
Local modification of a space-time by a Basel topos $\mathcal{B}$ changes smoothness structure:

It can be applied to

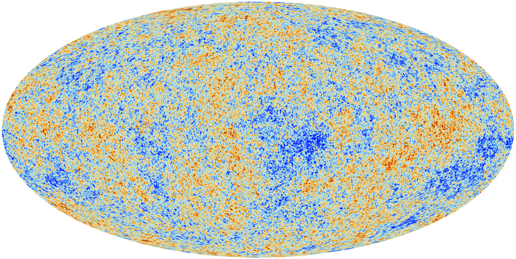
- cosmology
- QFT

Further work:

- if a Casson handle is infinite, it is *exotic*
- however, in $\mathcal{B}$ it becomes s-finite
- could it be non-exotic internally to $\mathcal{B}$?



- CMB, cosmological constant and model theory
- entropy of the early universe and language (of a theory)



Thank you!

Literature:

- I. Moerdijk, G. E. Reyes, Models for Smooth Infinitesimal Analysis, Springer, New York 1991
- E. Palmgren, Constructive nonstandard representations of generalized functions, Indagationes Mathematicae N.S. 11(1), 129-138
- T. Todorov, H. Vernaeve, Full algebra of generalized functions and non-standard asymptotic analysis, Logic and Analysis 1 (3-4):205-234
- R. Steinbauer, The Ultrarelativistic Reissner-Nordstrom field in the Colombeau algebra, J.Math.Phys. 38 (1997) 1614-1622
- J. Heinzle, R. Steinbauer, Remarks on the distributional Schwarzschild geometry, J.Math.Phys. 43 (2002) 1493-1508