Recent developments from Feynman integrals

Turning mathematics into precision predictions

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1. Introduction: Curves of genus 0 and 1
2. Higher genus
3. Higher dimensions

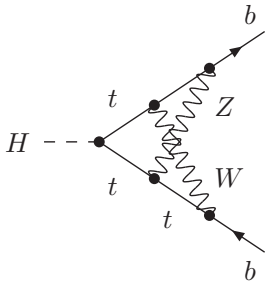
Scattering amplitudes

- We would like to make precise predictions for observables in scattering experiments from (quantum) field theory.
- Any such calculation will involve a scattering amplitude.
- Unfortunately we cannot calculate scattering amplitudes exactly.
- If we have a small parameter like a small coupling, we may use perturbation theory.
- We may organise the perturbative expansion of a scattering amplitude in terms of Feynman diagrams.

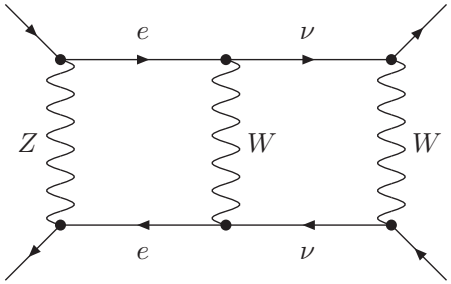
Scattering amplitude = sum of all Feynman diagrams

Applications

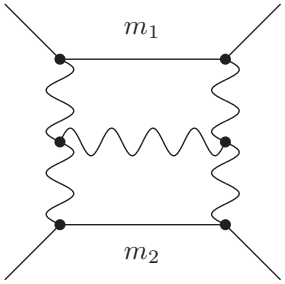
High-energy experiments: LHC



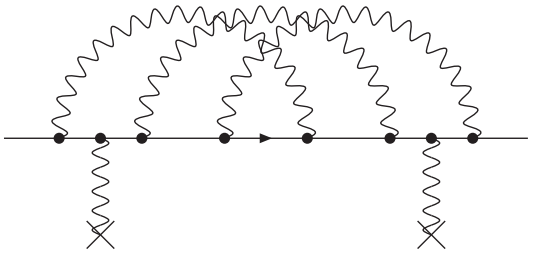
Low-energy experiments: Moller and P2



Gravitational waves:



Spectroscopy: Lamb shift



Standard techniques

- Dimensional regularisation ('t Hooft, Veltman '72, Bollini, Giambiagi '72, Ashmore '72):
  $D = 4 - 2\varepsilon$, used to regulate ultraviolet and infrared divergences.

- Integration-by-parts identities (Tkachov '81, Chetyrkin '81):
  leads to master integrals $I = (I_1, I_2, \ldots, I_{N_F})$.

- Method of differential equations (Kotikov '90, Remiddi '97, Gehrmann and Remiddi '99):

$$dI = A(x, \varepsilon)\, I$$

- Transformation to ε-factorised form (Henn '13):

$$dI = \varepsilon A(x)\, I$$

The method of differential equations

We want to calculate

$$I(\varepsilon, x)$$

as a Laurent series in ε.

1. *Find a differential equation with respect to the kinematic variables for the Feynman integral (always possible).*
2. *Transform the differential equation into an* ε*-factorised form (**bottle neck**).*
3. *Solve the latter differential equation with appropriate boundary conditions (always possible).*

Example for an ε-factorised form

$$dI = \varepsilon A(x) I, \qquad A(x) = C_1 \omega_1 + C_2 \omega_2$$

with differential one-forms

$$\omega_1 = \frac{dx}{x}, \qquad \omega_2 = \frac{dx}{x-1},$$

and matrices

$$C_1 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

Notation

$\mathbf{N_F} = N_{\text{Fibre}}$: Number of master integrals,
master integrals denoted by $I = (I_1, \ldots, I_{N_F})$.

$\mathbf{N_B} = N_{\text{Base}}$: Number of kinematic variables,
kinematic variables denoted by $x = (x_1, \ldots, x_{N_B})$.

$\mathbf{N_L} = N_{\text{Letters}}$: Number of letters,
differential one-forms denoted by $\omega = (\omega_1, \ldots, \omega_{N_L})$.

Fibre bundles

We have a **vector bundle**:

- Fibre spanned by the master integrals $I_1, \ldots, I_{N_F}$.

  (The master integrals $I_1(x), \ldots, I_{N_F}(x)$ can be viewed as local sections, and for each x they define a basis of the vector space in the fibre. In other words, they define a local frame.)
- Base space with coordinates $x = (x_1, \ldots, x_{N_B})$ corresponding to kinematic variables.
- Connection defined by the matrix A appearing in the differential equation.

Allowed **transformations**:

- a change of basis in the fibre,
- a coordinate transformation on the base manifold.

Iterated integrals

Definition

For ω_1, ..., ω_k differential 1-forms on a manifold M and $\gamma : [0,1] \to M$ a path, write for the pull-back of ω_j to the interval $[0,1]$

$$f_j(\lambda)\, d\lambda \quad = \quad \gamma^* \omega_j .$$

The **iterated integral** is defined by

$$I_\gamma(\omega_1, \ldots, \omega_k; \lambda) \quad = \quad \int_0^\lambda d\lambda_1 f_1(\lambda_1) \int_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) \ldots \int_0^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k) .$$

Chen '77

Section 1

Geometry

The base space

Question:

After a suitable coordinate transformation, can we relate the base space to a space known from mathematics?

The base space

- Assume we have $(n-3)$ variables $z_1, \ldots, z_{n-3}$ and differential one-forms

$$\begin{aligned}\omega_k \quad \in \quad & \{d\ln(z_1), d\ln(z_2), \ldots, d\ln(z_{n-3}), \\ & d\ln(z_1-1), \ldots, d\ln(z_{n-3}-1), \\ & d\ln(z_1-z_2), \ldots, d\ln(z_i-z_j), \ldots, d\ln(z_{n-4}-z_{n-3})\}\end{aligned}$$

- The iterated integrals $I_\gamma(\omega_1, \ldots, \omega_r; \lambda)$ are multiple polylogarithms.
- We require $z_i \notin \{0, 1, \infty\}$ and $z_i \neq z_j$:
  This defines the **moduli space** $\mathcal{M}_{0,n}$: The space of configurations of n points on a Riemann sphere modulo Möbius transformations.
- Usually the z_i are functions of the kinematic variables x and the arguments of the dlog-forms define the Landau singularities.

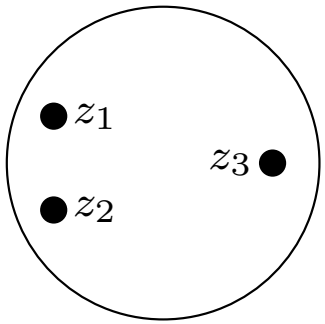
Differential one-forms on $\mathcal{M}_{0,n}$

Multiple polylogarithms:

$$\omega^{\mathrm{MPL}} = \frac{dz}{z-c}.$$

Take home message:

Feynman integrals, which evaluate to multiple polylogarithms are related to a Riemann sphere (a smooth complex algebraic curve of genus zero).



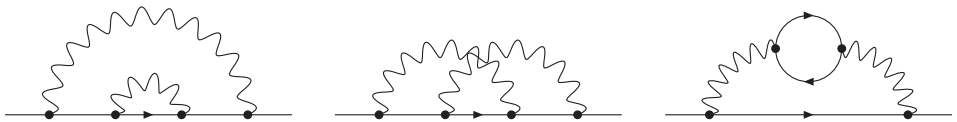
Beyond multiple polylogarithms

- Not every Feynman integral can be expressed in terms of multiple polylogarithms.
- Starting from two-loops, we encounter more complicated functions.
- The next-to-simplest Feynman integrals involve an **elliptic curve**.

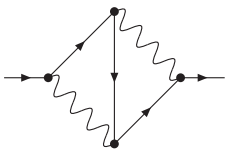
Elliptic curves

We do not have to go very far to encounter elliptic integrals in precision calculations: The simplest example is the two-loop electorn self-energy in QED:

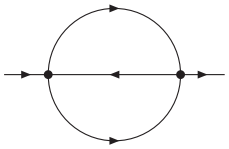
There are three Feynman diagrams contributing to the two-loop electron self-energy in QED with a single fermion:



All master integrals are (sub-) topologies of the **kite graph**:



One sub-topology is the **sunrise graph** with three equal non-zero masses:



(Sabry, '62)

Elliptic curves

Where is the elliptic curve?

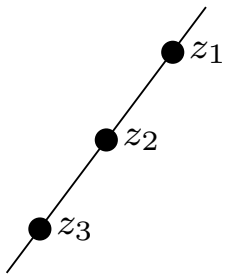
For the sunrise it's very simple: The second graph polynomial defines an elliptic curve in Feynman parameter space:

$$-p^2 a_1 a_2 a_3 + (a_1 + a_2 + a_3)(a_1 a_2 + a_2 a_3 + a_3 a_1) m^2 \quad = \quad 0.$$

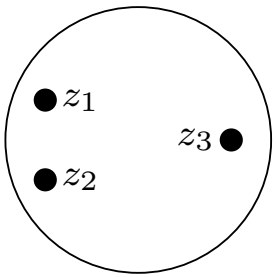
Moduli spaces

$\mathcal{M}_{g,n}$: Space of **isomorphism classes of** smooth (complex, algebraic) **curves of genus g with n marked points**.

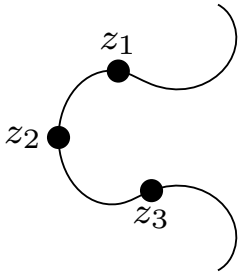
complex curve



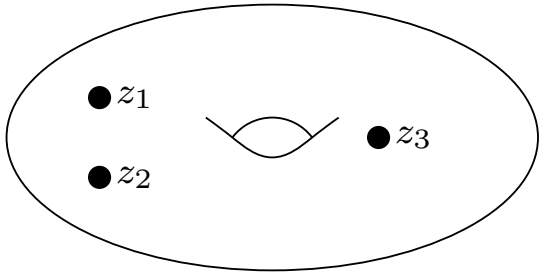
$\Leftrightarrow$



real surface



$\Leftrightarrow$



Coordinates

Genus 0: $\dim \mathcal{M}_{0,n} = n - 3$.
Sphere has a unique shape
Use Möbius transformation to fix $z_{n-2} = 1$, $z_{n-1} = \infty$, $z_n = 0$
Coordinates are $(\mathbf{z_1}, \ldots, \mathbf{z_{n-3}})$

Genus 1: $\dim \mathcal{M}_{1,n} = n$.
One coordinate describes the shape of the torus
Use translation to fix $z_n = 0$
Coordinates are $(\tau, \mathbf{z_1}, \ldots, \mathbf{z_{n-1}})$

Differential one-forms on $\mathcal{M}_{1,n}$

1. *From modular forms ($f_k(\tau)$ modular form):*

$$\omega_k^{\text{modular}} = 2\pi i \, f_k(\tau) d\tau$$

Adams, S.W. '17

2. *From the Kronecker function:*

$$\omega_k^{\text{Kronecker}} = (2\pi i)^{2-k} \left[g^{(k-1)}(z,\tau)\, dz + (k-1)\, g^{(k)}(z,\tau) \frac{d\tau}{2\pi i} \right]$$

Broedel, Duhr, Dulat, Tancredi, '17

Numerics

Physics is about numbers:

- Iterated integrals of modular forms and elliptic multiple polylogarithms can be evaluated numerically with arbitrary precision.
- Implemented in `GiNaC`.

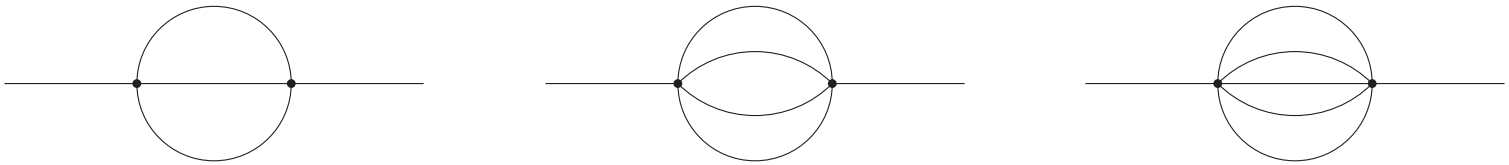
  Walden, S.W, '20

```
ginsh - GiNaC Interactive Shell (GiNaC V1.8.1)
  __,  _______  Copyright (C) 1999-2021 Johannes Gutenberg University Mainz,
 (__) *       | Germany.  This is free software with ABSOLUTELY NO WARRANTY.
  ._) i N a C | You are welcome to redistribute it under certain conditions.
<-------------' For details type 'warranty;'.

Type ?? for a list of help topics.
> Digits=50;
50
> iterated_integral({Eisenstein_kernel(3,6,-3,1,1,2)},0.1);
0.23675657575197179243274817775862177623438999192840338805367
```

Generalisations

- We understand by now very well Feynman integrals related to algebraic curves of genus 0 and 1. These correspond to iterated integrals on the moduli spaces $\mathcal{M}_{0,n}$ and $\mathcal{M}_{1,n}$.
- The obvious generalisation is the generalisation to algebraic curves of **higher genus** g, i.e. iterated integrals on the moduli spaces $\mathcal{M}_{g,n}$.
- However, we also need the generalisation from curves to surfaces and **higher dimensional objects**: The geometry of the banana graphs with equal non-vanishing internal masses



are Calabi-Yau manifolds.

Section 2

Higher genus curves

Hyperelliptic curves

Definition

A hyperelliptic curve is an algebraic curve of genus $g \geq 2$ whose defining equation takes the form

$$y^2 = P(z),$$

for some polynomial $P(z)$ of degree $(2g+1)$ or $(2g+2)$.

They generalise elliptic curves, whose defining equation takes the same form when $g = 1$.

We are interested in **Feynman integrals**, where the maximal cut takes the form

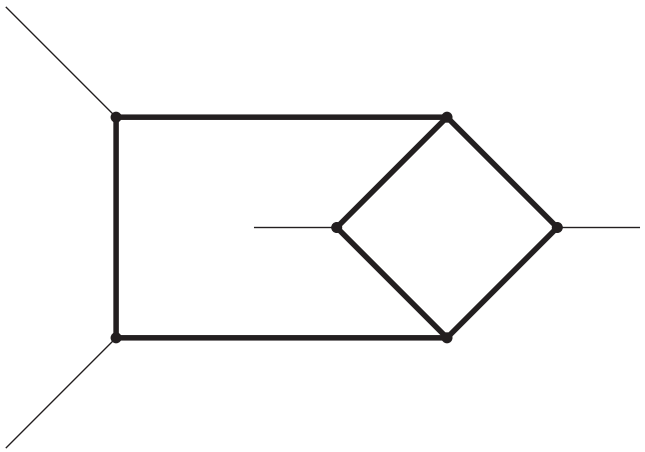
$$\int dz \, \frac{N(z)}{\sqrt{P(z)}}$$

Non-planar double boxes

Non-planar double boxes (with sufficient internal/external masses) provide examples of higher-genus Feynman integrals.

- In the loop momentum representation one obtains a genus 3 curve.

  Georgoudis, Zhang, '15
- In the Baikov representation one obtains a genus 2 curve.



Can we understand this?

Yes we can!

R. Marzucca, A. McLeod, B. Page, S.Pögel, S.W., '23

Extra involutions

- Any hyperelliptic curve $H : y^2 = P(z)$ has an involution symmetry $e_0 : y \to -y$.
- **The solution to this riddle**: The higher genus curve has an extra involution. In the simplest case, if $P(z)$ is of the form

$$P(z) \;=\; Q\left(z^2\right) \;=\; \left(z^2 - \alpha_1^2\right) \ldots \left(z^2 - \alpha_{g+1}^2\right)$$

  the extra involution is given by $e_1 : z \to -z$.
- There is an algorithm to detect the extra involution.
- To a hyperelliptic curve with an extra involution we can associate two curves through the substitution $w = z^2$

$$\begin{aligned} H_1 \quad &: \quad y_1^2 = Q(w) \\ H_2 \quad &: \quad y_2^2 = wQ(w) \end{aligned}$$

  of genus $\lfloor \frac{g}{2} \rfloor$ and $\lceil \frac{g}{2} \rceil$, respectively.

Lorentz invariance

Why is there an extra involution?

For our example we can trace it back to discrete Lorentz transformations (parity, time reversal):

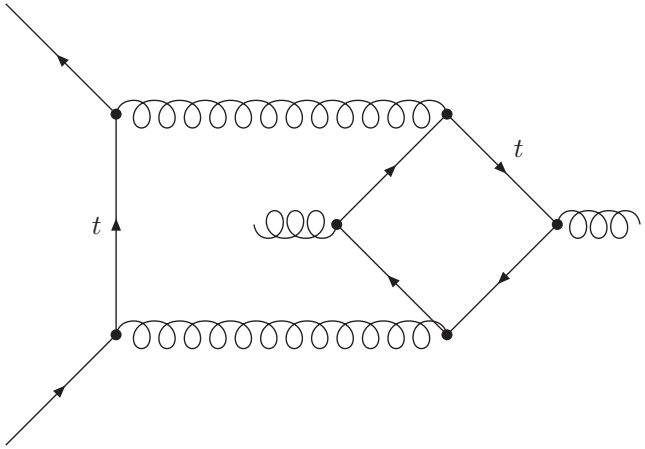
- In the Baikov representation everything is manifestly Lorentz invariant, the Baikov variables are Lorentz invariants:

$$z = k^2 - m^2.$$

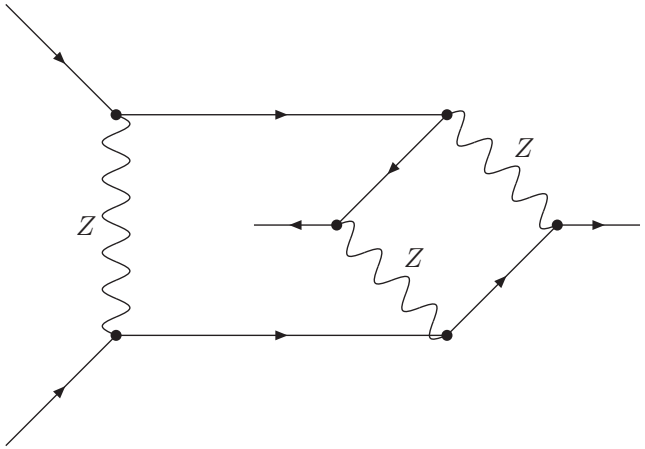
- In the loop momentum representation we choose a frame, we choose a parametrisation of the loop momenta, we choose an elimination order: The full Lorentz symmetry is not required to be trivially realised, but may manifest itself through extra symmetries of the curve.

Examples

- Top pair production at NNLO (genus drop from 3 to 2)



- Møller scattering at NNLO (genus drop from 3 to 2)



Section 3

Calabi-Yau manifolds

Calabi-Yau manifolds

Definition

A Calabi-Yau manifold of complex dimension n is a compact Kähler manifold M with vanishing first Chern class.

Theorem (conjectured by Calabi, proven by Yau)

An equivalent condition is that M has a Kähler metric with vanishing Ricci curvature.

Mirror symmetry

The **mirror map** relates a Calabi-Yau manifold A to another Calabi-Yau manifold B with Hodge numbers $h_B^{p,q} = h_A^{n-p,q}$.

Candelas, De La Ossa, Green, Parkes '91

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

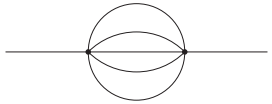
Calabi-Yau manifold A

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{2,1} & & 0 & \\ 1 & & h^{1,1} & & h^{1,1} & & 1 \\ & 0 & & h^{2,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

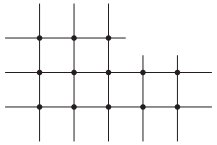
mirror image B

Fantastic Beasts and Where to Find Them

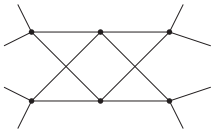
- Bananas



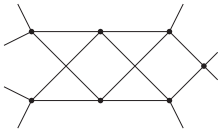
- Fishnets



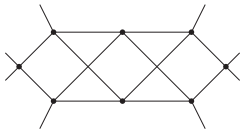
- Amoebas



- Tardigrades



- Paramecia



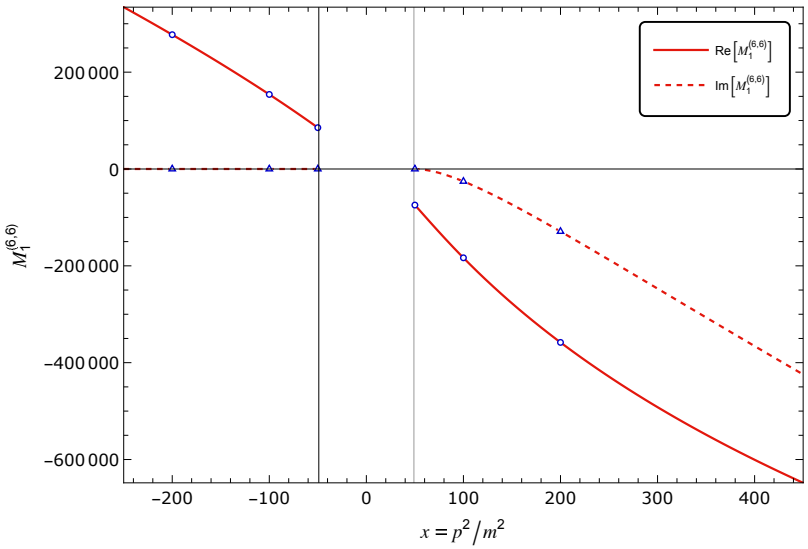
Aluffi, Marcolli, '09, Bloch, Kerr, Vanhove, '14
Bourjaily, McLeod, von Hippel, Wilhelm, '18
Duhr, Klemm, Loebbert, Nega, Porkert, '22

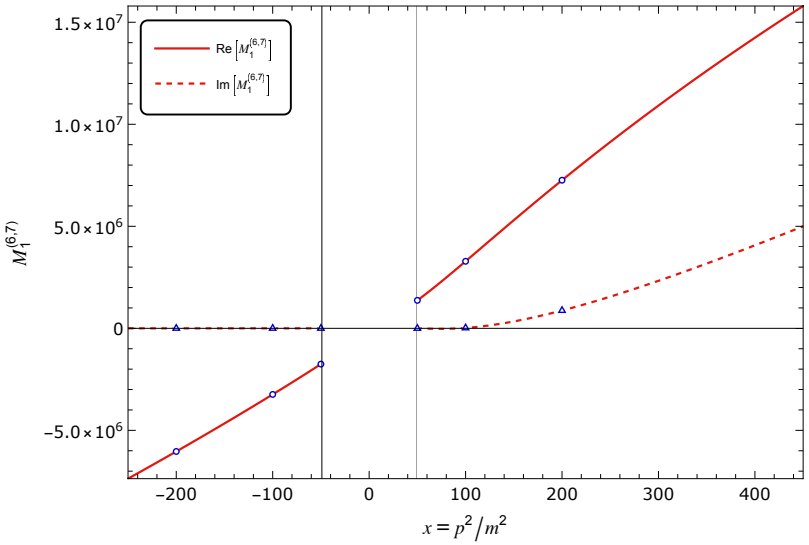
Bananas

- The l-loop banana integral with (equal) non-zero masses is related to a **Calabi-Yau $(l-1)$-fold**.
- An elliptic curve is a Calabi-Yau 1-fold, this is the geometry at two-loops.
- The system of differential equations for the equal mass l-loop banana integral can be transformed to an **ε-factorised form**.
  - Change of variables from $x = p^2/m^2$ to τ given by mirror map.
  - Transformation constructed from special local normal form of a Calabi-Yau operator.

    M. Bogner '13, D. van Straten '17
- Strong support for the conjecture that a transformation to an ε-factorised differential equation exists for all Feynman integrals.

Results: Six loops





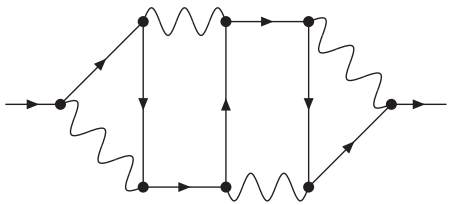
Expansion around $y = 0$ converges at six loops for $|p^2| > 49m^2$.
Agrees with results from `pySecDec`.

The geometry of this Feynman integral is a **Calabi-Yau five-fold**.

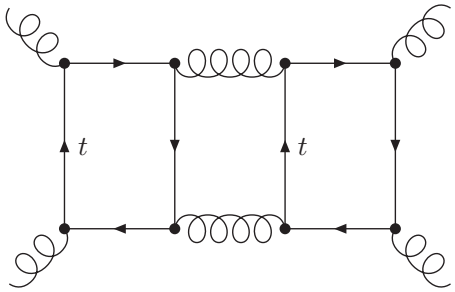
Pögel, Wang, S.W. '22

Examples

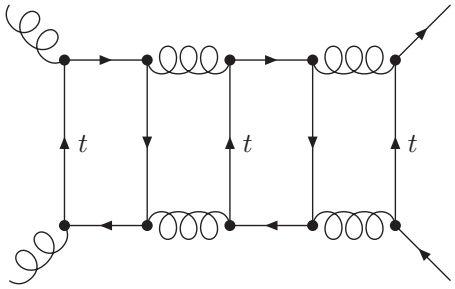
- Electron self-energy in QED
  (related to a Calabi-Yau 3-fold).



- Dijet production at N^3LO
  (related to a Calabi-Yau 2-fold).



- Top pair production at N^4LO
  (related to a Calabi-Yau 3-fold)



Conclusions

- Feynman integrals are needed for precision calculations in perturbative quantum field theory.
- Method of differential equations is a powerfull tool for computing Feynman integrals.
- It is helpful to relate a Feynman integral to a geometric object (spheres, elliptic curves, curves of higher genus, Calabi-Yau n-folds, ...). Algebraic geometry gives us information on the original Feynman integral.

Section 4

Back-up slides

The Kronecker function

Define the **first Jacobi theta function** $\theta_1(z,q)$ by (with $q = e^{2\pi i\tau}$)

$$\theta_1(z,q) = -i \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} e^{i\pi(2n+1)z}.$$

The **Kronecker function** $F(z,\alpha,\tau)$:

$$F(z,\alpha,\tau) = \theta_1'(0,q)\frac{\theta_1(z+\alpha,q)}{\theta_1(z,q)\theta_1(\alpha,q)} = \frac{1}{\alpha}\sum_{k=0}^{\infty} \mathbf{g}^{(\mathbf{k})}(\mathbf{z},\tau)\alpha^k$$

We are interested in the coefficients $g^{(k)}(z,\tau)$ of the Kronecker function.