## Third order corrections to semileptonic $b \rightarrow u$ decay: fermionic contributions <br> (common work with: Matteo Fael) <br> MTTD 2023 - Ustroń

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## Process: $b \rightarrow u$ decay

- The total rate for $b \rightarrow u$ decay is

$$
\begin{align*}
\Gamma\left(B \rightarrow X_{u} \ell \bar{\nu}_{\ell}\right) & =\Gamma_{0}\left[1+C_{F} \sum_{n \geq 1}\left(\frac{\alpha_{s}}{\pi}\right)^{n} X_{n}\right] \\
& +O\left(\frac{\Lambda_{\mathrm{QCD}}^{2}}{m_{b}^{2}}\right) \tag{1}
\end{align*}
$$

- $\Gamma_{0}=G_{F}^{2} m_{b}^{5}\left|V_{u b}\right|^{2} A_{\text {ew }} /\left(192 \pi^{3}\right)$
- $\alpha_{s} \equiv \alpha_{s}^{(5)}\left(\mu_{s}\right)=0.2186$ with $\mu_{s}$ the renormalization scale
- $A_{\text {ew }}=1.014$ leading electroweak correction [A. Sirin, Nucl. Phys. B 196 (1982), 88-92]
- $m_{b}$ is on-shell mass of the bottom quark
- Color factors in QCD: $C_{F}=4 / 3, C_{A}=3, T_{F}=1 / 2$
- We set $N_{L}=4$ massless quarks and $N_{H}=1$ (we ignore charm mass effects)
- $O\left(\frac{\Lambda_{\mathrm{QCD}}^{2}}{m_{b}^{2}}\right)$ [J. Chay, H. Georgi and B. Grinstein, 1990 I. I. Y. Bigi, M. A. Shifman, N. G. Uraltsev and A. I.

Vainshtein, 1993, A. V. Manohar and M. B. Wise, 1994, M. Gremm and A. Kapustin 1997, T. Becher, H. Boos and E.
Lunghi, 2007, A. Alberti, P. Gambino and S. Nandi, 2014, T. Mannel, A. A. Pivovarov and D. Rosenthal, 2015, T.
Mannel and A. A. Pivovarov, 2019]

## State of the art

- $X_{1}=-1.80975 C_{F}$ [Kinoshita, Sirlin, Phys. Rev. 113 (1959) 1652]
- $X_{2}=-15.975 C_{F}$ [T. van Rittergen, Phys. Lett. B 454, 353 (1999)]

A $X_{3}=(-202 \pm 20) C_{F}$ [Matteo Fael, Matthias Steinhauser, Kay Schönwald, 2011.13654]
B $X_{3}=(-195.3 \pm 9.8) C_{F}$ [Long-Bin Chen, Hai Tao Li, Zhao Li, Jian Wang, Yefan Wang, Quan-feng Wu, 2309.00762]
C $X_{3}=(-201.3 \pm 1.95) C_{F}$ [This work]
A From $b \rightarrow c l \bar{\nu}_{l}$, expansion $\delta=1-m_{c} / m_{b}$ up to $\delta^{12}$, plus extrapolation to $m_{c} / m_{b} \rightarrow 0$
B Compute leading color diagrams only
C We compute fermionic contributions exact and take bosonic leading color contributions

- We investigate right now different renormalization schemes to understand: Does the perturbative series have a good convergence?


## Calculational set up



- We apply the optical theorem. For example the leading order is a two-loop diagram
- It has a neutrino, a lepton and a up quark as internal particles at leading order
- At $\mathrm{N}^{3} \mathrm{LO}$ we have 5 -loops, and gluons and ghosts and bottom quarks appear as internal particles
- The weak interaction is treated as an effective vertex
- Our aim is to consider QCD corrections up to third order which adds three more loops.


## Diagram generation



- Originally the calculation is set up with QGRAF[p. Nogueira, J. Comput. Phys. 105, 279 (1993).] and Tapir [Gerlach, Herren, Lang, 2201.05618]
- Alternatively DiaGen generates the amplitude [uv, Michat Czakon, Marco Niggetiedt]
- Generates a script, which does the color algebra and Dirac algebra in Form [Jos Vermaseren]
- Expresses the amplitude in terms of minimal number of integrals
- Number of diagrams 1092 -> Number of integral families 97
- Timing: diagram generation 20 minutes / FORM procedure 1 day
- Generates input for the program Kira


## Integral family



- $q_{j}=k_{1}, \ldots, k_{L}, p_{1}, \ldots, p_{E}$
- $s_{i j}=q_{i} q_{j}, \quad i=1, \ldots, L, \quad j=i, \ldots, L+E$
- $\vec{s}=\left(\left\{s_{i}\right\},\left\{m_{i}^{2}\right\}\right)$, dimensional regularization parameter $D=4-2 \epsilon$
- The integral family definition is complete, if all $P_{i}$ are linearly independent in the $s_{i j}$
- $s_{11}=m_{1}^{2}+P_{1}, \quad s_{12}=\frac{1}{2}\left(m_{1}^{2}+P_{1}+P_{3}-P_{5}\right), \quad s_{22}=P_{3}$, $s_{13}=\frac{1}{2}\left(-m_{1}^{2}-P_{1}-p_{1} p_{1}+P_{2}\right), \quad s_{23}=\frac{1}{2}\left(-p_{1} p_{1}-P_{3}+P_{4}\right)$


## Integration-by-parts (IBP) identities

$$
\begin{gathered}
I\left(a_{1}, \ldots, a_{5}\right)=\int \frac{d^{D} k_{1} d^{D} l_{2}}{\left[k_{1}^{2}-m_{1}^{2}\right]^{a_{1}}\left[\left(p_{1}+k_{1}\right)^{2}\right]^{a_{2}}\left[k_{2}{ }^{2}\right]^{a_{3}}\left[\left(p_{1}+k_{2}\right)^{2}\right]^{a_{4}}\left[\left(k_{2}-k_{1}\right)^{2}\right]^{a_{5}}} \\
\int d^{D} \boldsymbol{k}_{1} \ldots d^{D} \boldsymbol{k}_{L} \frac{\partial}{\partial\left(\boldsymbol{k}_{i}\right)_{\mu}}\left(\left(q_{j}\right)_{\mu} \frac{1}{\left[P_{1}\right]^{a_{1}} \ldots\left[P_{N}\right]^{a_{N}}}\right) \text { [Chetyrkin, Tkachov, 1981] }=0 \\
c_{1}\left(\left\{a_{f}\right\}, \vec{s}, D\right) I\left(a_{1}, \ldots, a_{N}-\mathbf{1}\right)+\cdots+c_{m}\left(\left\{a_{f}\right\}, \vec{s}, D\right) I\left(a_{1}+\mathbf{1}, \ldots, a_{N}\right)=0
\end{gathered}
$$

$m$ number of terms generated by one IBP identity
Reduction: express all integrals with the same set of propagators but with different exponents $a_{f}$ as a linear combination of some basis integrals (master integrals)

- Gives relations between the scalar integrals with different exponents $a_{f}$
- Number of $L(E+L)$ IBP equations, for each choice of $i=1, \ldots, L$ and $j=1, \ldots, E+L$
- $a_{f}=$ symbols: Seek for recursion relations, LiteRed [Lee, 2012]
- $a_{f}=$ integers: Sample a system of equations, Laporta algorithm [Laporta, 2000]
- Seeds: $I\left(a_{1}, \ldots, a_{5}\right)=\left[P_{1}\right]^{a_{1}} \ldots\left[P_{N}\right]^{a_{N}}$


## Trick to simplify a reduction

Example integral $I(1,1,1,1,1,1,1,1,1,1,1,1,-5,0,0,0,0,0,0,0)$

- It is necessary to reduce 5 scalar products
- Very difficult with public tools out of the box Kira [Klappert, Lange, Maiembofer, Usovitsch, 1705.05610, 2008.06494], Reduze [von Manteuffel, Studerus, 1201.4330], FIRE 6 [Smirnov, Chuharev, 1901.07808], FiniteFlow [Peraro, 1905.08019]+LiteRed
- But it works for sure with Kira if we integrate out one-loop self-energy analytically
- $\int d^{D} k \frac{k^{\alpha_{1} \ldots k^{\alpha_{n}}}}{\left(-k^{2}\right)^{\lambda_{1}}\left[-(q-k)^{2}\right]^{\lambda_{2}}}=$

$$
\begin{aligned}
& \frac{i \pi^{D / 2}}{\left(-q^{2}\right)^{\lambda_{1}+\lambda_{2}+\epsilon-2}} \sum_{r=0}^{[n / 2]} A_{N T}\left(\lambda_{1}, \lambda_{2} ; r, n\right)\left(\frac{q^{2}}{2}\right)^{r}\left\{[g]^{r}[g]^{n-2 r}\right\}^{\alpha_{1} \ldots \alpha_{n}} \text { with } \\
& A_{N T}\left(\lambda_{1}, \lambda_{2} ; r, n\right)=\frac{\Gamma\left(\lambda_{1}+\lambda_{2}+\epsilon-2-r\right) \Gamma\left(n+2-\epsilon-\lambda_{1}-r\right) \Gamma\left(2-\epsilon-\lambda_{2}+r\right)}{\Gamma\left(\lambda_{1}\right) \Gamma\left(\lambda_{2}\right) \Gamma\left(4+n-\lambda_{1}-\lambda_{2}-2 \epsilon\right)}
\end{aligned}
$$

## Integral family with symbolic propagator power

- SLTOP5I992[1,1,1,1,1,1,1,1,1,1,1,1,0,0,0,0,0,0,-1,-1]
$=(19$ terms $)-3 \frac{\Gamma(2-d / 2) \Gamma(-1+d / 2) \Gamma(d / 2)}{\Gamma(-1+d)} \times$
xSLTOP5I992[-1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, -1, 0]
- New integral family has one-loop less (20 propagator power indices reduce to 14 indices), but one propagator is raised to a symbolic propagator power
- xSLTOP5I992[ $\left.b_{1}-1,1,1,1,1,1,1,1,1,1,0,-1,0\right], \quad b_{1}=((4-D) / 2)$
- Choose master integrals such that $b_{1}$ is without integer shifts
- Reintroduce $\frac{\Gamma(-2+D)}{\Gamma\left(1+\frac{-4+D}{2}\right)^{2} \Gamma\left(\frac{4-D}{2}\right)}$, when 4-loop $\rightarrow 5$-loop conversion
- Kira does support symbolic reduction for years
- In this work I improved symbolic reduction with Kira


## Tricks in symbolic reduction with Kira

$$
\begin{aligned}
& \int d^{D} \boldsymbol{k}_{\mathbf{1}} \ldots d^{D} \boldsymbol{k}_{L} \frac{\partial}{\partial\left(\boldsymbol{k}_{\boldsymbol{i}}\right)_{\mu}}\left(\left(q_{j}\right)_{\mu} \frac{1}{\left[P_{1}\right]^{a_{1}} \ldots\left[P_{N}\right]^{a_{N}}}\right) \text { [Chetyrkin, Tkachov, 1981] } \\
& =0 \\
& c_{1}\left(\left\{a_{f}\right\}, \vec{s}, D\right) I\left(a_{1}, \ldots, a_{N}-\mathbf{1}\right)+\cdots+c_{m}\left(\left\{a_{f}\right\}, \vec{s}, D\right) I\left(a_{1}+\mathbf{1}, \ldots, a_{N}\right)
\end{aligned}=0
$$

Number of IBP (identities) generators: $L(E+L)$

- The IBP generators are highly linearly dependent, especially at 5-loop
- I eliminate many of the operators
- I prefer to eliminate operators, which result in a positive shift to the symbolic power
- We eliminate integrals with no imaginary part
- Allowed seeds for $I\left(b_{1}+a_{1}, a_{2}, \ldots, a_{14}\right)$ are: $a_{2}, \ldots, a_{14}$ can take positive and negative values, but $a_{1}$ is only allowed to take negative values
- Especially the last point gives orders of magnitude better reduction results


## Amplitude reduction

- Amplitude reduction for the 5-loop process finished in 2 month
- Timing: worst case is 10 days on 12 cores
- All integrals from the squared amplitude are expressed through master integrals, which have either at most 2 dots or 2 scalar products
- Unfortunately I do not have any log files anymore, to go into the rich details of reduction specific properties


## 5-loop integral calculation with AMFlow [Lu. Xioo and ma,

Yan-Qing, 2201.11669]

- Auxiliary mass flow integral obtained formally with $i 0=-\eta$

$$
\begin{equation*}
I_{\vec{\nu}}(\eta)=\prod_{i=1}^{L} \int \frac{d^{D} k_{l}}{i \pi^{\frac{D}{2}}} \frac{P_{K+1}^{-\nu_{K+1}} \ldots P_{N}^{-\nu_{N}}}{\left(P_{1}-\eta\right)^{\nu_{1}} \ldots\left(P_{K}-\eta\right)^{\nu_{K}}} \tag{2}
\end{equation*}
$$

- With the limit

$$
\begin{equation*}
I_{\vec{\nu}}(\eta)=\lim _{\eta \rightarrow i 0^{-}} I_{\vec{\nu}}(\eta) \tag{3}
\end{equation*}
$$

- Strengths
- Systematic: in principle works for arbitrary integrals
- Differential equation solver scales linear with precision set
- Drawbacks
- Auxiliary integrals potentially involve many master integrals
- The tool was not tested at 5 -loop level, yet
- One of the bottlenecks was the determination of non-zero regions with the expansion-by-regions method
- Other critical error was eliminated in the differential equation solver ${ }^{12 / 19}$


## Modifications to AMFlow

- Problem
- To construct the differential equations, the IBP reductions are the bottleneck
- Integral family involves one more additional scale: auxiliary mass flow $\eta$.
- Reduction necessary for integrals with up to one dot and two scalar products
- Compared to the amplitude reduction we have less scalar products, but one more scale
- Solution
- To improve the AMFLow run time we have written our own IBP interface to Kira
- Automatic: do the 5-loop reductions by converting the integral families from 5-loop to 4-loop. Convert resulting 4-loop master integrals back to 5-loop.
- Differential equations are derived for these new 5 -loop master integra ${ }^{13} s^{19}$


## Numerical results

- All integrals which appear in the fermionic part of the amplitude are computed to a high precision, 40 digits of accuracy
- SLTOP5I992[1, 1, 1, 1, 1, 1, 0, 1, 0, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0]
$=(1.7532775277000914198075023186440731037-421.7578991818784250760648350179082675138 I)+$ $0.0833333333333333333333333333333333333333 / \epsilon^{5}+$
$0.1761601396243613080806199624656656537324 / \epsilon^{4}-$
(0.128122939737351305761954983116897702468 -
$1.832595714594046055769875306913043349115 I) / \epsilon^{3}-$
$(12.83243357898317568876458546444082974198+$
$6.90156090824811083759913541674986728464 I) / \epsilon^{2}-$
$(52.42494402414620647593686192734683910325+66.41098857415666926442856834196875614216 I) / \epsilon$



## Future: bosonic results

- Evaluations of master integrals of $\sim 30$ integral families are missing
- Estimated time for evaluation of one bosonic integral family is 30 days.
- Bottleneck is the IBP reduction. More improvement in Kira needed.
- Many bottlenecks in Kira are uncovered and are in process to be cleaned up
- Apply other strategies, e.g. Feynman parameter integration through differential equations [Hidding, Usovitsch, 2206.14790]
- This method is similar in spirit to auxiliary mass flow, but it tries to minimize in addition the number of master integrals.


## Summary and Outlook

- Advanced the state-of-the-art calculation of $\mathrm{N}^{3} \mathrm{LO} b \rightarrow u l \bar{\nu}_{l}$ process
- The publication of the complete fermionic corrections is in preparation
- The $\mathrm{N}^{3} \mathrm{LO}$ corrections uncertainty is reduced to percent level
- The calculation of bosonic corrections is halfway through, amplitude reduction is complete
- Remaining question will soon be answered: does the series have a good convergence?
- Improved symbolic IBP reductions with Kira
- Uncovered bottlenecks in Kira, which are in fixing process right now
- Advertised new tool DiaGen, soon available for automatic amplitude generation


## Feynman parameter integration through differential

equations [Hidide, usovisch, 2006. 14700]

- Initial integral

$$
\begin{equation*}
I_{\nu_{1} \ldots \nu_{K}}^{(1)}=\int \prod_{i=1}^{L} \frac{d^{d} k_{l}}{i \pi^{\frac{d}{2}}} \frac{D_{K+1}^{-\nu_{K+1}} \ldots D_{n}^{-\nu_{n}}}{\left(D_{1}+i \delta\right)^{\nu_{1}} \ldots\left(D_{K}+i \delta\right)^{\nu_{K}}} \tag{4}
\end{equation*}
$$

- We exploit the Feynman trick

$$
\begin{align*}
\frac{1}{D_{1}^{\nu_{i}} \ldots D_{K}^{\nu_{K}}} & =\frac{\Gamma(\nu)}{\Gamma\left(\nu_{1}\right) \ldots \Gamma\left(\nu_{n}\right)} \int_{0}^{\infty} d^{K} x \frac{x_{1}^{\nu_{1}-1} \ldots x_{n}^{\nu_{K}-1} \delta\left(1-\sum_{j=1}^{K} x_{j}\right)}{\left(x_{1} D_{1}+\ldots+x_{K} D_{K}\right)^{\nu}} \\
\frac{1}{D_{1}^{\nu_{1}} D_{2}^{\nu_{2}}} & =\frac{\Gamma\left(\nu_{1}+\nu_{2}\right)}{\Gamma\left(\nu_{1}\right) \Gamma\left(\nu_{2}\right)} \int_{0}^{1} d x_{1} \frac{x_{1}^{\nu_{1}-1}\left(1-x_{1}\right)^{\nu_{2}-1}}{\left(x_{1} D_{1}+\left(1-x_{1}\right) D_{2}\right)^{\nu_{1}+\nu_{2}}} \tag{5}
\end{align*}
$$

## Iterating the Feynman parameter integration

- Using (6), we may write

$$
\begin{equation*}
I_{\nu_{1} \ldots \nu_{K}}^{(1)}=\frac{\Gamma\left(\nu_{1}+\nu_{2}\right)}{\Gamma\left(\nu_{1}\right) \Gamma\left(\nu_{2}\right)} \int_{0}^{1} d x_{1} x_{1}^{\nu_{1}-1}\left(1-x_{1}\right)^{\nu_{2}-1} I_{\nu_{1}+\nu_{2}, \nu_{3} \ldots \nu_{K}}^{(2)} \tag{7}
\end{equation*}
$$

where $\nu_{1}$ and $\nu_{2}$ are assumed to be positive. If we iterate the recursion, we obtain

$$
\begin{align*}
I_{\nu_{1} \ldots \nu_{K}}^{(1)} & =\frac{\Gamma(\nu)}{\Gamma\left(\nu_{1}\right) \ldots \Gamma\left(\nu_{K}\right)}\left(\prod_{j=1}^{K-1} \int_{0}^{1} d x_{j} x_{j}^{\mu_{j}-1}\left(1-x_{j}\right)^{\nu_{j+1}-1}\right) I_{\nu}^{(K-1)} \\
& =\frac{\Gamma(\nu-l d / 2)}{\Gamma\left(\nu_{1}\right) \ldots \Gamma\left(\nu_{K}\right)}\left(\prod_{j=1}^{K-1} \int_{0}^{1} d x_{j} x_{j}^{\mu_{j}-1}\left(1-x_{j}\right)^{\nu_{j+1}-1}\right) \frac{\tilde{\mathcal{U}}^{\nu-(l+1) d / 2}}{\tilde{\mathcal{F}}^{\nu-l d / 2}} \tag{8}
\end{align*}
$$

where $\mu_{j}=\nu_{1}+\ldots+\nu_{j}$, and $\nu=\mu_{K}$

## System of differential equations with respect to the

## Feynman parameter

- For each family $\vec{I}^{(\chi, d)}$, of the form [Kotikov, 1991, Remiddi, 1997, Gehrmann, Remiddi, 2000]

$$
\begin{equation*}
\partial_{x_{\chi-1}} \vec{I}^{(\chi)}=M_{x_{\chi-1}} \vec{I}^{(\chi)} \tag{9}
\end{equation*}
$$

- Transport(DiffExp) boundary conditions to obtain a piecewise solution between $0<x_{\chi}<1$
- Integrate the expansions according to (7)
- The boundary condition for the first iteration is

$$
\begin{equation*}
I_{\nu}^{(K)}=\int \prod_{i=1}^{L} \frac{d^{d} k_{l}}{i \pi^{\frac{d}{2}}} \frac{1}{\left(D_{1 \ldots K}\right)^{\nu}}=\frac{\Gamma(\nu-L d / 2)}{\Gamma(\nu)} \frac{\tilde{U}^{\nu-(L+1) d / 2}}{\tilde{F}^{\nu-L d / 2}} \tag{10}
\end{equation*}
$$

- With kinematics set to numerial values, and all Feynman parameters set to $11 / 23$
- $D_{1 \ldots l}=x_{1} D_{1}+\ldots+x_{l} D_{l}$

