Simplifying systems of differential equations
The case of the Sunrise graph

Lorenzo Tancredi

TTP - KIT, Karlsruhe

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partly based on collaboration with *E. Remiddi*

[arXiv:1311.3342], [arXiv:1509.03330]

Introduction

1. Our understanding of Standard model is based on perturbative expansions in **Feynman Diagrams**
2. Feynman Diagrams can be expressed as collection of **Feynman Integrals**
3. **Analytical** or **numerical** calculation of Feynman Integrals of increasing complexity is of crucial importance for precise and reliable prediction to be compared with most recent measurements.

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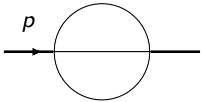
The two-loop massive sunrise graph

$$\mathcal{I}(n_1, n_2, n_3, n_4, n_5) \quad =$$



$$= \int \mathfrak{D}^d k\, \mathfrak{D}^d l\, \frac{(k \cdot p)^{n_4}\, (l \cdot p)^{n_5}}{(k^2 - m_1^2)^{n_1} (l^2 - m_2^2)^{n_2} ((k - l - p)^2 - m_3^2)^{n_3}}$$

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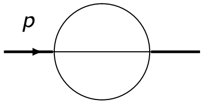

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How do we compute all these integrals?

Standard way to proceed

1. Use Integration-by-parts id.s (IBPs) to reduce them to **Master Integrals**
2. Use IBPs to derive **differential equations** (DE) satisfied by the MIs.
3. Try to solve these differential equations.

One finds:

a) Sunrise graph reduced to **4 MIs**
b) In general 4 **coupled** differential equations.

Different **choice** of the basis can **simplify** system of DE!

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How does this work for the Sunrise?

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$$S_1 = \int \frac{\mathfrak{D}^d k \mathfrak{D}^d l}{D_1 D_2 D_3}, \quad S_2 = \int \frac{\mathfrak{D}^d k \mathfrak{D}^d l}{D_1^2 D_2 D_3}, \quad S_3 = \int \frac{\mathfrak{D}^d k \mathfrak{D}^d l}{D_1 D_2^2 D_3}, \quad S_4 = \int \frac{\mathfrak{D}^d k \mathfrak{D}^d l}{D_1 D_2 D_3^2}$$

$\rightarrow$ 4 **coupled** DE in $d = 4$ and in $d = 2$ (and in *any even number of dimensions*...).

Choice 2

$$\widetilde{S}_1 = \int \mathfrak{D}^d k \mathfrak{D}^d l \frac{1}{D_1 D_2 D_3}, \quad \widetilde{S}_2 = \int \mathfrak{D}^d k \mathfrak{D}^d l \frac{1}{D_1^2 D_2 D_3},$$

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a) Result above found by **trial and error**.
$\rightarrow$ What is the irreducible number of coupled masters?

b) Initially answer through **Schouten Identities** in $d = 2$ dimensions
[Remiddi, Tancredi '13]

c) We will present here a generalisation of the method above.
Easier to apply and more powerful

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Study IBPs in **fixed number of dimensions**!

Fixing the values of d to an integer number in the IBPs can make the MIs degenerate $\rightarrow$ some of the MIs can become **linearly dependent** from the others!

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Imagine to produce IBPs for Sunrise graph with three different masses and, before solving them, fix $d = 2$.

a) This is equivalent to studying the IBPs as a *Laurent series* in $(d - 2)$
Feynman Integrals are in general **divergent** as $d \to n \in \mathbb{N}$

b) Upon solving the IBPs in $d = 2$ we find two new relations. This implies that in $d = 2$ there the MIs degenerate and only 2 Masters are **linearly independent**!

$$m_2^2\, P(s, m_1^2, m_2^2, m_3^2)\, S_3(2; s) = (m_1^2 - m_2^2)(m_1^2 + m_2^2 - m_3^2 - s)\, S_1(2; s)$$
$$+\, m_1^2 \left(m_1^4 - 3m_2^4 + 2m_1^2(m_2^2 - m_3^2 - s) + (m_3^2 - s)^2 + 2m_2^2(m_3^2 + s)\right) S_2(2; s)$$

$$m_3^2\, P(s, m_1^2, m_2^2, m_3^2)\, S_4(2; s) = (m_1^2 - m_3^2)(m_1^2 - m_2^2 + m_3^2 - s) S_1(2; s)$$
$$+\, m_1^2 \left(m_1^4 + m_2^4 - 3m_3^4 + 2m_2^2(m_3^2 - s) + 2m_3^2 s + p^4 - 2m_1^2(m_2^2 - m_3^2 + s)\right) S_2(2; s)\,,$$

where we defined the polynomial

$$P(s, m_1^2, m_2^2, m_3^2) = (-3m_1^4 + m_2^4 + (m_3^2 - s)^2 - 2m_2^2(m_3^2 + s) + 2m_1^2(m_2^2 + m_3^2 + s)).$$

Using this piece of information we can define a new basis of MIs in **d dimensions!**

$$S_1 = S_1(d;s)\,, \qquad S_2 = S_2(d;s) \qquad \textit{unchanged}$$

$$\begin{aligned} Z_3 = &\; m_2^2\, P(s, m_1^2, m_2^2, m_3^2)\, S_3(d;s) - (m_1^2 - m_2^2)(m_1^2 + m_2^2 - m_3^2 - s)\, S_1(d;s) \\ &- m_1^2 \left(m_1^4 - 3m_2^4 + 2m_1^2(m_2^2 - m_3^2 - s) + (m_3^2 - s)^2 + 2m_2^2(m_3^2 + s) \right) S_2(d;s) \end{aligned}$$

$$\begin{aligned} Z_4 = &\; m_3^2\, P(s, m_1^2, m_2^2, m_3^2)\, S_4(d;s) - (m_1^2 - m_3^2)(m_1^2 - m_2^2 + m_3^2 - s) S_1(d;s) \\ &- m_1^2 \left(m_1^4 + m_2^4 - 3m_3^4 + 2m_2^2(m_3^2 - s) + 2m_3^2 s + p^4 - 2m_1^2(m_2^2 - m_3^2 + s) \right) S_2(d;s)\,, \end{aligned}$$

Such that by definition

$$Z_3 \approx Z_4 \approx \mathcal{O}\left((d-2)\right)$$

Differential equations for the new basis assume **block form** as $d \to 2$

$$\frac{d}{d\,p^2}\begin{pmatrix} S_1 \\ S_2 \\ Z_3 \\ Z_4 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} S \\ S_1 \\ Z_2 \\ Z_3 \end{pmatrix} + \mathcal{O}\left((d-2)\right)$$

The two new master integrals are completely decoupled in the limit $d \to 2$!

Order by order in $(d-2)$ we can derive a **second order differential equation** satisfied by $S(d;p^2)$. [see Weinzierl et al., '11]

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Solution of second order differential equation in terms of **Elliptic Polylogarithms** [see Luise Adams' talk]

The method can be easily generalized

a) In **different numbers of dimensions** $d = n \in \mathbb{N}$

b) For any **other topology** irrespective of number of loops or masses!

As a (*natural*) extension **the three-loop banana graph**

$I_4(d; n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9) =$

$$= \int \mathfrak{D}^d k_1 \, \mathfrak{D}^d k_2 \, \mathfrak{D}^d k_3 \, \frac{(k_1 \cdot p)^{n_5} (k_2 \cdot p)^{n_6} (k_3 \cdot p)^{n_7} (k_1 \cdot k_2)^{n_8} (k_1 \cdot k_3)^{n_9}}{(k_1^2 - m_1^2)^{n_1} (k_2^2 - m_2^2)^{n_2} (k_3^2 - m_3^2)^{n_3} ((k_1 + k_2 + k_3 - p)^2 - m_4^2)^{n_4}}$$

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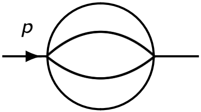

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The case of **all equal masses**

All masses equal $m_4 = m_3 = m_2 = m_1 = m \rightarrow 3$ MIs in d dimensions

$$\mathcal{I}_1(d;s) = I_1(d;1,1,1,1,0,0,0,0,0)\,, \quad \mathcal{I}_2(d;s) = I_1(d;2,1,1,1,0,0,0,0,0)\,,$$
$$\mathcal{I}_3(d;s) = I_1(d;3,1,1,1,0,0,0,0,0)\,.$$

They remain independent if $d=2$ $\rightarrow$ The scalar master integrals fulfil a **third-order differential equation**!

$$D_d^{(3)}\,\mathcal{I}_1(d;s) = 0\,,$$

$$D_d^{(3)} = \frac{d^3}{d\,s^3} + \frac{3\left(64m^4 + 10(d-5)m^2 s - (d-4)s^2\right)}{s(s-4m^2)(s-16m^2)}\,\frac{d^2}{d\,s^2}$$
$$+ \frac{(d-4)(11d-36)s^2 - 64(d-4)d\,m^4 - 4\,(216 + d(7d-88))\,m^2\,s}{4\,s^2(s-4m^2)(s-16m^2)}\,\frac{d}{d\,s}$$
$$+ \frac{(3-d)(3d-8)\left(2(d+2)m^2 + (d-4)s\right)}{4\,s^2\,(s-4m^2)(s-16m^2)}$$

The case of **two different masses** - two different configurations

$$I_2^A(d; n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9) = I_4(d; n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9)\Big|_{m_3=m_2=m_1=m_a, m_4=m_b}$$

$$I_2^B(d; n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9) = I_4(d; n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9)\Big|_{m_2=m_1=m_a, m_4=m_3=m_b}$$

A) In configuration A 5 independent MIs in d dimensions

$$\mathcal{I}_1^A(d;s) = I_2^A(d; 1,1,1,1,0,0,0,0,0)\,, \quad \mathcal{I}_2^A(d;s) = I_2^A(d; 2,1,1,1,0,0,0,0,0)\,,$$
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$$\mathcal{I}_5^A(d;s) = I_2^A(d; 2,2,1,1,0,0,0,0,0)$$

B) In configuration B 6 independent MIs in d dimensions

$$\mathcal{I}_1^B(d;s) = I_2^B(d; 1,1,1,1,0,0,0,0,0)\,, \quad \mathcal{I}_2^B(d;s) = I_2^B(d; 2,1,1,1,0,0,0,0,0)\,,$$
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Simplifying the system of differential equations in $d = 2$

A) Solving IBPs in $d = 2$ we find that 4 MIs remain independent

$$m_a^2(s - 5m_a^2 + m_b^2)\,\mathcal{I}_5^A(2;s) = + \frac{3m_a^2 + m_b^2 - s}{12\,m_a^2}\,\mathcal{I}_1^A(2;s) + \frac{51m_a^4 + (m_b^2 - s)^2 - 6m_a^2(m_b^2 + 2s)}{12\,m_a^2}\,\mathcal{I}_2^A(2;s) + \frac{m_b^2(m_b^2 - s)}{6\,m_a^2}\,\mathcal{I}_3^A(2;s) + \frac{21m_a^4 + (m_b^2 - s)^2 - 6m_a^2(m_b^2 + s)}{6}\,\mathcal{I}_4^A(2;s)\,.$$

B) Similarly we find here two relations such that again only 4 MIs remain independent

$$\mathcal{I}_5^B(s;2) = a_1\,\mathcal{I}_1(2;s) + a_2\,\mathcal{I}_2(2,s) + a_3\,\mathcal{I}_3(2,s) + a_4\,\mathcal{I}_4(2,s)$$

$$\mathcal{I}_6^B(s;2) = b_1\,\mathcal{I}_1(2;s) + b_2\,\mathcal{I}_2(2,s) + b_3\,\mathcal{I}_3(2,s) + b_4\,\mathcal{I}_4(2,s)$$

As for the two-loop sunrise, these relations can be used to decouple 1 (2) MIs from the system of differential equations.

4 differential equations remain **coupled**, corresponding to a **fourth-order differential equation** for the scalar amplitude.

Simplifying the system of differential equations in $d = 2$

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$$m_a^2(s - 5m_a^2 + m_b^2)\,\mathcal{I}_5^A(2;s) = \\ + \frac{3m_a^2 + m_b^2 - s}{12\,m_a^2}\,\mathcal{I}_1^A(2;s) + \frac{51m_a^4 + (m_b^2 - s)^2 - 6m_a^2(m_b^2 + 2s)}{12\,m_a^2}\,\mathcal{I}_2^A(2;s) \\ + \frac{m_b^2(m_b^2 - s)}{6\,m_a^2}\,\mathcal{I}_3^A(2;s) + \frac{21m_a^4 + (m_b^2 - s)^2 - 6m_a^2(m_b^2 + s)}{6}\,\mathcal{I}_4^A(2;s)\,.$$

B) Similarly we find here two relations such that again only 4 MIs remain independent

$$\mathcal{I}_5^B(s;2) = a_1\,\mathcal{I}_1(2;s) + a_2\,\mathcal{I}_2(2,s) + a_3\,\mathcal{I}_3(2,s) + a_4\,\mathcal{I}_4(2,s)$$
$$\mathcal{I}_6^B(s;2) = b_1\,\mathcal{I}_1(2;s) + b_2\,\mathcal{I}_2(2,s) + b_3\,\mathcal{I}_3(2,s) + b_4\,\mathcal{I}_4(2,s)$$

As for the two-loop sunrise, these relations can be used to decouple 1 (2) MIs from the system of differential equations.

4 differential equations remain **coupled**, corresponding to a **fourth-order differential equation** for the scalar amplitude.

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The **general case** with 4 different masses

In the most general case one finds 11 different master integrals in d dimensions

$$\begin{aligned}
\mathcal{I}_1(d;s) &= I_4(d;1,1,1,1,0,0,0,0,0)\,, \quad \mathcal{I}_2(d;s) = I_4(d;2,1,1,1,0,0,0,0,0)\,,\\
\mathcal{I}_3(d;s) &= I_4(d;1,2,1,1,0,0,0,0,0)\,, \quad \mathcal{I}_4(d;s) = I_4(d;1,1,2,1,0,0,0,0,0)\,,\\
\mathcal{I}_5(d;s) &= I_4(d;1,1,1,2,0,0,0,0,0)\,, \quad \mathcal{I}_6(d;s) = I_4(d;3,1,1,1,0,0,0,0,0)\\
\mathcal{I}_7(d;s) &= I_4(d;2,2,1,1,0,0,0,0,0)\,, \quad \mathcal{I}_8(d;s) = I_4(d;2,1,2,1,0,0,0,0,0)\\
\mathcal{I}_9(d;s) &= I_4(d;2,1,1,2,0,0,0,0,0)\,, \quad \mathcal{I}_{10}(d;s) = I_4(d;1,2,2,1,0,0,0,0,0)\\
&\mathcal{I}_{11}(d;s) = I_4(d;1,2,1,2,0,0,0,0,0)\,.
\end{aligned}$$

Repeating the exercise of solving the IBPs in $d=2$ we find 5 independent relations such that **6 master integrals** remain independent!

$\rightarrow$ corresponds to a *sixth-order differential equation for the scalar amplitude*

Conclusions

a) Differential equations are one of the most important tools for calculation of Feynman integrals

b) For complicated Feynman graph one is often faced with **large systems of coupled differential equations**

c) Coupled differential equations can be simplified in the limit $d \to n \in \mathbb{N}$ by decoupling some of the equations in that limit.

d) Information needed for the decoupling can be found **studying the IBPs** in the limit $d \to n \in \mathbb{N}$!

Given a topology, what is the **irreducible** number of **coupled master integrals**?

Thanks!