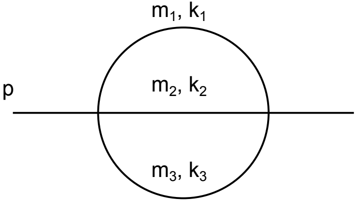
The Sunrise Integral and Elliptic Polylogarithms

Luise Adams*, Christian Bogner, Stefan Weinzierl



Johannes Gutenberg University Mainz

Matter to the Deepest, Ustroń, 14th September 2015

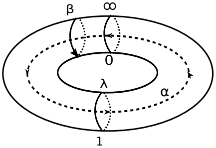
Elliptic Curves and Elliptic Integrals

An elliptic curve can be written with the help of the Weierstrass equation:

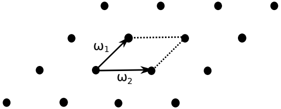
$$Y^2 Z + a_1 XYZ + a_3 YZ^2 = X^3 + a_2 X^2 Z + a_4 XZ^2 + a_6 Z^3$$

and is topologically equivalent to a torus;

two integrals along the paths α, β are in the first homology group of the torus $\Rightarrow$ periods ω_1, ω_2



elliptic integral = path integral along elliptic curve E;
only well-defined mod $\Lambda = \{n_1\omega_1 + n_2\omega_2 : n_1, n_2 \in \mathbb{Z}\}$
which defines a lattice



$E(\mathbb{C}) \xrightarrow{\text{elliptic integrals}}$ torus $\mathbb{C}\backslash\Lambda$: elliptic integral gives an isomorphism from $E(\mathbb{C})$ to $\mathbb{C}\backslash\Lambda$

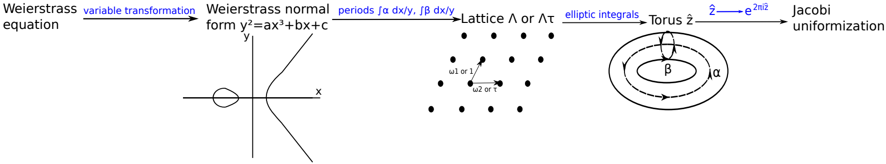
modified lattice Λ_τ generated by 1 and $\tau = \omega_2/\omega_1$:

$$\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z} \qquad \text{and it is} \qquad \Lambda_\tau = \Lambda_{\tau+k}, k \in \mathbb{Z} \qquad \text{with } q = e^{2\pi i\tau}$$

Under the exponential map $J : \mathbb{C} \to \mathbb{C}^*, z \to e^{2\pi iz} = w$ the lattice Λ_τ in $\mathbb{C}$ is mapped to $q^{\mathbb{Z}}$ in $\mathbb{C}^*$ $\quad\to$ analytic isomorphism $E_\tau = \mathbb{C}/\Lambda_\tau \to \mathbb{C}^*/q^{\mathbb{Z}}$

The representation of the elliptic curve in $\mathbb{C}^*/q^{\mathbb{Z}}$ is called the Jacobi uniformization of the curve.

Chain of mappings between the different representations of an elliptic curve



Classical and Elliptic Polylogarithms

Multiple polylogarithms are special Z-sums with two different representations:

1) as nested sums: $\mathrm{Li}_{m_1,m_2,\dots,m_k}(x_1,\dots,x_k) = \sum_{\infty \geq i_1 \geq i_2 \geq \dots \geq i_k \geq 0} \frac{x_1^{i_1}}{i_1^{m_1}} \frac{x_2^{i_2}}{i_2^{m_2}} \dots \frac{x_k^{i_k}}{i_k^{m_k}}$

2) as iterated integrals:

With $G(z_1,\dots,z_k;y) = \int_0^y \frac{dt_1}{t_1 - z_1} \int_0^{t_1} \frac{dt_2}{t_2 - z_2} \dots \int_0^{t_{k-1}} \frac{dt_k}{t_k - z_k}$ and

$G(0,\dots,0;y) := \frac{1}{k!}(\log y)^k$ we can define

$G_{m_1,\dots,m_k}(z_1,\dots,z_k;y) = G(\underbrace{0,\dots,0}_{m_1-1},z_1,\dots,z_{k-1},\underbrace{0,\dots,0}_{m_k-1},z_k;y)$ and obtain

$$\mathrm{Li}_{m_1,m_2,\dots,m_k}(x_1,x_2,\dots,x_k) = (-1)^k G_{m_1,m_2,\dots,m_k}\left(\frac{1}{x_1},\frac{1}{x_1 x_2},\dots,\frac{1}{x_1 x_2 \dots x_k}\right) (\star)$$

$$\Leftrightarrow G_{m_1,m_2,\dots,m_k}(z_1,z_2,\dots,z_k;y) = (-1)^k \mathrm{Li}_{m_1,m_2,\dots,m_k}\left(\frac{y}{z_1},\frac{z_1}{z_2},\dots,\frac{z_{k-1}}{z_k}\right)$$

generalise eq. $(\star)$ (for $\mathbb{P}^1\backslash\{0,1,\infty\}$) to a punctured elliptic curve $\mathcal{E}^\times$

$\rightsquigarrow$ LHS: elliptic polylogarithms, RHS: iterated integrals on an elliptic curve

Elliptic polylogarithms 'live' on an elliptic curve; they are **iterated integrals** on the configuration space $\mathcal{E}^{(n)}$ of an elliptic curve with n marked points
[Brown, Levin 2013], [Levin, 1997]

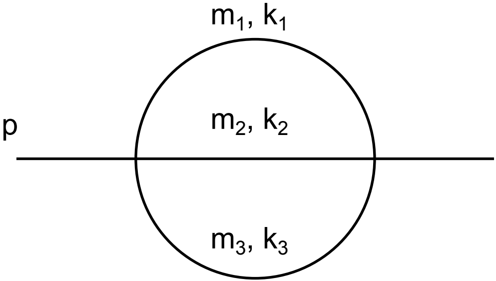
Basic idea: Average multivalued functions on a punctured elliptic curve $E^{\times} = E \backslash \{0\}$ with respect to the multiplication by q arriving at

$$\rightsquigarrow \sum_{m_1, m_2, \ldots, m_r \in \mathbb{Z}} u_1^{m_1} u_2^{m_2} \ldots u_r^{m_r} \mathrm{Li}_{n_1, n_2, \ldots, n_r} \left(\frac{q^{m_1} t_1}{q^{m_2} t_2}, \frac{q^{m_2} t_2}{q^{m_3} t_3}, \ldots, \frac{q^{m_{r-1}} t_{r-1}}{q^{m_r} t_r}, q^{m_r} t_r \right)$$

The parameters u_i dampen the singularities of the polylogs;

Compute the pole structure in the u_i coordinates, regularise the function with $u_i = \exp(2\pi i \alpha_i)$
$\Rightarrow$ the coefficients of the Taylor expansion around $\alpha_i = 0, i = 0, 1, \ldots, r$ yield the multiple elliptic polylogarithms

II. The Sunrise Integral in $D = 2$ space-time dimensions



The two-loop sunrise integral in D dimensions reads
[Caffo, Czyz, Laporta, Remiddi 1998; Laporta, Remiddi 2004] :

$$S_{\nu_1\nu_2\nu_3}(D, p^2, m_1, m_2, m_3) =$$
$$= (\mu^2)^{\nu-D} \int \frac{d^D k_1}{i\pi^{\frac{D}{2}}} \int \frac{d^D k_2}{i\pi^{\frac{D}{2}}} \frac{1}{(-k_1^2 + m_1^2)^{\nu_1}(-k_2^2 + m_2^2)^{\nu_2}(-(p - k_1 - k_2)^2 + m_3^2)^{\nu_3}}$$

In its Feynman parameterisation the integral reads $(\nu = \nu_1 + \nu_2 + \nu_3)$:

$$S_{\nu_1\nu_2\nu_3}(D, t) = \frac{\Gamma(\nu - D)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} (\mu^2)^{\nu-D} \int_\sigma \omega \; x_1^{\nu_1 - 1} x_2^{\nu_2 - 1} x_3^{\nu_3 - 1} \frac{\mathcal{U}^{\nu - \frac{3}{2}D}}{\mathcal{F}^{\nu - D}}$$

with the differential two-form

$$\omega = x_1 \; dx_2 \wedge dx_3 + x_2 \; dx_3 \wedge dx_1 + x_3 \; dx_1 \wedge dx_2$$

and the integration region

$$\sigma = \{[x_1 : x_2 : x_3] \in \mathbb{P}^2 | x_i \geq 0, i = 1, 2, 3\}$$

and the first and second graph polynomial $(t = p^2)$
$\mathcal{U} = x_1x_2 + x_2x_3 + x_1x_3, \; \mathcal{F} = -x_1x_2x_3 t + (x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2)(x_1x_2 + x_2x_3 + x_3x_1).$

In D dimensions the integral S_{111} satisfies a differential equation of order four:

$$\Bigg\{\underbrace{P_4\frac{d^4}{dt^4} + P_3\frac{d^3}{dt^3} + P_2\frac{d^2}{dt^2} + P_1\frac{d}{dt} + P_0}_{=:L_4(D)}\Bigg\} S_{111}(D,t) = \mu^2\left[c_{12}T_{12} + c_{23}T_{23} + c_{13}T_{13}\right]$$

where the P_i's and c_{ij}'s are polynomials in D, t and the masses and the T_{ij}'s are products of tadpoles [Adams, Bogner, Weinzierl, 2015]

In $D = 2 - 2\epsilon$ the integral S_{111} and the differential operator $L_4^{(0)}$ can be expanded in a Laurent series leading to a factorisable differential operator in ϵ^0:

$$L_{1,a}^{(0)}(2)\ L_{1,b}^{(0)}(2)\ L_2^{(0)}(2)\ S_{111}^{(0)}(2,t) = -32\mu^2 t^2(15t^2 + 14M_{100}t + 77\Delta)$$

$$\rightsquigarrow \left\{p_2(t)\frac{d^2}{dt^2} + p_1(t)\frac{d}{dt} + p_0(t)\right\} S_{111}^{(0)}(2,t) = p_3(t)$$

[Müller-Stach, Weinzierl, Zayadeh, 2013]

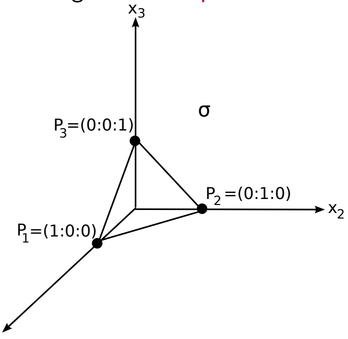
in ϵ^1: $\quad L_{1,a}^{(0)}(2)\ L_{1,b}^{(0)}(2)\ L_2^{(0)}(2)\ S_{111}^{(1)}(2,t) = I_1(t)$ [later more]

First consider the equation

$$\mathcal{F} = 0$$

which defines together with the choice of an origin $\mathcal{O}$ an elliptic curve.

Choose one of the intersection points of the integration region σ with the variety defined by $\mathcal{F} = 0$ as origin, e.g. $P_3 = [0 : 0 : 1]$.



Transform into Weierstrass normal form

$$\hat{E} : y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3 \qquad \text{with} \qquad \mathcal{O} = [0 : 1 : 0]$$

Working with $z=1$ one can factorise the RHS to

$$y^2 = 4(x-e_1)(x-e_2)(x-e_3) \qquad \text{with} \qquad e_1+e_2+e_3=0$$

where the roots e_i depend on t, the masses and the invariants

$$g_2 = -4(e_1e_2+e_2e_3+e_3e_1), \qquad g_3 = 4e_1e_2e_3$$

With the assumption $0 \le m_1 \le m_2 \le m_3$ it is

$$e_2 \le e_3 < 0 < e_1$$

and one can find the periods of the elliptic curve in Weierstrass form:

$$\Psi_1 = 2\int_{e_2}^{e_3} \frac{dx}{y} = \frac{4\mu^2}{\sqrt[4]{D}} K(k), \qquad \Psi_2 = 2\int_{e_1}^{e_3} \frac{dx}{y} = \frac{4i\mu^2}{\sqrt[4]{D}} K(k')$$

where $K(x)$ denotes the complete elliptic integral of the first kind

$$K(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}}$$

with the (complementary) modulus $k^{(\prime)}$: $k = \sqrt{\frac{e_3-e_2}{e_1-e_2}}$ and
$k' = \sqrt{1-k^2} = \sqrt{\frac{e_1-e_3}{e_1-e_2}}$

The periods Ψ_1, Ψ_2 are the solutions of the homogeneous differential equation! [Adams, Bogner, Weinzierl (2013, 2014)]

Consider the ratio of the two periods τ and the nome q:

$$\tau = \mathrm{i}\frac{K(k')}{K(k)}, \qquad q = \mathrm{e}^{\mathrm{i}\pi\tau}$$

and you can make a transformation of the variable $t \to q$ [Bloch, Vanhove, 2013]
With variation of the constants one finds for a special inhomogeneous solution:

$$S_{special} = -\mu^2 \frac{\Psi_1(q)}{\pi^2} \int_0^q \frac{dq'}{q'} \int_0^{q_1} \frac{dq''}{q''} \frac{p_3(q'')\Psi_1(q'')^3}{p_2(q'')W(q'')^2}$$

with the Wronski determinant $W = \Psi_1 \frac{d}{dt}\Psi_2 - \Psi_2 \frac{d}{dt}\Psi_1$

Aim: Express the special solution in terms of the homogeneous solutions and generalised (elliptic) polylogarithms

Consider the elliptic curve E_i defined by $\mathcal{F} = 0$ with origin P_i

$\rightarrow$ transform into Weierstrass normal form $\hat{E}$ with origin $Q_{i,i} = [0:1:0]$

$\rightarrow$ Periods Ψ_1, Ψ_2 of $\hat{E}$ define a lattice Λ

$\rightarrow$ map from $\hat{E}$ to the torus $\mathbb{C}/\Lambda$ with the elliptic integral

$$[x:y:1] \rightarrow \hat{z} = \frac{1}{\Psi_1} \int_x^\infty \frac{dx}{\sqrt{4(x-e_1)(x-e_2)(x-e_3)}}$$

$\rightarrow$ intersection points $Q_{j,k} \rightarrow \hat{z}_i = \frac{1}{2}\frac{F(u_i, k)}{K(k)}$ with $u_i = \sqrt{\frac{e_1 - e_2}{x_{j,k} - e_2}}$ ((i,j,k) as cyclic permutation of (1,2,3) and $F(z, x)$ as the incomplete elliptic integral of the first kind

$$F(z, x) = \int_0^z \frac{dt}{\sqrt{(1 - t^2)(1 - x^2 t^2)}}$$

($Q_{i,j} \rightarrow [x_{i,j} : y_{i,j} : 1]$ is the image of the point $P_i \in E_j$ on $\hat{E}$)

$\rightarrow$ map to the Jacobi uniformization $C^*/q^{2\mathbb{Z}}$ with

$$\hat{z} \rightarrow w = e^{2\pi i \hat{z}}$$

$\rightarrow$ points $\hat{z}_i \rightarrow w_i$ with $w_i = \exp\left[i\pi \frac{F(u_i, k)}{K(k)}\right]$

The classical polylogarithms are defined by

$$\mathrm{Li}_n(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^n},$$

while the real and imaginary part of $\mathrm{Li}_n(e^{i\varphi})$ is given by the Clausen and Glaisher functions:

$$\mathrm{Cl}_n(\varphi) = \begin{cases} \frac{1}{2i}\left[\mathrm{Li}_n(e^{i\varphi}) - \mathrm{Li}_n(e^{-i\varphi})\right], \\ \frac{1}{2}\left[\mathrm{Li}_n(e^{i\varphi} + \mathrm{Li}_n(e^{-i\varphi}))\right], \end{cases} \qquad \mathrm{Gl}_n(\varphi) = \begin{cases} \frac{1}{2}\left[\mathrm{Li}_n(e^{i\varphi}) + \mathrm{Li}_n(e^{-i\varphi})\right], & n \text{ even} \\ \frac{1}{2i}\left[\mathrm{Li}_n(e^{i\varphi}) - \mathrm{Li}_n(e^{-i\varphi})\right], & n \text{ odd.} \end{cases}$$

Consider the following generalisation of polylogarithms:

$$\mathrm{ELi}_{n;m}(x, y; q) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{x^j}{j^n}\frac{y^k}{k^m} q^{jk} = \sum_{k=1}^{\infty} \mathrm{Li}_n(xq^k)\frac{y^k}{k^m}$$

and define the weight to be $w = n + m$ and

$$E_{n;m}(x, y; q) =$$

$$\begin{cases} \frac{1}{i}\left[\frac{1}{2}\mathrm{Li}_n(x) - \frac{1}{2}\mathrm{Li}_n(x^{-1}) + \mathrm{ELi}_{n;m}(x, y; q) - \mathrm{ELi}_{n;m}(x^{-1}, y^{-1}; q)\right], & n + m \text{ even} \\ \frac{1}{2}\mathrm{Li}_n(x) + \frac{1}{2}\mathrm{Li}_n(x^{-1}) + \mathrm{ELi}_{n;m}(x, y; q) + \mathrm{ELi}_{n;m}(x^{-1}, y^{-1}; q), & n + m \text{ odd} \end{cases}$$

The result in the arbitrary mass case in two dimensions is then given by

$$S_{111}^{(0)} = \frac{\Psi_1(q)}{\pi} \left[\mathrm{E}_{2;0}(w_1(q), -1; -q) + \mathrm{E}_{2;0}(w_2(q), -1; -q) + \mathrm{E}_{2;0}(w_3(q), -1; -q) \right]$$

[Adams, Bogner, Weinzierl, 2014]

with the functions w_i which are the images of the intersection points, not chosen as origin under the chain of mappings

$$E_i \to \hat{E} \to \mathbb{C}/\Lambda \to \mathbb{C}^*/q^{2\mathbb{Z}}$$

as arguments of the elliptic dilogarithms

In the equal mass case the solution simplifies to

$$S_{111}^{(0)} = 3\frac{\Psi_1(q)}{\pi} \mathrm{E}_{2;0} \left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}, -1; -q \right).$$

[Bloch, Vanhove, 2013]

III. The Sunrise Integral in $D = 4$ space-time dimensions

The Laurent expansion of the two-dimensional solution starts at ϵ^0:

$$S_{111}(2,t) = S_{111}^{(0)}(2,t) + \epsilon S_{111}^{(1)}(2,t) + \mathcal{O}[\epsilon^2]$$

The Laurent expansion of the four-dimensional solution starts at ϵ^{-2}:

$$S_{111}(4,t) = S_{111}^{(-2)}(4,t)\frac{1}{\epsilon^2} + S_{111}^{(-1)}(4,t)\frac{1}{\epsilon} + S_{111}^{(0)}(4,t) + \epsilon S_{111}^{(1)}(4,t) + \mathcal{O}[\epsilon^2]$$

In the basis $(i = \{1,2,3\})$

$$\mu^2 \frac{\partial}{\partial t} S_{111}(D,t) = S_{222}(D+2,t),\ \ \mu^2 \frac{\partial}{\partial m_i^2} S_{111}(D,t) = -S_{1+\delta_{1i}\ 1+\delta_{2i}\ 1+\delta_{3i}}(D,t)$$

we find with the dimensional shift relations [Tarasov, 1996; 1997]

$$S_{111}^{(0)}(4,t) = \frac{1}{\mu^4}\tilde{L}_3^{(-1)}(2)\ S_{111}^{(1)}(2,t) + \frac{1}{\mu^4}\tilde{L}_3^{(0)}(2)\ S_{111}^{(0)}(2,t) + \tilde{R}$$

where $\tilde{L}_3^{(k)} = C_0^{(k)} + \sum_{i=1}^{3} C_i^{(k)} m_i^2 \frac{\partial}{\partial m_i^2}$ and the $C_0^{(k)}, C_i^{(k)}$ are polynomials in t and the masses and $\tilde{R}$ contains simpler terms depending on the masses and t as well as $\log\left(\frac{m_i^2}{\mu^2}\right)$-terms $\rightsquigarrow$ still missing: $S_{111}^{(1)}(2,t)$

Now consider the second-order differential equation for $S^{(1)}_{111}(2,t)$ for the equal mass case [Laporta, Remiddi, 2004] with the first-order differential operator $A^{(1)}_{em}(t,m)$:

$$L^{(0)}_{2,em}\ S^{(1)}_{111}(2,t) = 12\mu^2 \log\left(\frac{m^2}{\mu^2}\right) + A^{(1)}_{em}(t,m)\ S^{(0)}_{111}(2,t)$$

We make an obvious ansatz for $S^{(1)}_{111}(2,t)$ consisting of a part proportional to $S^{(0)}_{111}(2,t)$ and a remainder:

$$S^{(1)}_{111}(2,t) = \tilde{S}^{(1)}_{111}(2,t) + F_1(t)\ S^{(0)}_{111}(2,t)$$

With this ansatz it is possible to solve the above differential equation for $\tilde{S}^{(1)}_{111}(2,t)$

$$\tilde{S}^{(1)}_{111}(2,t) = \text{homogeneous solutions} + \tilde{S}^{(1)}_{111,\text{special}}(2,t)$$

and finally to find an expression for $S^{(1)}_{111}(2,t)$.

With the q-expansion of $\tilde{S}^{(1)}_{111,special}$ and the full solution $S^{(1)}_{111}(2,t) = \frac{\Psi_1}{\pi} E^{(1)}$ we find for $E^{(1)}$ where $S^{(0)}_{111}(2,t) = \frac{\Psi_1}{\pi} E^{(0)}$ a quite long expression with (elliptic) polylogarithms consisting of a large part of homogeneous weight three and a small one of mixed weight three and four:

1) Part of homogeneous weight three

- term proportional to $E^{(0)}$
- term containing ordinary polylogarithms of the form $\mathrm{Li}_{2,1}, \mathrm{Li}_3, \log \times \mathrm{Li}_2$
- term with the following weight three - elliptic polylogarithms and the w_i's and ± 1 as arguments

$$\mathrm{E}_{0,1;-2,0;4}(x_1, x_2; y_1, y_2; -q) =$$
$$= \frac{1}{\mathrm{i}} \Big\{ \left[\mathrm{ELi}_{2;0}(x_1, y_1; -q) - \mathrm{ELi}_{2;0}\left(x^{-1}, y^{-1}; -q\right)\right] \times \frac{1}{2}\left[\mathrm{Li}_1(x_2) + \mathrm{Li}_1\left(x_2^{-1}\right)\right]$$
$$\sum_{j_1=1}^{\infty}\sum_{k_1=1}^{\infty}\sum_{j_2=1}^{\infty}\sum_{k_2=1}^{\infty} \frac{k_1^2}{j_2(j_1k_1 + j_2k_2)^2}\left(x_1^{j_1}y_1^{k_1} - x_1^{-j_1}y_1^{-k_1}\right)\left(x_2^{j_2}y_2^{k_2} - x_2^{-j_2}y_2^{-k_2}\right)$$
$$\times(-q)^{j_1k_1 + j_2k_2}\Big\}$$

2) Part with mixed weight three and four: $\mathrm{E}_{3;1}(w_j, -1; -q)$

- occurs as remainder of the $\log(-q)$-terms in
$S^{(1)}_{111}(2,t) = \tilde{S}^{(1)}_{111}(2,t) + F_1(t) S^{(0)}_{111}(2,t)$ and
$\tilde{S}^{(1)}_{111}(2,t) = c_1 \Psi_1 + c_2 \Psi_1 \log(-q) + \tilde{S}^{(1)}_{111,special}(2,t)$ which in principle cancel out

the function $\mathrm{E}_{3;1}$ results from an integration with respect to $d\log(q) = dq/q$ leading purely to terms of weight four because $\log(q)$ should be counted a weight two:

$$\int_0^q \mathrm{ELi}_{n;m}(x, y; q') d\log(q') = \mathrm{ELi}_{n+1;m+1}(x, y; q)$$

Final result for $S^{(0)}_{111}(4,t)$:

$$S^{(0)}_{111}(4,t) = \frac{1}{\mu^4} \tilde{L}_3^{(-1)}(2)\; S^{(1)}_{111}(2,t) + \frac{1}{\mu^4} \tilde{L}_3^{(0)}(2)\; S^{(0)}_{111}(2,t) + \tilde{R}$$

with $\tilde{L}_3^{(k)} = C_0^{(k)} + \sum_{i=1}^{3} C_i^{(k)} m_i^2 \frac{\partial}{\partial m_i^2}$ and $S^{(0)}_{111}(2,t)$ and $S^{(1)}_{111}(2,t)$ containing the elliptic polylogarithms

Conclusions

- ϵ^0-part of the sunrise integral in two space-time dimensions $S_{111}^{(0)}(2, t)$ expressible as sum of three elliptic dilogarithms with numbers as arguments which are the images of the intersection points of the elliptic curve and the integration region under a chain of mappings between the different representations of the elliptic curve

- ϵ^0-part in four space-time dimensions $S_{111}^{(0)}(4, t)$ depends on the solution in $D = 2$ dimensions denoted by $S_{111}^{(0)}(2, t)$ and $S_{111}^{(1)}(2, t)$, mass derivatives thereof and simpler terms

Example of the calculation of an elliptic polylogarithm: Li_1

Consider the generating series where u^{-1} is chosen such that the convergence is ensured

$$E(z;u) = \sum_{m\in\mathbb{Z}} u^m \mathrm{Li}_1(q^m z)$$

for $m \ll 0$: $\xrightarrow{\text{asymptotic}} u^m \log(q^m z)$ (bounded for $u > 1$)
for $m \gg 0$: $\longrightarrow u^m q^m z$ (bounded for $u < |q|^{-1}$)
$\Rightarrow$ absolute convergent for $1 < u < |q|^{-1}$ with pole at $u = 1$

change of variables: $u = e^{2\pi i\alpha}, z = e^{2\pi i\xi}$ implies a pole at $\alpha = 0$ ($\hat{=}\ u = 1$)
regularise function

$$E^{\text{reg}}(\xi,\alpha) = E(\xi,\alpha) - \frac{1}{\alpha}$$

Compute Taylor expansion of E^{reg} around $\alpha = 0$; coefficients yield multivalued functions on a punctured allpitic curve

[Brown, Levin, 2013]