

Precision tools for future colliders

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Matter To The Deepest  
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# Outline

1 Intro & Motivation

2 Tools Overview

3 Feynman parametrization (FP) based approaches

- SD and MB basic idea
- Numerical evaluation
- Construction of MB representations
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# Introduction

- The standard model is basically right and precision tests agree largely with it.
- If there is new physics, it must be hidden
- Precision = Discovery

## Collider Physics at the Precision Frontier G. Heinrich, 2009.00516

	analytic	<u>numerical</u>
pole cancellation control of integrable singularities fast evaluation <a href="#">extension to more scales/loops</a> <a href="#">automation</a>	exact analytic continuation yes difficult difficult	with numerical uncertainty less straightforward depends promising less difficult

*"New directions in science are launched by new tools much more often than by new concepts."*

F. Dyson

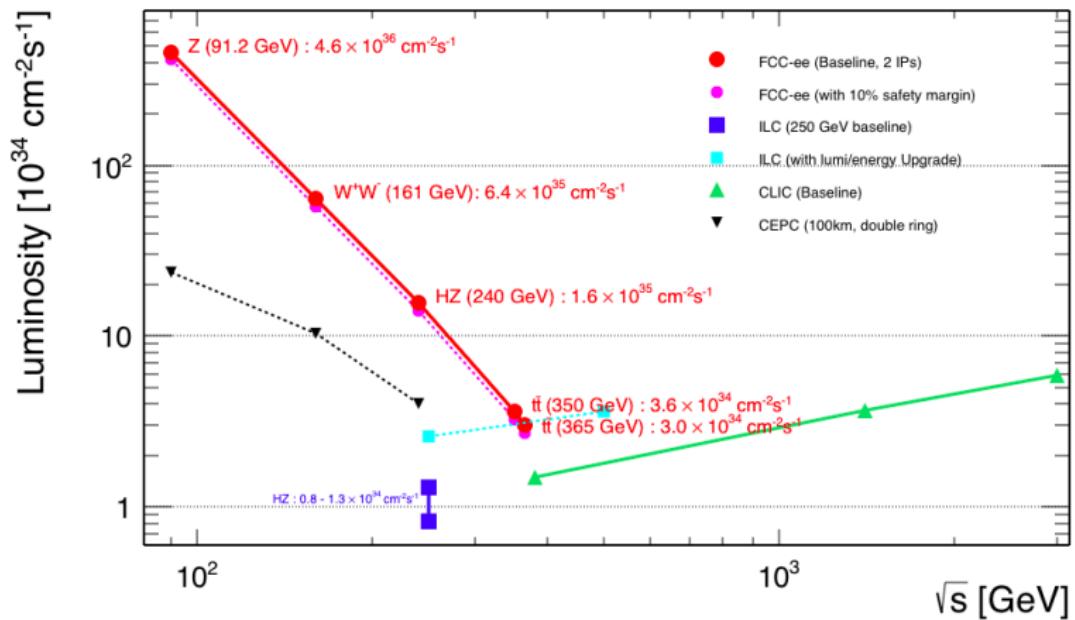


Figure: The prospective centre-of-mass energy reaches and luminosity targets of high-energy  $e^+e^-$  collider projects.

	$\delta\Gamma_Z$ [MeV]	$\delta R_l$ [ $10^{-4}$ ]	$\delta R_b$ [ $10^{-5}$ ]	$\delta \sin_{eff}^{2,l} \theta$ [ $10^{-6}$ ]
Present EWPO theoretical uncertainties				
EXP-2018	2.3	250	66	160
TH-2018	0.4	60	10	45
EWPO theoretical uncertainties when FCC-ee will start				
EXP-FCC-ee	0.1	10	2 ÷ 6	6
TH-FCC-ee	0.07	7	3	7

Table: Comparison for selected precision observables of present experimental measurements (EXP-2018), current theory errors (TH-2018), FCC-ee precision goals at the end of the Tera-Z run (EXP-FCC-ee) and rough estimates of the theory errors assuming that electroweak 3-loop corrections and the dominant 4-loop EW-QCD corrections  $\mathcal{O}(\alpha\alpha_s^2)$ ,  $\mathcal{O}(N_f\alpha^2\alpha_s)$ ,  $\mathcal{O}(N_f^2\alpha^3)$  are available at the start of FCC-ee (TH-FCC-ee). Based on discussion in [1809.01830](#).

Towards 3-loop results:

$Z \rightarrow b\bar{b}$			
Number of topologies	1 loop	2 loops	3 loops
	1	5	50
Number of diagrams	15	1114	120187
Fermionic loops	0	150	17580
Bosonic loops	15	964	102607
QCD / EW	1 / 14	98 / 1016	10405 / 109782

Table: The number of Z decay Feynman diagrams needed to be calculated to meet FCC-ee experimental accuracy. Tadpoles, products of lower loop diagrams and symmetrical diagrams are not included.

# Tools Overview (Direct numerical approach)

- Sector decomposition (SD) method:

- **FIESA** [2016], [A.V.Smirnov]
- **pySecDec** [2022], [Expansion by regions with pySecDec],

- The Mellin-Barnes (MB) method:

- MB [M.Czakon, 2006] (**AMBRE 2** [J.Gluza, et. al., 2011], **AMBRE 3** [I.Dubovik, et. al., 2015], **MBresolve** [A.V.Smirnov, V.A.Smirnov, 2009] )
- **MBnumerics** [J.Usovitsch, I.Dubovik, T.Riemann, 2015]

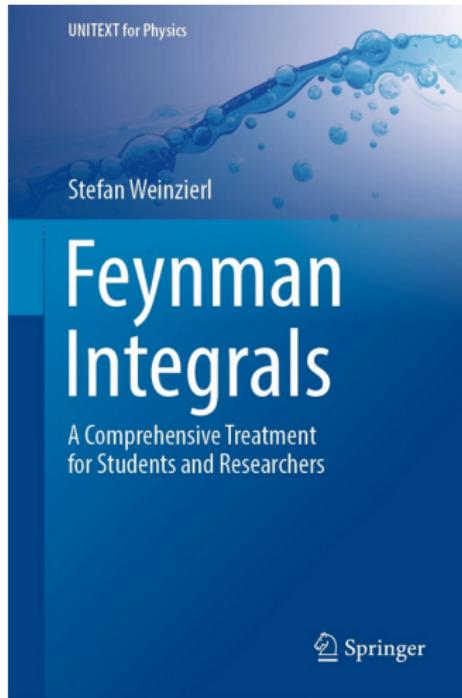
- Differential equations (DEs) method:

- **DiffExp** [F. Moriello, 2019; M. Hidding, 2021],
- **AMFlow** [ X. Liu, Y.-Q. Ma, 2022],
- **SeaSyde** [T. Armadillo, R. Bonciani, S. Devoto, N. Rana, A. Vi, 2022]

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- Integration-By-Parts (IBPs) - crucial for DEs

- **Kira** [F. Lange, P. Maierhöffer, J. Usovitsch, 2021]
- **Reduze** [C. Studerus, 2010]
- **FIRE** [A. Smirnov, 2008]
- **LiteRed** [R. Lee, 2014]



<https://arxiv.org/abs/2201.03593>

<https://doi.org/10.1007/978-3-030-99558-4>

# Feynman parametrization (FP) based approaches

- Feynman parametrization (textbook knowledge) - starting point for SD and MB

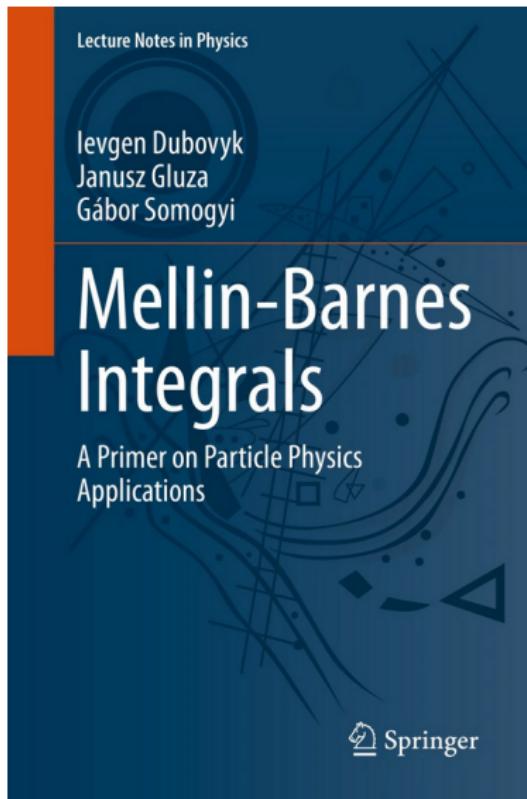
$$G(X) = \frac{(-1)^{N_\nu} \Gamma\left(N_\nu - \frac{d}{2}L\right)}{\prod_{i=1}^N \Gamma(n_i)} \int \prod_{j=1}^N dx_j x_j^{n_j-1} \delta(1 - \sum_{i=1}^N x_i) \frac{U(x)^{N_\nu-d(L+1)/2}}{F(x)^{N_\nu-dL/2}}$$

$U, F$  - Symanzik polynomials, K. Symanzik, Dispersion Relations and Vertex Properties in Perturbation Theory, Progress of Theoretical Physics 20(5) (1958) 690 - 702, <https://doi.org/10.1143/PTP.20.690>

- $U$  and  $F$  are homogeneous in the Feynman parameters,  $U$  is of degree  $L$ ,  $F$  is of degree  $L + 1$
- $U$  is linear in each Feynman parameter. If all internal masses are zero, then also  $F$  is linear in each Feynman parameter -  $F_0; F = F_0 + U \sum_{i=1}^N x_i m_i^2$
- In expanded form each monomial of  $U$  has coefficient +1
- SD (iterated sector decomposition)

$$G_{lk} = \int_0^1 \prod_{j=1, j \neq l}^N dt_j t_j^{a_j - b_j \epsilon} \frac{U_{lk}(t)^{N_\nu-d(L+1)/2}}{F_{lk}(t)^{N_\nu-dL/2}}$$

$$U_{lk}(t) = 1 + u(\mathbf{t}), \quad F_{lk}(t) = -s_0 + \sum_{\beta} s_{\beta} f(\mathbf{t})$$



<https://arxiv.org/abs/2211.13733>  
<https://doi.org/10.1007/978-3-031-14272-7>

- MB

- MB master formula

$$\frac{1}{(A_1 + \dots + A_n)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{(2\pi i)^{n-1}} \int_{-i\infty}^{i\infty} dz_1 \dots dz_{n-1} \times \prod_{i=1}^{n-1} A_i^{z_i} A_n^{-\lambda - z_1 - \dots - z_{n-1}} \prod_{i=1}^{n-1} \Gamma(-z_i) \Gamma(\lambda + z_1 + \dots + z_{n-1})$$

Number of MB integrations (complexity) depends on number of terms in graph polynomials

- MB integral in the most general form

$$I = \frac{1}{(2\pi i)^r} \int_{-i\infty + z_{10}}^{+i\infty + z_{10}} \dots \int_{-i\infty + z_{r0}}^{+i\infty + z_{r0}} \prod_i^r dz_i f_S(Z) \frac{\prod_{j=1}^{N_n} \Gamma(\Lambda_j)}{\prod_{k=1}^{N_d} \Gamma(\Lambda_k)} f_\psi(Z)$$

$f_S(Z)$  depends on:  $Z$  – some subset of integration variables  
 $S$  – kinematic parameters and masses

example :  $f_S(Z) = \left( \frac{M_Z^2}{-s} \right)^z$

$\Lambda_i$  : linear combinations of  $z_i$ , e.g.,  $\Lambda_i = \sum_l \alpha_{il} z_l + \gamma_i$

## Numerical evaluation

- causal  $i\epsilon$  prescription of the Feynman propagators

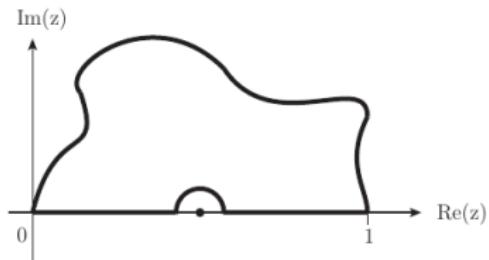
$$\frac{1}{q^2 - m^2 + i\epsilon}$$

- SD

$$F_{lk}(t) = -s_0 + \sum_\beta s_\beta f(\mathbf{t}) - i\epsilon$$

$$\vec{z}(\vec{x}) = \vec{x} - i\vec{\tau}(\vec{x})$$

$$\tau_k = x_k - \lambda x_k (1 - x_k) \frac{\partial F(\vec{x})}{\partial x_k}$$



- MB

$$z = z_0 + it, \quad t \in (-\infty, \infty)$$

in Minkowskian case  $s \rightarrow s + i\epsilon$  ( $s > 0$ )

$$\left( \frac{M_Z^2}{-s} \right)^z = e^{z \ln(-\frac{M_Z^2}{s} + i\epsilon)} \longrightarrow e^{i t \ln \frac{M_Z^3}{s}} e^{-(\pi - \delta)t}, \quad s > 0$$

# Asymptotic behavior

$$\Gamma(z)|_{|z| \rightarrow \infty} = \sqrt{2\pi} e^{-z} z^{z - \frac{1}{2}} \left[ 1 + \frac{1}{12z} + \frac{1}{288z^2} + \dots \right]$$

$z = z_0 + it$ ,  $t \in (-\infty, \infty)$ ,  $|z| \rightarrow \infty \Leftrightarrow t \rightarrow \pm\infty$

$$\frac{\prod_{j=1}^{N_n} \Gamma(\Lambda_j)}{\prod_{k=1}^{N_d} \Gamma(\Lambda_k)} \longrightarrow e^{-\pi|\mathbf{t}|} \frac{1}{|t|^\alpha} \text{ (AMBREv1/2/3)}$$

$e^{-\pi|\mathbf{t}|}$  and  $e^{-\pi t}$  cancel each other when  $t \rightarrow -\infty$  and oscillations are **NOT** damped any more by an exponential factor

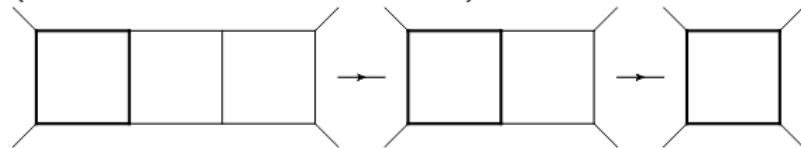
- if we are lucky and  $\alpha \geq 1$  ( $\alpha \geq 2$ ): technical tricks + **proper integrator**
- MBnumerics
- contours deformation (restoring of the exponential damping factor)

# Numerical Integration Libraries

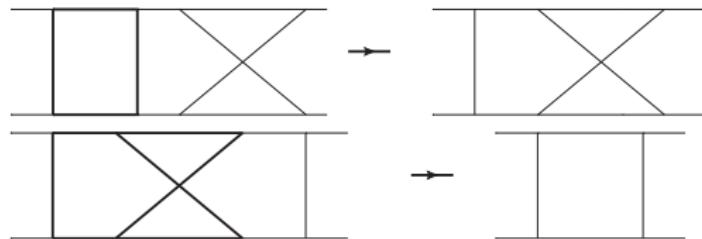
- Classic: Vegas/Suave/Divonne/Cuhre ([Cuba](#)), Cquad ([Gsl](#)).
- New:
  - [QMC](#)(pySecDec) Randomized Quasi-Monte Carlo
    - [Z. Li et al '16, S. Borowka et al '18]
      - \* Integration error  $\sim \frac{1}{N_{\text{samples}}}$  (classical MC  $\sim \frac{1}{\sqrt{N_{\text{samples}}}}$ );
      - \* Works on CPUs and GPUs (with CUDA);
  - [Vegas+](#) [G. P. Lepage '20]
    - Further development of the classical algorithm;
  - [TT](#)(FIESA) Tensor-Train Numerical Integration. [L. Vysotsky et al '21]
    - TT decomposition of a tensor approximating the integrand, multivariate quadrature formulas;
  - [i-flow](#) (pySecDec) [C. Gao et al '20, R. Winterhalder et al '22]
    - First attempts to use the machine learning approach.

## AMBRE versions overview

- iteratively to each subloop – loop-by-loop approach (LA): mostly for planar (AMBREv1.3.1 & AMBREv2.1.1)



- in one step to the complete U and F polynomials – global approach (GA): general (AMBREv3.1.1)
- combination of the above methods – Hybrid approach (HA)  
(next versions ?)



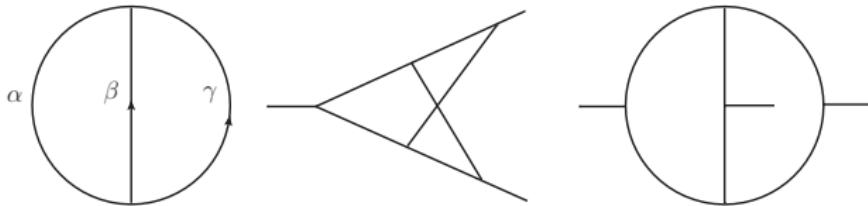
Examples, description, links to basic tools and literature:

<http://prac.us.edu.pl/~gluza/ambre/>

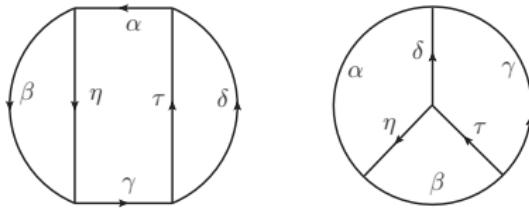
AMBREv3.m:

- topology based factorization - chain diagrams, Kinoshita '74

2-loop:



3-loop:



transformation/rescaling of Feynman parameters:

$$\{\vec{x}\}_i : \quad x_k \rightarrow v_i \xi_{ik} \times \delta \left( 1 - \sum_{k=1}^{\eta_i} \xi_{ik} \right),$$

where  $i$  denotes chain index and  $k \in [1, \eta_i]$ , with  $\eta_i$  - number of propagators in chain.  
 $\delta$ -function keeps number of variables unchanged.

For **any** 2-loop diagram:

$$U_{\text{2-loop}} = v_1 v_2 + v_2 v_3 + v_1 v_3$$

For **any** "ladder" 3-loop diagram:

$$U_{\text{3-loop(I)}} = v_1 v_2 v_3 + v_1 v_2 v_4 + v_2 v_3 v_4 + v_1 v_2 v_5 + v_1 v_3 v_5 + v_2 v_3 v_5 + v_1 v_4 v_5 + v_3 v_4 v_5$$

For **any** "mercedes" 3-loop diagram:

$$\begin{aligned} U_{\text{3-loop(II)}} = & v_1 v_2 v_3 + v_1 v_2 v_4 + v_1 v_3 v_4 + v_1 v_2 v_5 + v_1 v_3 v_5 + v_2 v_3 v_5 + v_2 v_4 v_5 + v_3 v_4 v_5 \\ & + v_1 v_2 v_6 + v_2 v_3 v_6 + v_1 v_4 v_6 + v_2 v_4 v_6 + v_3 v_4 v_6 + v_1 v_5 v_6 + v_3 v_5 v_6 + v_4 v_5 v_6 \end{aligned}$$

Cheng–Wu theorem: Delta function in the Feynman parameters representation can be replaced by  $\delta\left(\sum_{i \in \Omega} x_i - 1\right)$  where  $\Omega$  is an arbitrary subset of the lines  $1, \dots, L$ , when the integration over the rest of the variables, i.e. for  $i \notin \Omega$ , is extended to the integration from zero to infinity.

- 2-loop:  $\delta(1 - v_1 - v_2)$ ,  $U(\vec{v}) = v_3 + v_1 v_2$   
no additional MB integrations from  $U$ , similar to 1-loop cases
- 3-loop:  $\delta(1 - v_1 - v_2 - v_3)$ 
  - "ladder" - 2 additional MB integrations
  - "mercedes" - 4 additional MB integrations

To get minimal dimensionality:

- 1-loop:  $U(\vec{x}) \equiv 1$  whenever it's possible
- 2- and 3-loop: expression for  $F$  polynomial is not expanded

$$F = F_0 + U \sum_{i=1}^N x_i m_i^2$$

- Barnes lemmas

$x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4$	3-dim representation
$(x_1 + x_2)(x_3 + x_4)$	2-dim representation
$(x_1 + x_2)(x_3 + x_4) \rightarrow$ $[x_1 \rightarrow v_1\xi_{11}, x_2 \rightarrow v_1\xi_{12}, \delta(1 - \xi_{11} - \xi_{12});$ $x_3 \rightarrow v_2\xi_{21}, \dots] \rightarrow v_1v_2$	0-dim representation
$(x_1 + x_2)(x_3 + x_4) + \text{BL}$	0-dim representation *)

$$\begin{aligned} *) \quad & (x_1 + x_2)^p \rightarrow \int dz_1 x_1^{z_1} x_2^{p-z_1} \Gamma(-z_1) \Gamma(-p+z_1) \\ & \rightarrow \int dz_1 \Gamma(-z_1) \Gamma(-p+z_1) \Gamma(z_1+1) \Gamma(p-z_1+1) / \Gamma(p+2) \end{aligned}$$

BL can be also applied without factorization, but this requires special transformation of  $z_i$  variables, see e.g., `barnesroutines.m` [D. Kosower, 2009]

$$\int_{-i\infty}^{i\infty} dz \Gamma(a+z) \Gamma(b+z) \Gamma(c-z) \Gamma(d-z) = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}$$

## Minimal Dimensionality vs Integrability

Minimal Dimensionality:

$$G(X) \sim \frac{U(x)^{N_\nu - d(L+1)/2}}{\left(F_0(x) + U(x) \sum_i m_i^2 x_i\right)^{N_\nu - dL/2}} \sim \prod_i (m_i^2 x_i)^{z_i} \frac{U(x)^{N_\nu - d(L+1)/2 + \sum_i z_i}}{F_0(x)^{N_\nu - dL/2 + \sum_i z_i}}$$

- number of additional integrations (beyond massless case)  $\sim$  number of massive propagators
- we lost all information about the threshold behaviour (as well, as in case of LA or HA)

Full expansion in  $F(x, s)$  (two-loop case with one kinematical invariant):

$$F(x, s) = -s x_i x_j x_k + \left( \sum_i m_i^2 - s \right) x_i x_j x_k + m_i^2 x_i x_j x_k$$

- thresholds are clearly separated, below thresholds integrals are "euclidean", expansion is also required for cases with massive external lines (e.g., QED)
- integrals have higher dimensionality
- for a wide class of diagrams the "euclidean" behavior (**NO cancelation of exponential damping factor**) and can be straightforwardly integrated;  $e^{-N\pi|t|} \frac{1}{|t|^\alpha}$ ,  $N > 1$
- requires new "factorization" approach (individually for different  $F$  polynomials)

## Efficient using of BL

We start by encoding the  $z$ -dependence of gamma functions in matrix form

$$M_\Gamma Z = \begin{bmatrix} \alpha_{ij}(\text{numerator}) \\ \dots \\ \alpha_{ij}(\text{denominator}) \end{bmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix}.$$

$M_\Gamma$  is a rectangular  $(N_n + N_d) \times r$  matrix,  $Z$  is an  $r$ -vector of integration variables  $z_i$ . Now, any linear variable transformation can be represented as

$$M_\Gamma Z = M_\Gamma U U^{-1} Z = M_\Gamma U Z', \quad Z' = U^{-1} Z,$$

with a non-singular  $r \times r$  transformation matrix  $U$ .  $M_\Gamma U$  encodes a new  $z$  structure of gamma functions. Barnes lemmas can be applied if columns in  $M_\Gamma U$  have specific structure.

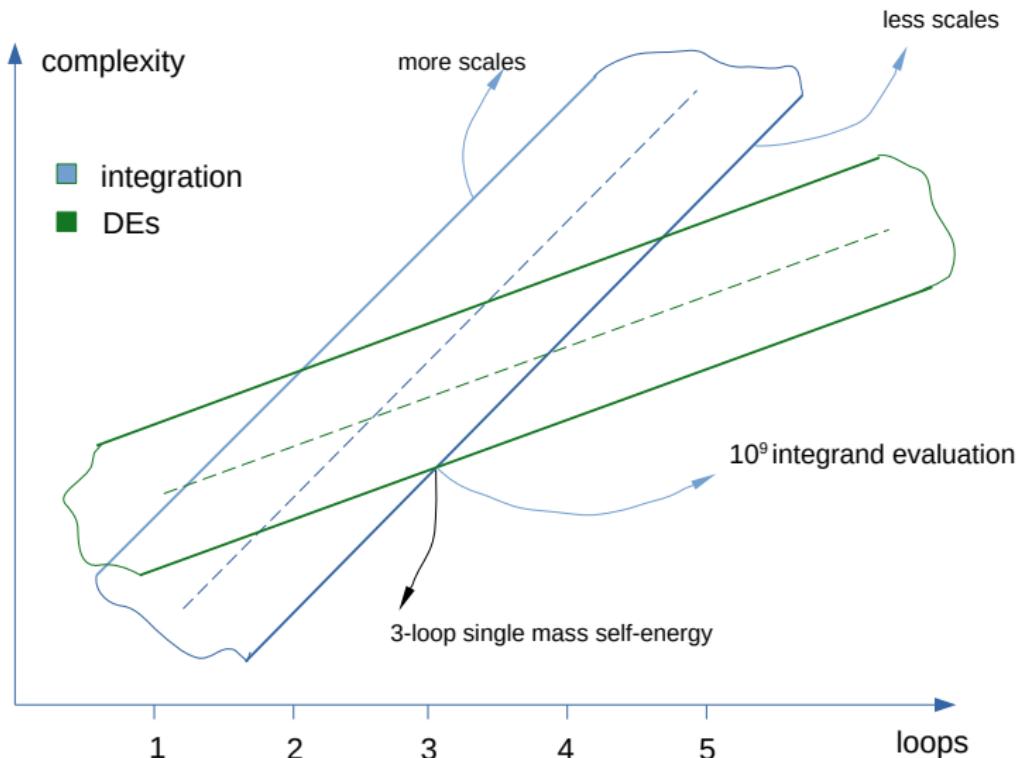
- For 1st Barnes lemma elements in a column from  $N_n + 1$  to  $N_n + N_d$  must be equal to 0 and elements from 1 to  $N_n$  must contain the set  $\{1, 1, -1, -1\}$  while all others must be also equal to 0:

$$M_\Gamma X = \{B_1\}.$$

$X$  is an unknown  $r$ -vector representing a column in  $U$  and  $\{B_1\}$  is a set of all possible right hand sides. For general  $N_n$  one has  $\frac{3N_n!}{4!(N_n - 4)!}$  different r.h.s.

- Each solution represents a column in the matrix  $U$  and can be placed on any position starting from the diagonal matrix.

# Limitations and Perspectives



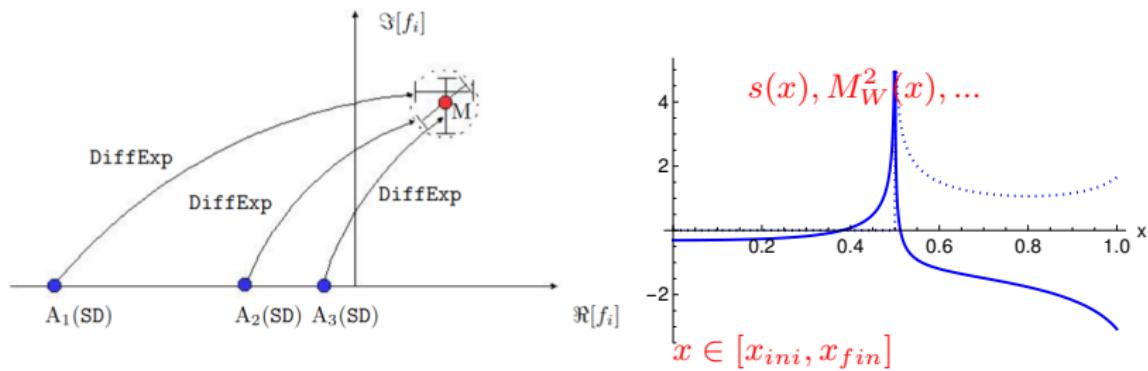
# pySecDec - status at NNNLO

- 3-loop EW self-energies (SE) correction
    - ta - diagrams with top quark + photon
    - th - diagrams with top quark + W/Z/H
    - lh - diagrams with light quarks + W/Z/H
- Mercedes (Merc)  
 Non-planar (NP)  
 Planar (PL)

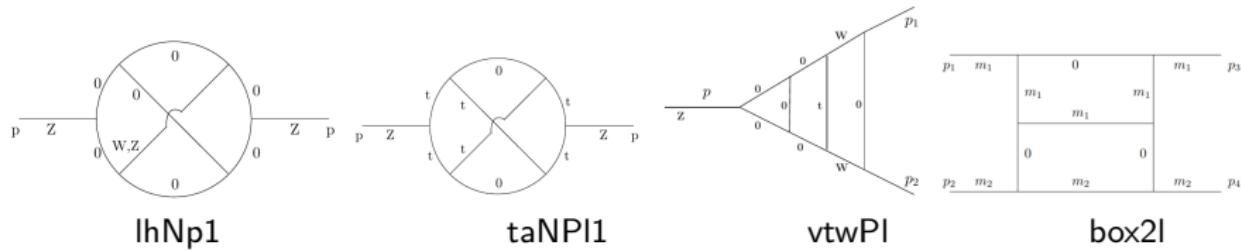
Type	Propagators	Masses	Errors (Merc,NP,PL)	Time(Merc,NP,PL)
ta	6	4	$10^{-8}$ , -, -	6s, -, -
	7	5	$10^{-9}$ , $10^{-9}$ , $10^{-8}$	20s, 9s, 9s
	8	6	$10^{-9}$ , $10^{-9}$ , $10^{-9}$	12s, 9s, 5s
th	6	5	$10^{-8}$ , -, -	9s, -, -
	7	6 (H)	$10^{-9}$ , $10^{-9}$ , $10^{-9}$	37s, 19s, 19s
	8	7	$10^{-10}$ , $10^{-10}$ , $10^{-10}$	48s, 16s, 10s
	8	3	$10^{-2}$ , $10^{-3}$ , $10^{-2}$	3h, 4h, 2h
lh	6	1	$10^{-5}$ , $10^{-8}$ , -	15min, 9s, -
	7	1	$10^{-3}$ , $10^{-3}$ , $10^{-3}$	2h, 2h, 3h
	8	1	NAN, $10^{-1}$ , $10^{-1}$	4h, 6h, 7h

I. Dubovyk, A. Freitas, JG, K. Grzanka, M. Hidding, J. Usovitsch,  
 'Evaluation of multi-loop multi-scale Feynman integrals for precision physics',  
[https://arxiv.org/abs/2201.02576 \(PRD 2022\)](https://arxiv.org/abs/2201.02576)

- KIRA (IBPs) - finite number of MIs;
- Reduze by A. von Manteuffel, E. Panzer, R. M. Schabinger - choice of MIs basis for boundary conditions without  $\epsilon$  singularities;
- pySecDec - calculation of boundary conditions in Euclidean points;
- DiffExp - transport of solutions to the Minkowskian points.



With present SD and MB tools we can not get satisfactory results for 3-loop EWPOs at the  $e^+e^-$  Z-resonance peak.



$$\begin{aligned}
 I_{\text{box2l}}[2, 1, 1, 1, 1, 1, 1, 0, 0, s, t, m_1^2, m_2^2] &= +0.000328707579/\epsilon^2 \\
 &- (0.0014129475 - 0.0020653306 i)/\epsilon \\
 &- (0.005702737 - 0.000485980 i) + \mathcal{O}(\epsilon), \\
 &\text{55 MIs, } s = 2, t = 5, m_1^2 = 4, m_2^2 = 16.
 \end{aligned}$$

MIs with high accuracy: AMFlow, CERN 2022, <https://indico.cern.ch/event/1140580/>

## Summary and Outlook

### ➤ What we have

- Auxiliary mass flow method fully automated the computation of boundary conditions for differential equations.
- AMFlow is the first public tool which can compute arbitrary Feynman loop integrals, at arbitrary kinematic point, to arbitrary precision.

### ➤ What we need

- Powerful reduction techniques are urgently needed to construct differential equations, both for  $\eta$  and for dynamical variables.
- A guide for choosing better master integrals in general cases is needed, which may strongly simplify the differential equations.

AMFlow method,  $\eta = \infty \longrightarrow \eta = 0^+$  analytic continuation (auxiliary mass flow)

A set of Jan 27 2022 papers by Zhi-Feng Liu, Yan-Qin Ma and Xiao Liu:

<https://inspirehep.net/literature/2020677>, <https://inspirehep.net/literature/2020676>,

<https://inspirehep.net/literature/2020880> and 1711.09572

<https://inspirehep.net/literature/1639025>.

$$\tilde{I}_{\vec{\nu}}(\eta) = \int \left( \prod_{i=1}^L \frac{d^D \ell_i}{i \pi^{D/2}} \right) \frac{\tilde{\mathcal{D}}_{K+1}^{-\nu_{K+1}} \cdots \tilde{\mathcal{D}}_N^{-\nu_N}}{\tilde{\mathcal{D}}_1^{\nu_1} \cdots \tilde{\mathcal{D}}_K^{\nu_K}}.$$

$$\tilde{\mathcal{D}}_1 = \ell_1^2 - m^2 + i\eta$$

$$I_{\vec{\nu}} = \lim_{\eta \rightarrow 0^+} \tilde{I}_{\vec{\nu}}(\eta)$$

$$i \frac{\partial}{\partial \eta} \vec{\tilde{J}}(\eta) = A(\eta) \vec{\tilde{J}}(\eta)$$

**Key point:** boundary conditions at  $\eta \rightarrow \infty$  are single mass scale bubble integrals, solved iteratively.

# Conclusions

- Many groups present rapid progress.
- SD is a must have tool for numerical evaluation of FI.
- MB:
  - for certain classes of Feynman integrals, MB method gives very compact and well integrable representations, but in general, the method is not universal
  - linear transformations of integration variables in MB representations are a key ingredient for the simplification of MB integrals (Barnes lemmas and more)
  - proper handling of (pseudo)thresholds allows obtaining "always Euclidean" representation
  - is suitable for analytical methods of FI solution
  - is promising tool and needs further improvements
- Looking on the progress of numerical DEs maybe we see last days of "classical" integration methods.

Thank you!