Deformed Amplituhedron

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Motivation

- The amplituhedron [N. Arkani-Hamed and J. Trnka, 2013] provides the all-loop integrand for the planar $\mathcal{N} = 4$ sYM theory.
- Despite being UV finite it can still have IR divergneces.
- Requires regularization $\Rightarrow$ dimensional regularization.
- Computing amplitudes can be challenging in the dimensional regularization.
- Dimensional regularization breaks the geometric picture.
- Can we regulate the IR divergences directly at the amplituhedron level?

Amplituhedron

The four-point Amplituhedron consists of four external momentum twistors Z_1, Z_2, Z_3, Z_4 and L lines $\{(AB)_i\}$, satisfying

$$\langle (AB)_i 12\rangle > 0,\ \langle (AB)_i 23\rangle > 0,$$
$$\langle (AB)_i 34\rangle > 0,\ \langle (AB)_i 14\rangle > 0\,,$$

Sign flip condition

$$\langle (AB)_i 13\rangle < 0,\ \ \langle (AB)_i 24\rangle < 0\,.$$

Additional positivity conditions:

$$\langle (AB)_i (AB)_j\rangle > 0$$

where

$$\langle 1234\rangle \equiv \det(Z_1 Z_2 Z_3 Z_4)$$

From geometry to the integrand

A differential form (**canonical form**) with logarithmic singularities on the boundary of the positive geometry.

one-simplex: $[a, b]$

Logarithmic singularities on the boundaries: $a : \frac{dx}{x-a}\,, \quad b : \frac{dx}{x-b}$

Canonical form:

$$\Omega_{[a,b]} = \frac{dx}{x-a} - \frac{dx}{x-b} = \frac{a-b}{(x-a)(x-b)} dx$$

$$= dlog(x-a) - dlog(x-b) = dlog\left(\frac{x-a}{x-b}\right)$$

Residues on the boundaries:

$$res_{x=a}(\Omega_{[a,b]}) = 1\,, \quad res_{x=b}(\Omega_{[a,b]}) = -1$$

Deformation in two dimensions

Amplituhedron:

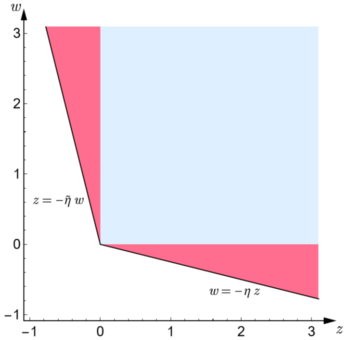
$$z > 0\,, w > 0$$

$$\omega = \frac{dwdz}{wz}$$

Deformation:

$$\eta z + w > 0\,, \tilde{\eta} w + z > 0$$

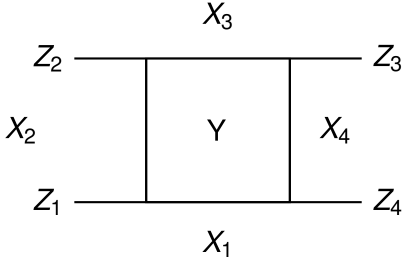
$$\omega = \frac{dwdz(1-\eta\tilde{\eta})}{(\eta z + w)(\tilde{\eta} w + z)}$$



Four point one-loop amplituhedron

Let us introduce the notation:

$$X_1 = Z_1 Z_2\,, X_2 = Z_2 Z_3\,, X_3 = Z_3 Z_4\,, X_4 = Z_1 Z_4\,, Y = Z_A Z_B\,.$$



massless on-shell kinematics:

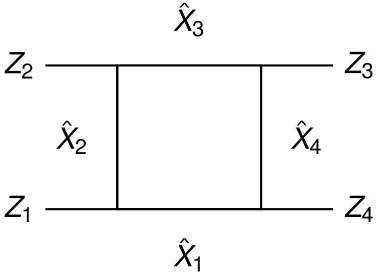
$$X_i^2 = (X_i, X_i) = 0\,, (X_i X_{i+1}) = 0$$

Number of boundaries: $(4, 10, 12, 6)$

Deformation

The deformed Amplituhedron shifts external kinematics with two deformation parameters x, y,

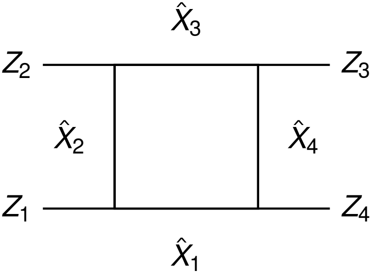
$$\hat{X}_1 = X_1 + x X_3\,,$$
$$\hat{X}_2 = X_2 + y X_4\,,$$
$$\hat{X}_3 = X_3 + x X_1\,,$$
$$\hat{X}_4 = X_4 + y X_2\,.$$



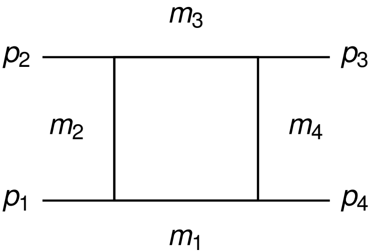
Adjacent conditions still hold: $(\hat{X}_i \hat{X}_{i+1}) = 0$
Massive propagators: $\hat{X}_i^2 \neq 0$

Thanks to the deformation parameters all collinear configurations are removed as boundaries!
Number of boundaries: $(4, 6, 4, 2)$

Kinematics



$\Leftrightarrow$



$$p_i^2 = m_i^2 + m_{i+1}^2$$

Dual conformal cross ratios:

$$u = \frac{(\hat{X}_1\hat{X}_3)^2}{\hat{X}_1^2\hat{X}_3^2} = \frac{1}{4}\left(x + \frac{1}{x}\right)^2 = \frac{(-s + m_1^2 + m_3^2)^2}{4m_1^2 m_3^2}\,,$$

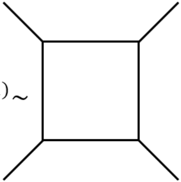
$$v = \frac{(\hat{X}_2\hat{X}_4)^2}{\hat{X}_2^2\hat{X}_4^2} = \frac{1}{4}\left(y + \frac{1}{y}\right)^2 = \frac{(-t + m_2^2 + m_4^2)^2}{4m_2^2 m_4^2}\,.$$

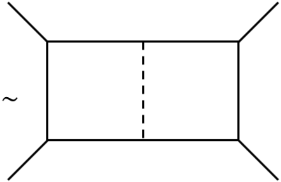
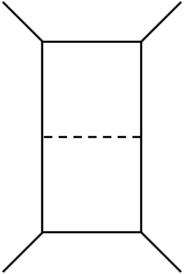
Two-loop deformed amplitude

We consider the amplitude, normalized by its tree-level contribution,

$$M = 1 - g^2 M^{(1)} + g^4 M^{(2)} + \mathcal{O}(g^6)\,,$$

with $g^2 = g_{YM}^2 N_c/(16\pi^2)$

$M^{(1)} \sim$ 

$M^{(2)} \sim$  $+$ 

One-Loop contribution

$$M = 1 - g^2 M^{(1)} + g^4 M^{(2)} + \mathcal{O}(g^6)$$

Deformed one-loop integral:

$$M^{(1)} = \int_Y \frac{(1-x^2)\,(1-y^2)}{(\hat{X}_1 Y)(\hat{X}_2 Y)(\hat{X}_3 Y)(\hat{X}_4 Y)}$$

Can be easily evaluated using Feynman parameters:

$$M^{(1)}(x,y) = \int_0^\infty \frac{d^4\alpha}{GL(1)} \frac{(1-x^2)\,(1-y^2)}{[(\alpha_1 x + \alpha_3)(\alpha_1 + \alpha_3 x) + (\alpha_2 y + \alpha_4)(\alpha_2 + \alpha_4 y)]^2} =$$

$$2\log(x)\log(y)$$

Can also be established with the differential equations method!

Two-loop amplitude

$$M = 1 - g^2 M^{(1)} + g^4 M^{(2)} + \mathcal{O}(g^6)\,,$$

$$M^{(2)} = -Q\left(x^2\right) - Q\left(y^2\right) + Q\left(x^2 y^2\right) + J_3\left(x^2\right)\log\left(y^2\right) + J_3\left(y^2\right)\log\left(x^2\right)\,,$$

with

$$Q(z) = 3\mathrm{Li}_4(z) - 3\log(z)\mathrm{Li}_3(z) + \frac{3}{2}\log^2(z)\mathrm{Li}_2(z) + \frac{1}{2}\log^3(z)\log(1-z) + \frac{3\pi^4}{10} + \frac{\pi^2}{4}\log^2(z) + \frac{3}{16}\log^4(z) + \log^2(z)\,\mathrm{Li}_2(1-z) + 4\pi^2\mathrm{Li}_2\left(-\sqrt{z}\right) - \log(z)\,\mathrm{Li}_3\left(1-\frac{1}{z}\right) - \log(z)\,\mathrm{Li}_3(1-z)\,,$$

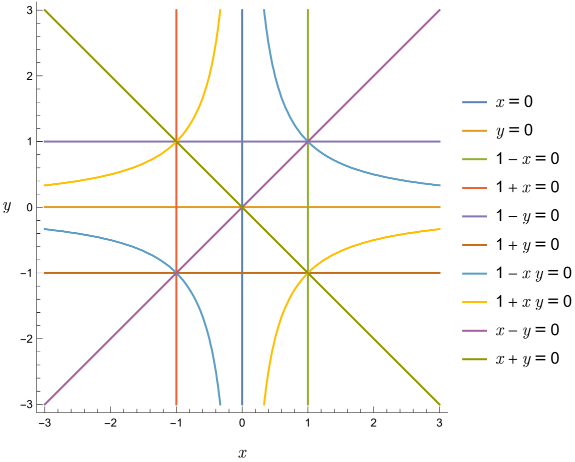
and

$$J_3(z) = \frac{1}{4}\log^3(z) + \log(z)\,\mathrm{Li}_2(1-z) - 2\mathrm{Li}_3(1-z) - 2\mathrm{Li}_3\left(1-\frac{1}{z}\right)\,.$$

Only classical polylogarithms no $\mathrm{Li}_{2,2}$ or $(\mathrm{Li}_2)^2$.

Symbol alphabet

$$\mathcal{A} = \{x,\, 1-x,\, 1+x,\, y,\, 1-y,\, 1+y,\, x-y,\, x+y,\, 1-xy,\, 1+xy\}$$



High-energy limit

This corresponds to $x, y \to 0$, or equivalently, $s, t \to \infty$, keeping $x/y = t/s$ fixed.

$$\lim_{x,y\to 0} \log M = -\frac{1}{2}\Gamma_{\mathrm{cusp}}(g) \log x \log y + \Gamma_{\mathrm{collinear}}(g) (\log x + \log y) + C(g) \,,$$

$\Gamma_{\mathrm{cusp}} = 4g^2 - 8\zeta_2 g^4 + \mathcal{O}(g^6)$
$\Gamma_{\mathrm{collinear}}(g) = -4\zeta_3 g^4 + \mathcal{O}(g^6)$ and $C(g) = -3/10\pi^4 g^4 + \mathcal{O}(g^6)$.

Analogous to formulas in
dimensional regularization [Z. Bern, I. Dixon and V. Smirnov, 2005] and on the Coulomb branch [L. Alday et al., 2010]

Regge limit

This corresponds to $y \to 0$, keeping x fixed ($t \to \infty$).
We find that the leading terms in the Regge limit are given by

$$\lim_{y \to 0} M(x = e^{i\phi}, y) = r(\phi, \theta_0; g)\, y^{\Gamma_{\rm cusp}(\phi;\theta_0;g)} + \mathcal{O}(y^0)\,,$$

where

$$\Gamma_{\rm cusp}(\phi, \theta; g) = g^2 \xi(-2\log x) + g^4 \Bigg\{ \xi \frac{4}{3} \log x \left(\pi^2 + \log^2 x\right) +$$
$$\xi^2 \left[4\mathrm{Li}_3\left(x^2\right) - 4\mathrm{Li}_2\left(x^2\right)\log(x) - \frac{4}{3}\log^3(x) - \frac{2}{3}\pi^2 \log(x) - 4\zeta_3 \right] \Bigg\}$$

and

$$\xi = \frac{\cos\theta - \cos\phi}{i \sin\phi} = \frac{1 + x^2 - 2x\cos\theta}{1 - x^2}$$

provided that we set $\xi = 1$

Summary

- We generalized the four-particle Amplituhedron geometry of planar sYM such that the amplitude $M(x, y)$ is infrared finite and depends on two dual conformal parameters x, y.
- We obtained analytic result for the two-loop deformed amplitude.
- In different kinematic limits we obtained behaviour similar to that on the Coulomb branch.
- We expect that this new setup will lead to substantial progress in making the connection between geometry and integrated functions.

Extra slides

Embedding formalism

- Embedding in the projective space $\mathbb{C}^4 \to \mathbb{CP}^5$: $x^\mu \to X^a = (x^\mu, X^-, X^+)$, with $X^a \simeq \alpha X^a$ $(\alpha \neq 0)$.
- Scalar product

$$(XY) = 2x_\mu y^\mu + X^+ Y^- + X^- Y^+$$

- The integration measure is defined such that

$$\int_Y \frac{1}{(YQ)^4} = \frac{1}{\Gamma(4)} \frac{1}{[\frac{1}{2}(QQ)]^2} \, .$$

Differential equations in four-dimensions

[S. Caron-Huot and J. Henn, 2014]

- Working with finite integrals in $D = 4$ simplifies the differential equations.
- We work in the embedding formalism where a dual conformal symmetry is apparent.
- Derivatives of dual conformal integrals with respect to kinematic variables are dual conformal. This is also true for the integration-by-parts identities (IBP), Thus, we can work only with a subset of integrals.
- In $D = 4$ different loop orders can be connected using the four-dimensional Laplace-type equation.
- Differential equation matrix in a triangular form. Basis functions of uniform transcendental weight.

One-loop differential equations

- We consider the integrals belonging to the family

$$G_{a_1,a_2,a_3,a_4} := \int_Y \frac{1}{(\tilde{X}_1 Y)^{a_1} (\tilde{X}_2 Y)^{a_2} (\tilde{X}_3 Y)^{a_3} (\tilde{X}_4 Y)^{a_4}} \quad \text{with} \quad \sum_{i=1}^{4} a_i = 4$$

- We use the following derivatives:

$$\partial_x = \frac{1}{(-1+x)(1+x)} (xO_{1,1} - O_{1,3} - O_{3,1} + xO_{3,3})$$

where $O_{i,j} = (X_i \partial_{X_j})$. An analogous definition holds for ∂_y.

- We solve the differential equations iteratively

$$\partial_x G_{1,1,1,1} = \frac{2xG_{1,1,1,1} - 2G_{0,1,2,1}}{1-x^2}$$

One-loop differential equations

- System of the differential equations:

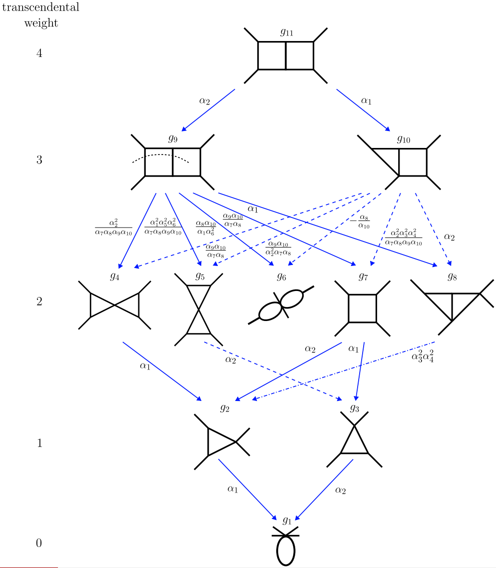
$$\begin{aligned} g_1 &= 4xy\, G_{2,2,0,0}\,, \\ g_2 &= -2x(1-y^2)\, G_{0,1,2,1}\,, \\ g_3 &= -2(1-x^2)y\, G_{1,2,1,0}\,, \\ g_4 &= (1-x^2)(1-y^2)\, G_{1,1,1,1}\,. \end{aligned}$$

$$d\vec{g} = d\begin{pmatrix} 0 & 0 & 0 & 0 \\ \log(y) & 0 & 0 & 0 \\ \log(x) & 0 & 0 & 0 \\ 0 & \log(x) & \log(y) & 0 \end{pmatrix}\vec{g}$$

- Integrated out result:

$$\begin{aligned} g_1 &= 2\,, \\ g_2 &= 2\log(y)\,, \\ g_3 &= 2\log(x)\,, \\ g_4 &= 2\log(x)\log(y)\,. \end{aligned}$$

Two-loop differential equations



IBP vectors

- The generation of an IBP relation is based on the fundamental identity

$$0 = \int_Y \frac{\partial}{\partial Y^a} \delta\left(\frac{1}{2} Y^2\right) Q^a(Y)\,,$$

- To generate IBP relations, we will be considering the IBP vectors of the form

$$Q^a_{ij,1} \equiv (Y_1 X_j) X^a_i - (Y_1 X_i) X^a_j\,, \quad Q^a_{ij,2} \equiv (Y_2 X_j) X^a_i - (Y_2 X_i) X^a_j\,,$$

  for $(i, j) \in \{1, 2, 3, 4\}$.
- In general we require orthogonality between Y_k for $k = 1, 2$ and IBP vectors Q_{ij}, i.e., $(Y_k, Q_{ij}) = 0$.
- In the two-loop case we can consider additional vectors

$$Q^a_{i,2} \equiv (Y_1 X_i) Y^a_2 - (Y_1 Y_2) X^a_i\,, \quad Q^a_{i,1} \equiv (Y_2 X_i) Y^a_1 - (Y_2 Y_1) X^a_i\,.$$

Double box

Two-loop contribution

$$M^{(2)}(x,y) = I^{\mathrm{db}}(x,y) + I^{\mathrm{db}}(y,x)\,,$$

Integral representation

$$I^{\mathrm{db}}(x,y) = \int_{Y_1}\int_{Y_2} \frac{(1-x^2)^2(1-y^2)}{(\hat{X}_1 Y_1)(\hat{X}_2 Y_1)(\hat{X}_3 Y_1)(Y_1 Y_2)(\hat{X}_1 Y_2)(\hat{X}_3 Y_2)(\hat{X}_4 Y_2)}$$

Two-loop box

$$I^{\mathrm{db}}(x,y) = -Q(y^2) + \frac{1}{2}Q\left(\frac{y^2}{x^2}\right) + \frac{1}{2}Q(x^2y^2) + J_3(x^2)\log(y^2)$$