Calculation and application of off-shell amplitudes

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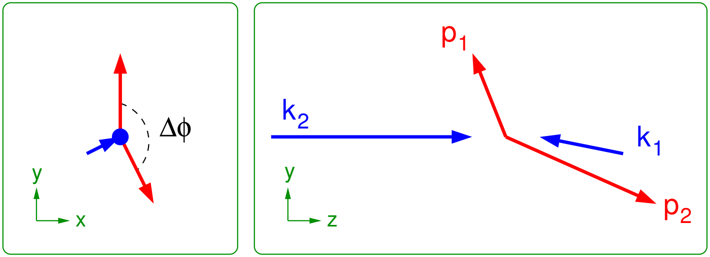
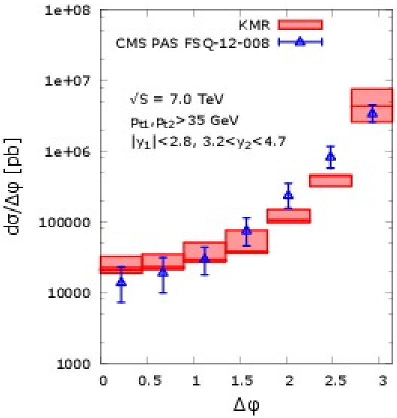
Outline

- forward-central dijet decorrelations
- factorization
- off-shell amplitudes and gauge invariance
- BCFW recursion for off-shell amplitudes

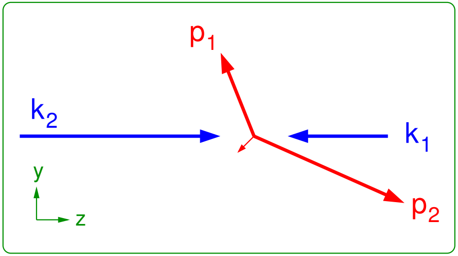
Forward-central dijet decorrelations

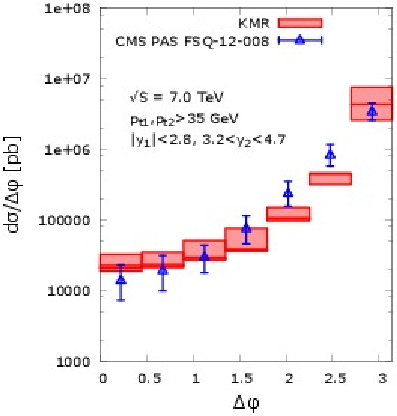
AvH, Kutak, Kotko, Sapeta 2014





Forward-central dijet decorrelations





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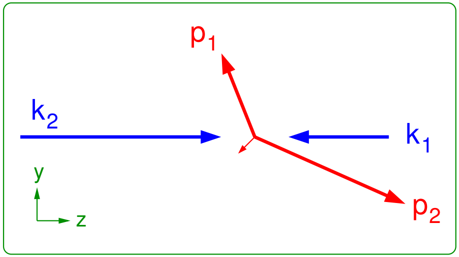
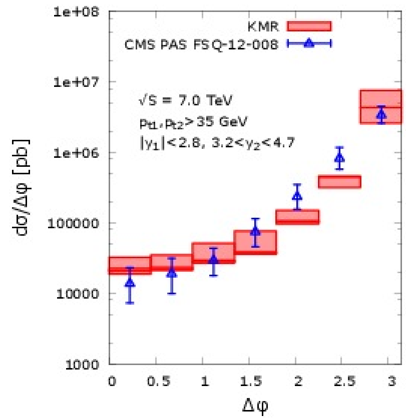
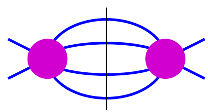




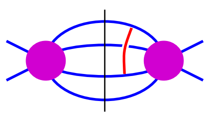
Leading Order:

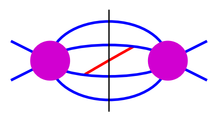
$$\hat{\sigma}^{\mathrm{LO}}_{a,b\to n} = \int d\Phi_n \left|\mathcal{M}^{(0)}_{a,b\to n}\right|^2 \mathcal{O}^{\mathrm{LO}}_n$$



Next-to-Leading Order:

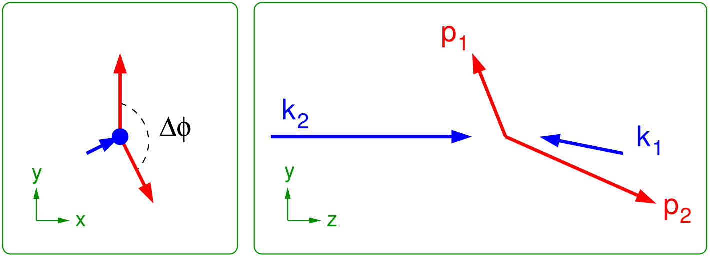
$$\hat{\sigma}^{\mathrm{NLO}}_{a,b\to n} = \int d\Phi_n \, 2\Re\left(\mathcal{M}^{(0)}_{a,b\to n}\,\mathcal{M}^{(1)}_{a,b\to n}\right) \mathcal{O}^{\mathrm{LO}}_n + \int d\Phi_{n+1} \left|\mathcal{M}^{(0)}_{a,b\to n+1}\right|^2 \mathcal{O}^{\mathrm{NLO}}_{n+1}$$

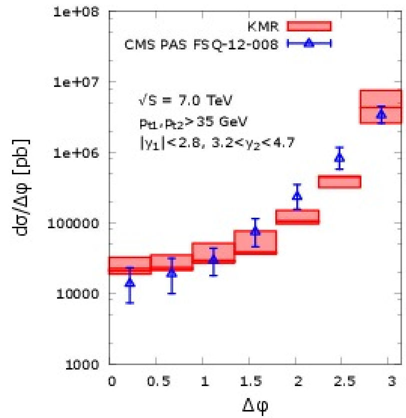




Forward-central dijet decorrelations

AvH, Kutak, Kotko, Sapeta 2014



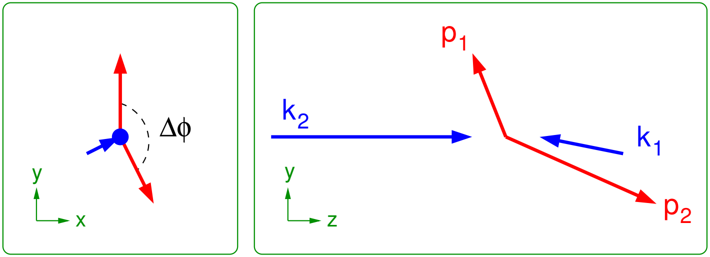


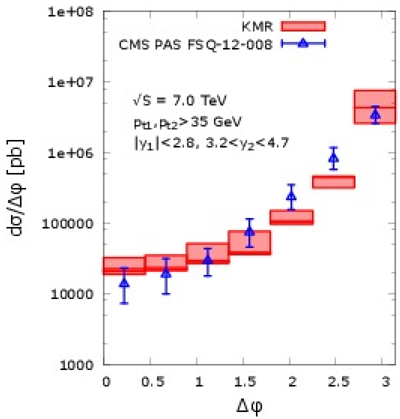
Hybrid factorization:

$$d\sigma_{AB\to X} = \int dk_T^2 \int dx_A \int dx_B \sum_b \mathcal{F}_{g^*/A}(x_A, k_T, \mu)\, f_{b/B}(x_B, \mu)\, d\hat{\sigma}_{g^*b\to X}(x_A, x_B, k_T, \mu)$$

Forward-central dijet decorrelations

AvH, Kutak, Kotko, Sapeta 2014





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$$x_B \gg x_A$$

$$\left|\vec{p}_1 + \vec{p}_2\right| = \left|\vec{k}_T\right|$$

Factorization

To separate a perturbatively calculable partonic process from universal models for the hadrons in hadron scattering.

Different factorization formulas are applicable for different kinematical regions in terms of the hard scale P_T, the transverse momentum inbalance k_T, and the saturation scale Q_s.

Collinear factorization

$$d\sigma_{AB\to X} = \int dx_A \sum_a \int dx_B \sum_b f_{a/A}(x_A,\mu)\, f_{b/B}(x_B,\mu)\, d\hat{\sigma}_{ab\to X}(x_A,x_B,\mu)$$

Central scattering: $x_A \approx x_B \sim 1$, and $P_T \gg k_T \gg Q_s$.
Partonic cross section $d\hat{\sigma}_{ab}$ is calculated with on-shell initial-state partons.

High Energy Factorization

Catani, Ciafaloni, Hautmann 1991
Ellis, Collins 1991

$$d\sigma_{AB\to X} = \int dk_T^2 \int dx_A \int dx_B \sum_b \mathcal{F}_{g^*/A}(x_A,k_T,\mu)\, f_{b/B}(x_B,\mu)\, d\hat{\sigma}_{g^*b\to X}(x_A,x_B,k_T,\mu)$$

Eg. forward-central scattering: $x_B \gg x_A$, and $P_T \sim k_T \gg Q_s$.
Unintegrated gluon density $\mathcal{F}_{g^*/A}(x_A,k_T,\mu)$ evolved following BFKL or similar.
Partonic cross section $d\hat{\sigma}_{g^*b}$ is calculated with an off-shell initial-state gluon.

Factorization

To separate a perturbatively calculable partonic process from universal models for the hadrons in hadron scattering.

Generalized TMD factorization — Dominguez, Marquet, Xiao, Yuan 2011

$$d\sigma_{AB\to X} = \int dk_T^2 \int dx_A \sum_i \int dx_B \sum_b \phi_{gb}^{(i)}(x_A, k_T, \mu)\, f_{b/B}(x_B, \mu)\, d\hat{\sigma}_{gb\to X}^{(i)}(x_A, x_B, k_T, \mu)$$

For $x_A \ll 1$ and $P_T \gg k_T \sim Q_s$.
Applicable to $p - A$ (dilute-dense) collisions.
TMD gluon distributions $\phi_{gb}^{(i)}(x_A, k_T, \mu)$ satisfy non-linear evolution equations, and admit saturation.
Partonic cross section $d\hat{\sigma}_{gb}^{(i)}$ depends on color-structure i, and is calculated with on-shell initial-state partons.

Improved generalized TMD factorization — Kotko, Kutak, Marquet, Petreska, Sapeta, AvH 2015

Model interpolating between High Energy Factorization and Generalized TMD factorization: $P_T \gtrsim k_T \gtrsim Q_s$.
Partonic cross section $d\hat{\sigma}_{gb}^{(i)}$ depends on color-structure i,
and is calculated with off-shell initial-state partons.

A general factorization formula

General formula for cross section with $\pi^* \in \{g^*, q^*, \bar{q}^*\}$:

$$\sigma\big(h_1(p_1)h_2(p_2) \to Y\big) = \sum_{a,b} \int d^4k_1\, F_{1,a}(k_1) \int d^4k_2\, F_{2,b}(k_2)\, \frac{\hat{\sigma}\big(\pi_a^*(k_1)\pi_b^*(k_2) \to Y\big)}{4\sqrt{(k_1\cdot k_2)^2 - k_1^2 k_2^2}}$$

Collinear factorization: $F_{i,a}(k) = \frac{1}{2N_c}\int_0^1 \frac{dx}{x}\, f_{i,a}(x,\mu)\,\delta^4(k - x\,p_i)$

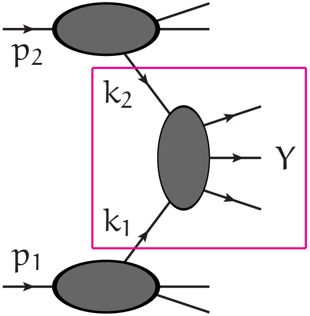
k_T-factorization: $F_{i,a}(k) = \frac{1}{N_c}\int \frac{d^2k_T}{2\pi}\int_0^1 \frac{dx}{x}\, \mathcal{F}_{i,a}(x,k_T,\mu)\,\delta^4(k - x\,p_i - k_T)$

$$\hat{\sigma} = \int d\Phi(1,2 \to 3,4,\ldots,n)\, \big|\mathcal{M}(1,2,\ldots,n)\big|^2\, \mathcal{O}(p_1,p_2,\ldots,p_n)$$

phase space includes summation over color and spin

squared amplitude calculated perturbatively

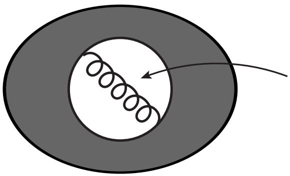
observable includes phase space cuts, or jet algorithm



Gauge invariance

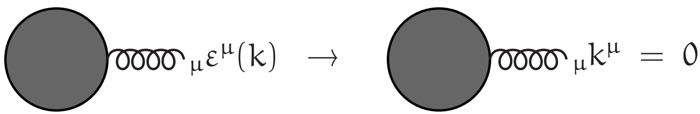
In order to be physically relevant, any scattering amplitude following the constructive definition given before must satisfy the following

Freedom in choice of gluon propagator:



$$\begin{cases} \dfrac{-i}{k^2}\left[g^{\mu\nu} - (1-\xi)\dfrac{k^\mu k^\nu}{k^2}\right] \\[2ex] \dfrac{-i}{k^2}\left[g^{\mu\nu} - \dfrac{k^\mu n^\nu + n^\mu k^\nu}{k\cdot n} + (n^2 + \xi k^2)\dfrac{k^\mu k^\nu}{(k\cdot n)^2}\right] \end{cases}$$

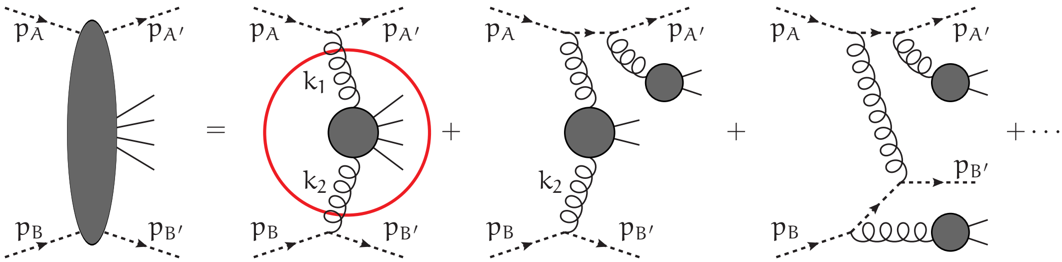
Ward identity:



- Only holds if all external particles are on-shell.
- k_T-factorization requires off-shell initial-state momenta.
- How to define amplitudes with off-shell intial-state momenta?

Amplitudes with off-shell partons

AvH, Kutak, Kotko 2013, AvH, Kutak, Salwa 2013: Embed the process in an on-shell process with auxiliary partons



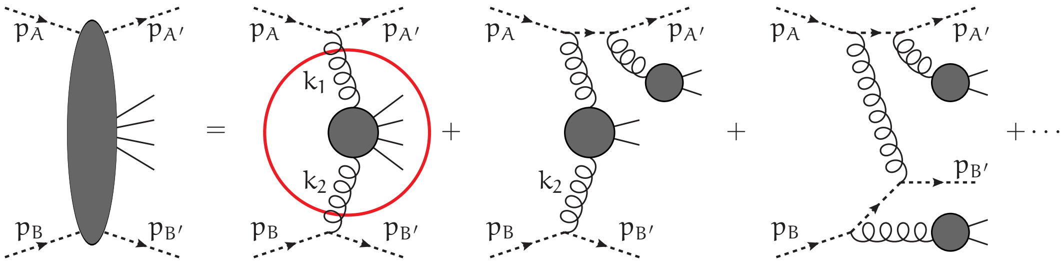
Hadron momenta p_1, p_2:

$$p_1 \cdot p_A = p_1 \cdot p_{A'} = p_1 \cdot k_1 = 0$$

$$p_2 \cdot p_B = p_2 \cdot p_{B'} = p_2 \cdot k_2 = 0$$

Amplitudes with off-shell partons

AvH, Kutak, Kotko 2013, AvH, Kutak, Salwa 2013: Embed the process in an on-shell process with auxiliary partons and eikonal Feynman rules.



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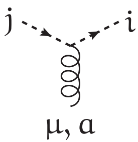
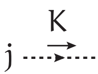


$= -i\, T^a_{i,j}\, p_1^\mu$



$= \delta_{i,j} \dfrac{i}{p_1 \cdot K}$

Amplitudes with off-shell partons

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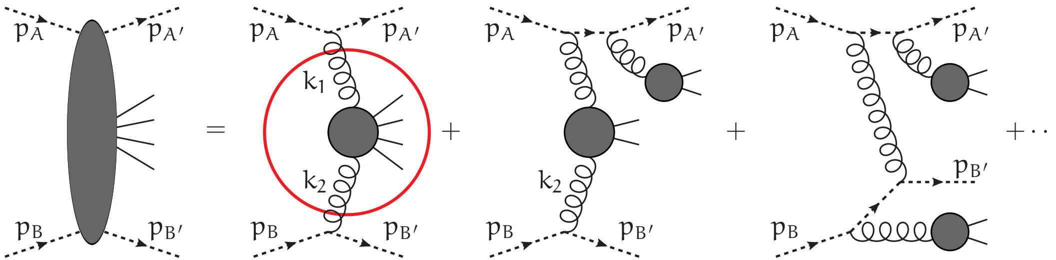

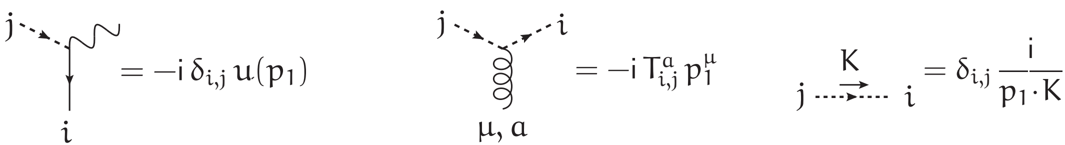

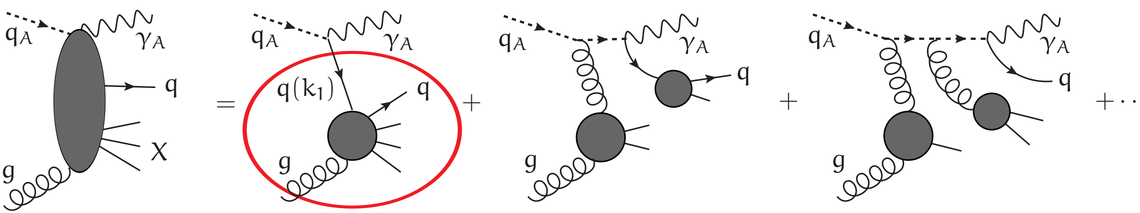

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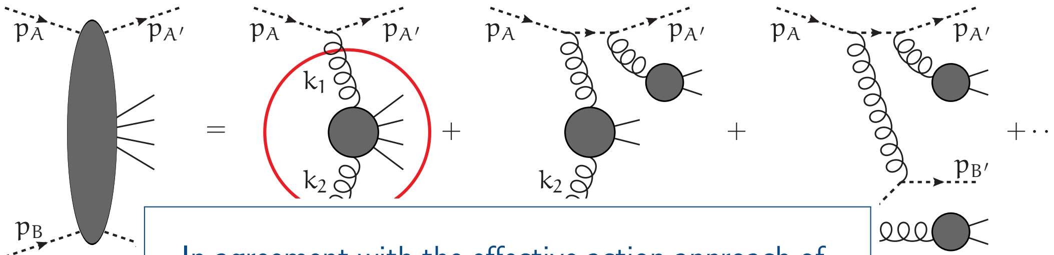
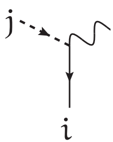
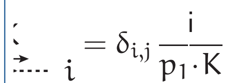
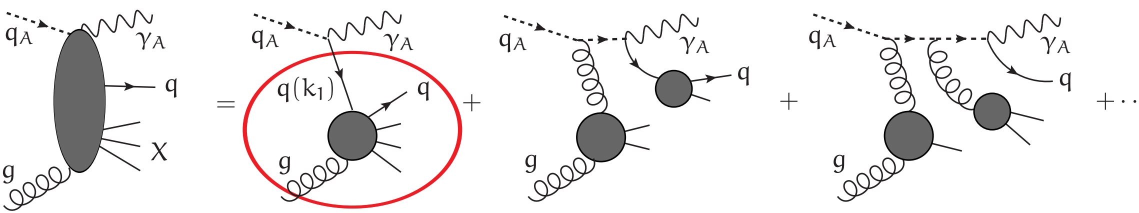




In agreement with the effective action approach of
Lipatov 1995, Lipatov, Vyazovsky 2000
and the Wilson-line approach of
Kotko 2014





MHV formula

$$\mathcal{A}(i^-, j^-, (\text{the rest})^+) = \frac{\langle p_i p_j \rangle^4}{\langle p_1 p_2 \rangle \langle p_2 p_3 \rangle \cdots \langle p_{n-2} p_{n-1} \rangle \langle p_{n-1} p_n \rangle \langle p_n p_1 \rangle}$$

BCFW recursion for on-shell amplitudes

Multi-gluon amplitudes have much simpler expressions than one would expect from the Feynman graphs, in particular the MHV amplitudes:

$$\mathcal{A}(i^-, j^-, (\text{the rest})^+) = \frac{\langle p_i p_j \rangle^4}{\langle p_1 p_2 \rangle \langle p_2 p_3 \rangle \cdots \langle p_{n-2} p_{n-1} \rangle \langle p_{n-1} p_n \rangle \langle p_n p_1 \rangle}$$

BCFW recursion (Britto, Cachazo, Feng, Witten 2005) allows for easy construction of such simple expressions

- it is a recursion of *on-shell amplitudes*, rather than off-shell Green functions
- it is most efficiently applied as a recursion of *expressions*
- it is easily proven using Cauchy's theorem

For a rational function f of a complex variable z which vanishes at infinity, we have

$$\oint_R \frac{dz}{2\pi i} \frac{f(z)}{z} \overset{R\to\infty}{=} 0 \qquad \Rightarrow \qquad f(0) = \sum_i \frac{\text{Residue}(f \ @ \ z = z_i)}{-z_i}$$

This is applied to amplitudes by turning them into functions of a complex variable by analytical continuation of the momenta to complex values.

BCFW recursion for on-shell amplitudes

Choose two gluons $1, n$ and construct the *shift vector*

$$e^\mu = \frac{1}{2}\langle p_1|\gamma^\mu|p_n]$$

which satisfies

$$p_1 \cdot e = p_n \cdot e = e \cdot e = 0$$

and *shift* the momenta of gluons $1, n$ following

$$\hat{p}_1^\mu(z) = p_1^\mu + z e^\mu \qquad \hat{p}_n^\mu(z) = p_n^\mu - z e^\mu$$

so that

$$\hat{p}_1^\mu(z) + \hat{p}_n^\mu(z) = p_1^\mu + p_n^\mu \qquad \hat{p}_1(z) \cdot \hat{p}_1(z) = \hat{p}_n(z) \cdot \hat{p}_n(z) = 0$$

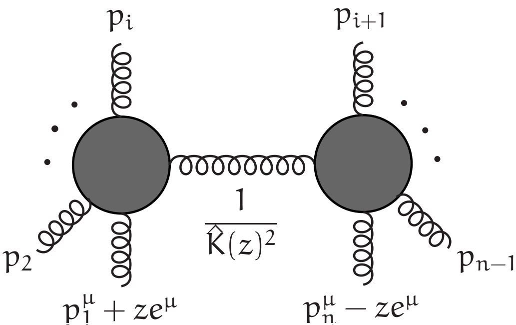
Applying Cauchy's theorem on

$$\hat{\mathcal{A}}(z) = \mathcal{A}\big(\hat{p}_1(z), p_2, \ldots, p_{n-1} \hat{p}_n(z)\big)$$

leads to the relation

$$\mathcal{A}(1^+, 2, \ldots, n-1, n^-) = \sum_{i=2}^{n-1} \sum_{h=+,-} \mathcal{A}(\hat{1}^+, 2, \ldots, i, -\hat{K}_{1,i}^h) \frac{1}{K_{1,i}^2} \mathcal{A}(\hat{K}_{1,i}^{-h}, i+1, \ldots, n-1, \hat{n}^-)$$

BCFW recursion for on-shell amplitudes



$$\hat{K}^\mu(z) = p_1^\mu + \cdots + p_i^\mu + ze^\mu$$
$$= -p_{i+1}^\mu - \cdots - p_n^\mu + ze^\mu$$

$$\hat{K}(z)^2 = 0 \quad \Leftrightarrow \quad z = -\frac{(p_1 + \cdots + p_i)^2}{2(p_2 + \cdots + p_i)\cdot e}$$

$$\mathcal{A}(1^+, 2, \ldots, n{-}1, n^-) = \sum_{i=2}^{n-1} \sum_{h=+,-} \mathcal{A}(\hat{1}^+, 2, \ldots, i, -\hat{K}_{1,i}^h) \frac{1}{K_{1,i}^2} \mathcal{A}(\hat{K}_{1,i}^{-h}, i{+}1, \ldots, n{-}1, \hat{n}^-)$$

Amplitudes with off-shell gluons

An n-gluon amplitude is a function of n momenta $k_1, k_2, \ldots, k_n$ and n *directions* $p_1, p_2, \ldots, p_n$, satisfying the conditions

$$k_1^\mu + k_2^\mu + \cdots + k_n^\mu = 0 \qquad \text{momentum conservation}$$

$$p_1^2 = p_2^2 = \cdots = p_n^2 = 0 \qquad \text{light-likeness}$$

$$p_1 \cdot k_1 = p_2 \cdot k_2 = \cdots = p_n \cdot k_n = 0 \qquad \text{eikonal condition}$$

With the help of an auxiliary four-vector q^μ with $q^2 = 0$, we define

$$k_T^\mu(q) = k^\mu - x(q) p^\mu \quad \text{with} \quad x(q) \equiv \frac{q \cdot k}{q \cdot p}$$

Construct k_T^μ explicitly in terms of p^μ and q^μ:

$$k_T^\mu(q) = -\frac{\kappa}{2} \frac{\langle p|\gamma^\mu|q]}{[pq]} - \frac{\kappa^*}{2} \frac{\langle q|\gamma^\mu|p]}{\langle qp\rangle} \quad \text{with} \quad \kappa = \frac{\langle q|\not{k}|p]}{\langle qp\rangle} \;,\; \kappa^* = \frac{\langle p|\not{k}|q]}{[pq]}$$

$k^2 = -\kappa\kappa^*$ is independent of q^μ, but also individually κ and κ^* are independent of q^μ.

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$k^2 = -\kappa\kappa^*$ is independent of q^μ, but also individually κ and κ^* are independent of q^μ.

$$\frac{\langle q|\not{k}|p]}{\langle qp\rangle} = \frac{\langle q|\not{k}|p]\langle pr\rangle}{\langle qp\rangle\langle pr\rangle} = \frac{\langle q|\not{k}\not{p}|r\rangle}{\langle qp\rangle\langle pr\rangle} = \frac{\langle q|2k\cdot p - \not{p}\not{k}|r\rangle}{\langle qp\rangle\langle pr\rangle} = -\frac{\langle qp\rangle[p|\not{k}|r\rangle}{\langle qp\rangle\langle pr\rangle} = \frac{\langle r|\not{k}|p]}{\langle rp\rangle}$$

Amplitudes with off-shell gluons

An n-gluon amplitude is a function of n momenta $k_1, k_2, \ldots, k_n$ and n *directions* $p_1, p_2, \ldots, p_n$, satisfying the conditions

$$\begin{aligned} k_1^\mu + k_2^\mu + \cdots + k_n^\mu &= 0 && \text{momentum conservation} \\ p_1^2 = p_2^2 = \cdots = p_n^2 &= 0 && \text{light-likeness} \\ p_1 \cdot k_1 = p_2 \cdot k_2 = \cdots = p_n \cdot k_n &= 0 && \text{eikonal condition} \end{aligned}$$

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$k^2 = -\kappa\kappa^*$ is independent of q^μ, but also individually κ and κ^* are independent of q^μ.

Besides the spinors of directions and light-like momenta, κ and κ^* will show up in expressions for off-shell amplitudes.

Shifting off-shell momenta

The shift of on-shell momenta is equivalent to a shift of the spinors

$$\hat{p}_1^\mu = p_1^\mu + \frac{z}{2}\langle 1|\gamma^\mu|n] \quad \Leftrightarrow \quad |\hat{1}\rangle = |1\rangle \qquad |\hat{1}] = |1] + z|n]$$
$$\hat{p}_n^\mu = p_n^\mu - \frac{z}{2}\langle 1|\gamma^\mu|n] \quad \Leftrightarrow \quad |\hat{n}\rangle = |n\rangle - z|1\rangle \qquad |\hat{n}] = |n]$$

For off-shell momenta k_1^μ, k_n^μ the shift vector is defined in terms of the light-like directions p_1^μ, p_n^μ and the shift of the momentum is equivalent to the shift of κ or κ^*

$$\hat{k}_1^\mu = k_1^\mu + \frac{z}{2}\langle 1|\gamma^\mu|n] \;=\; x_1(p_n)p_1^\mu - \frac{\kappa_1 - [1n]z}{2}\frac{\langle 1|\gamma^\mu|n]}{[1n]} - \frac{\kappa_1^*}{2}\frac{\langle n|\gamma^\mu|1]}{\langle n1\rangle}$$
$$\hat{k}_n^\mu = k_n^\mu - \frac{z}{2}\langle 1|\gamma^\mu|n] \;=\; x_n(p_1)p_n^\mu - \frac{\kappa_n}{2}\frac{\langle n|\gamma^\mu|1]}{[n1]} - \frac{\kappa_n^* + \langle 1n\rangle z}{2}\frac{\langle 1|\gamma^\mu|n]}{\langle 1n\rangle}$$

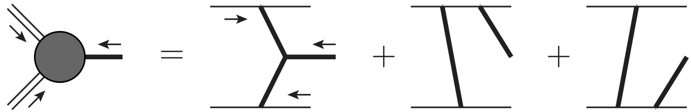
Besides momentum conservation, the shift also preserves the eikonal conditions

$$p_1\cdot\hat{k}_1 = 0 \qquad p_n\cdot\hat{k}_n = 0$$

Propagator denominators for "external" of-shell gluons, instead of polarization vectors, give correct powers of z for vanishing behavior at infinity.

Feynman rules

Planar graphs for the process $\emptyset \longrightarrow g^* g^* g$:

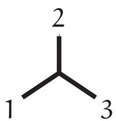


The Feynman rules in the Feynman gauge:

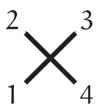
$$= \frac{-\eta^{\mu\nu}}{K^2}$$

$$= \frac{1}{2p \cdot K}$$

$$= \sqrt{2}\, p^{\mu}$$



$$= \frac{1}{\sqrt{2}} \left[(K_1 - K_2)^{\mu_3} \eta^{\mu_1 \mu_2} + (K_2 - K_3)^{\mu_1} \eta^{\mu_2 \mu_3} + (K_3 - K_1)^{\mu_2} \eta^{\mu_3 \mu_1} \right]$$

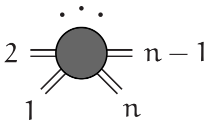


$$= \frac{-1}{2} \left[2\,\eta^{\mu_1 \mu_3} \eta^{\mu_2 \mu_4} - \eta^{\mu_1 \mu_2} \eta^{\mu_3 \mu_4} - \eta^{\mu_1 \mu_4} \eta^{\mu_2 \mu_3} \right]$$

where p^{μ} is the direction associated with the eikonal line.

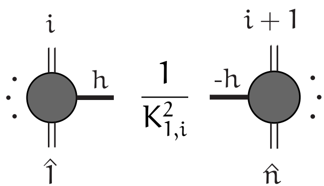
BCFW recursion for off-shell amplitudes

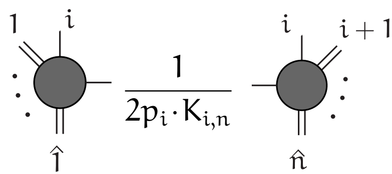
The BCFW recursion formula becomes

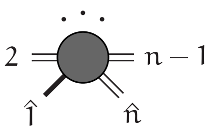


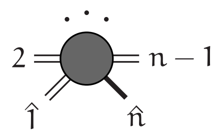
$$= \sum_{i=2}^{n-2}\sum_{h=+,-} A_{i,h} + \sum_{i=2}^{n-1} B_i + C + D\,,$$

where

$A_{i,h} =$ 

$B_i =$ 

$C = \frac{1}{\kappa_1}$ 

$D = \frac{1}{\kappa_n^*}$ 

The hatted numbers label the shifted external gluons.

On-shell limit

For each off-shell gluon j, we can identify the following terms in the amplitude

$$\mathcal{A}(k_j) = \frac{1}{\kappa_j^*}\,U(k_j) + \frac{1}{\kappa_j}\,V(k_j) + W(k_j)$$

The actual amplitude needs a factor proportional to $\sqrt{-k_j^2}$, we choose κ_j^*:

$$\kappa_j^*\mathcal{A}(k_j) = U(k_j) + \frac{\kappa_j^*}{\kappa_j}\,V(k_j) + \kappa_j^*\,W(k_j)$$

The ratio κ_j^*/κ_j does not vanish in the on-shell limit, and an angle dependence remains.

$$\left|\kappa_j^*\mathcal{A}(k_j)\right|^2 \xrightarrow{k_j^2\to 0} \left|U(p_j)\right|^2 + \left|V(p_j)\right|^2 + e^{2i\varphi_j}U(p_j)V(p_j)^* + e^{-2i\varphi_j}U(p_j)^*V(p_j)$$

Interference terms vanish upon integration over φ.

- the $-$ helicity can be associated with U, i.e. $1/\kappa_j^*$
- the $+$ helicity can be associated with V, i.e. $1/\kappa_j$

Some 4-gluon amplitudes

$$\mathcal{A}(1^*,2^+,3^+,4^*) = \frac{1}{\kappa_4^*\kappa_1^*}\,\frac{\langle 41\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 34\rangle} \quad , \quad \mathcal{A}(1^*,2^+,3^*,4^+) = \frac{1}{\kappa_1^*\kappa_3^*}\,\frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

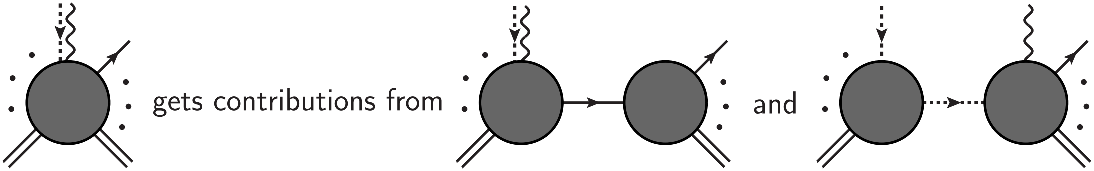
$$\begin{aligned}\mathcal{A}(1^*,2^+,3^-,4^*) = {} & \frac{1}{\kappa_1^*\kappa_4}\,\frac{-\langle 1|\not{p}_3+\not{k}_4|4]^4}{\langle 2|\not{k}_1|4]\langle 1|\not{k}_4|3]\langle 12\rangle[43](p_3+k_4)^2} \\ & + \frac{1}{\kappa_1}\,\frac{\langle 34\rangle^3[14]^3}{\langle 4|\not{k}_4+\not{k}_1|1]\langle 2|\not{k}_1|4]\langle 4|\not{k}_1|4]\langle 23\rangle} + \frac{1}{\kappa_4^*}\,\frac{[21]^3\langle 14\rangle^3}{\langle 4|\not{k}_4+\not{k}_1|1]\langle 1|\not{k}_4|3]\langle 1|\not{k}_4|1][32]}\end{aligned}$$

$$\begin{aligned}\mathcal{A}(1^*,2^-,3^*,4^+) = {} & \frac{\langle 13\rangle^3[13]^3}{\langle 34\rangle\langle 41\rangle\langle 1|\not{k}_3+\not{p}_4|3]\langle 3|\not{k}_1+\not{p}_4|1][32][21]} \\ & + \frac{1}{\kappa_1^*\kappa_3}\,\frac{\langle 12\rangle^3[43]^3}{\langle 2|\not{k}_3|4]\langle 1|\not{k}_3+\not{p}_4|3](k_3+p_4)^2} + \frac{1}{\kappa_1\kappa_3^*}\,\frac{\langle 23\rangle^3[14]^3}{\langle 2|\not{k}_1|4]\langle 3|\not{k}_1+\not{p}_4|1](k_1+p_4)^2}\end{aligned}$$

BCFW recursion with (off-shell) quarks

AvH, Serino 2015

- on-shell case treated in Luo, Wen 2005
  - only gluons are shifted
  - restrictions on allowed combinations of helicities and shift vectors
  - not always possible to have minimal number of terms by shifting adjacent gluons
- any off-shell parton can be shifted: propagators of "external" off-shell partons give the correct power of z in order to vanish at infinity
- different kinds of contributions in the recursion



- many of the MHV amplitudes come out as expected
- some more-than-MHV amplitudes do not vanish, but are sub-leading in k_T

$$\mathcal{A}(1^+,2^+,\ldots,n^+,\bar{q}^*,q^-) = \frac{-\langle\bar{q}q\rangle^3}{\langle 12\rangle\langle 23\rangle\cdots\langle n\bar{q}\rangle\langle\bar{q}q\rangle\langle q1\rangle}$$

Conclusions

- Factorization prescriptions with explicit k_T dependence in the pdfs ask for hard matrix elements with off-shell initial-state partons.
- They allow for Dyson-Schwinger recursion and BCFW recursion, both for off-shell gluons and off-shell quarks.
- Analytic expressions have been worked out for
  - up to 5-gluon amplitudes with up to 2 of them off-shell Bury, AvH 2015
  - up to 5-parton amplitudes with 1 $q\bar{q}$-pair and any 1 parton off-shell
- A numerical code has been published employing BCFW recursion to calculate multi-gluon matrix elements for up to 6 gluons with up to 2 of them off-shell
  http://bitbucket.com/hameren/amp4hef
- A numerical code has been published employing Dyson-Schwinger recursion to calculate (essentially) any tree-level (off-shell) amplitude, within a complete Monte Carlo environment
  http://bitbucket.com/hameren/avhlib