Soft photon emission and the LBK theorem

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Matter To The Deepest 2023

September 18, 2023

Why soft photons?

Ratio of observed soft photon over expected from soft bremsstrahlung. [C. Wong (2014)]

Experiment	Collision Energy	Photon k_T	Obs/Brem Ratio
K^+p, CERN, WA27, BEBC (1984)	70 GeV	$k_T < 60$ MeV	4.0 $\pm$0.8
K^+p, CERN, NA22, EHS (1993)	250 GeV	$k_T < 40$ MeV	6.4 $\pm$1.6
π^+p, CERN, NA22, EHS (1997)	250 GeV	$k_T < 40$ MeV	6.9 $\pm$1.3
π^-p, CERN, WA83, OMEGA (1997)	280 GeV	$k_T < 10$ MeV	7.9 $\pm$1.4
π^+p, CERN, WA91, OMEGA (2002)	280 GeV	$k_T <$20 MeV	5.3 $\pm$0.9
pp, CERN, WA102, OMEGA (2002)	450 GeV	$k_T <$20 MeV	4.1 $\pm$0.8
$e^+e^- \to$ hadrons, CERN, LEP, DELPHI with hadron production (2010)	$\sim$91 GeV(CM)	$k_T <$60 MeV	4.0
$e^+e^- \to \mu^+\mu^-$, CERN, LEP, DELPHI with no hadron production (2008)	$\sim$91 GeV(CM)	$k_T <$60 MeV	1.0

- Excess of observed soft photons, but only for processes involving hadrons.
- Future upgrades on the ALICE detector (ALICE 3 expected by $\sim$2035) will be able to measure ultra-soft photons, up to 1MeV.
- An efficient implementation for computing soft photon emission is needed.

Future experiments

ALICE 3 ($\sim$2035) [ALICE collaboration (2022)]

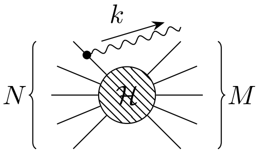
Observables	Kinematic range
Heavy-flavour hadrons	$p_T \to 0$, $\lvert\eta\rvert < 4$
Dielectrons	$p_T \approx 0.05$ to $3\,\mathrm{GeV}/c$, $M_{ee} \approx 0.05$ to $4\,\mathrm{GeV}/c^2$
Photons	$p_T \approx 0.1$ to $50\,\mathrm{GeV}/c$, $-2 < \eta < 4$
Quarkonia and exotica	$p_T \to 0$, $\lvert\eta\rvert < 1.75$
Ultrasoft photons	$p_T \approx 1$ to $50\,\mathrm{MeV}/c$, $3 < \eta < 5$
Nuclei	$p_T \to 0$, $\lvert\eta\rvert < 4$

Table 3: Overview of key physics objects and the respective kinematic ranges of interest for ALICE 3.

- Exploration of real and virtual soft photons
- $pp \to pp\pi^+\pi^- + \gamma$ and $pp \to ppJ/\psi + \gamma$ processes

Soft photon emission: Eikonal (LP) approximation

Emission of a soft photon from a general process $N \to M + \gamma$:



$$\mathcal{A}_j = Q_j \bar{v}(p_j) \not{\varepsilon}^*(k) \frac{\not{k} - \not{p}_j + m}{(p_j - k)^2 - m^2} \mathcal{H}_j(p_1, \ldots, p_j - k, \ldots, p_{N+M})$$

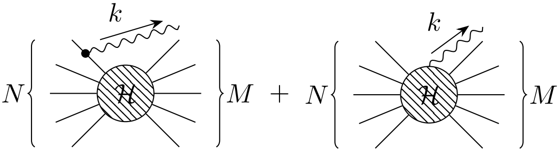
In the limit of soft photons $(k \to 0)$ [F.E. Low - *Phys.Rev.* (1958)]

$$\mathcal{A}_j^{\mathrm{LP}} = Q_j \frac{p_j \cdot \varepsilon^*(k)}{p_j \cdot k} \mathcal{H}(p)$$

Summing over all possible photon emissions,

$$\mathcal{A}^{\mathrm{LP}} = \left(\sum_j \eta_j Q_j \frac{p_j \cdot \varepsilon^*(k)}{p_j \cdot k} \right) \mathcal{H}(p), \qquad \eta = \begin{cases} +1 & \text{for anti-fermions} \\ -1 & \text{for fermions} \end{cases}$$

Soft photon emission: NLP (Low-Burnett-Kroll Theorem)



$$\mathcal{A} = \varepsilon^*_\mu \left(\mathcal{A}^\mu_{\text{ext}} + \mathcal{A}^\mu_{\text{int}}\right) \Longrightarrow k_\mu \left(\mathcal{A}^\mu_{\text{ext}} + \mathcal{A}^\mu_{\text{int}}\right) = 0 \Longrightarrow k_\mu \mathcal{A}^\mu_{\text{int}} = -k_\mu \mathcal{A}^\mu_{\text{ext}}$$

Considering only tree-level diagrams $\mathcal{A}^\mu_{\text{int}}$ is fully determined by $\mathcal{A}^\mu_{\text{ext}}$: [S.L. Adler, Y. Dothan - *Phys.Rev.* (1966)]

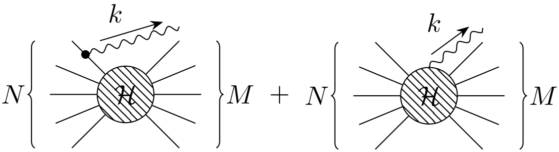
$$\mathcal{A}^\mu_{\text{LP+NLP}}(p,k) = \sum_j \frac{\eta_j Q_j}{k \cdot p_j} \left[p^\mu_j + i\eta_j k_\nu \frac{\hat{\sigma}^{\mu\nu}_j}{2} + (k \cdot p_j) G^{\mu\nu}_j \hat{D}_{j\nu} \right] \mathcal{H}$$

$$G^{\mu\nu}_j = g^{\mu\nu} - \frac{p^\mu_j k^\nu}{p_j \cdot k}$$

- $\hat{\sigma}^{\mu\nu}_j \mathcal{H}$: Substitute $u(p_j) \to \sigma^{\mu\nu} u(p_j)$ or equivalent for anti-fermions and final-state fermions.
- $\hat{D}_{j\nu}\mathcal{H}$: Differentiate the amputated part of $\mathcal{H}$ with respect to p^ν_j:

$$\bar{\mathcal{H}}_j u(p_j) \to \frac{\partial \bar{\mathcal{H}}_j}{\partial p^\nu_j} u(p_j)$$

Soft photon emission: NLP (Low-Burnett-Kroll Theorem)



The expression is simplified considering the unpolarized process and computing $\overline{|\mathcal{A}|}^2$. [T.H. Burnett, N.M. Kroll - *Phys.Rev.Lett.* (1967)]
This is because of the relation

$$ik_\nu\left[\sigma^{\mu\nu}, \not{p}_j \pm m\right] = -2(k\cdot p_j)G^{\mu\nu}\frac{\partial(\not{p}_j \pm m)}{\partial p_j^\nu}$$

which allows to combine all the NLP terms together;

$$\overline{|\mathcal{A}|}^2_{\mathrm{LP+NLP}} = -\sum_{i,j}\frac{(\eta_i Q_i p_i)\cdot(\eta_j Q_j p_j)}{(p_i\cdot k)(p_j\cdot k)}\left[1 + \frac{(p_j\cdot k)p_{i\mu}}{p_i\cdot p_j}G_j^{\mu\nu}\frac{\partial}{\partial p_j^\nu}\right]\overline{|\mathcal{H}|}^2$$

Conservation of 4-momentum

Conservation of 4-momentum: LP approximation

- Low's theorem relates the amplitude $\mathcal{A}$ for $N \to M + \gamma$ to the amplitude $\mathcal{H}$ for $N \to M$.

$$\mathcal{A}_{\mathrm{LP}}(p,k) = \left(\sum_i \eta_i Q_i \frac{p_i \cdot \varepsilon^*(k)}{p_i \cdot k} \right) \mathcal{H}(p)$$

- It is not possible to impose conservation of 4-momentum for both amplitudes simultaneously. Low's theorem relates a physical amplitude to an unphysical one.
- Even worse, Feynman amplitudes are ill-defined for arbitrary 4-momenta: $\tilde{\mathcal{M}} = \mathcal{M} + \Delta$ is physically equivalent to $\mathcal{M}$ if $\Delta(p)$ vanishes when $\sum_i p_i = 0$.
- Low's theorem gives a relation between a well-defined quantity and an ill-defined one!

Conservation of 4-momentum: LP approximation

The amplitude $\mathcal{A}$ must have a unique, well-defined value if 4-momentum is conserved: $\sum_i p_i = k$.

Because $\mathcal{H}(p)$ is not well defined we have an ambiguity on $\mathcal{A}$ given by

$$\left(\sum_i \eta_i Q_i \frac{p_i \cdot \varepsilon^*(k)}{p_i \cdot k}\right) \Delta(p)$$

$\Delta(p)$ must vanish at the surface $\sum_i p_i = 0$, so

$$\sum_j p_j \to 0 \Longrightarrow \Delta(p) \to 0$$

which means $\Delta(p) = \mathcal{O}(k)$ and the ambiguity in $\mathcal{A}$ is a NLP correction.

Low's theorem can be used unambiguously at LP.

Conservation of 4-momentum: NLP approximation

$$\overline{|\mathcal{A}|}^2_{\mathrm{LP+NLP}} = -\sum_{i,j} \frac{(\eta_i Q_i p_i)\cdot(\eta_j Q_j p_j)}{(p_i\cdot k)(p_j\cdot k)}\left[1 + \frac{(p_j\cdot k)p_{i\mu}}{p_i\cdot p_j} G_j^{\mu\nu}\frac{\partial}{\partial p_j^\nu}\right]\overline{|\mathcal{H}|}^2$$

In general, we proved the following:

For any function $\Delta(p)$ that vanish in the surface $\sum_i p_i = 0$, gauge invariance implies that

$$\sum_{i,j} \frac{(\eta_i Q_i p_i)\cdot(\eta_j Q_j p_j)}{(p_i\cdot k)(p_j\cdot k)}\left[1 + \frac{(p_j\cdot k)p_{i\mu}}{p_i\cdot p_j} G_j^{\mu\nu}\frac{\partial}{\partial p_j^\nu}\right]\Delta(p) = \mathcal{O}(1)$$

So, Low's theorem gives a well-defined result also at NLP for any amplitude $\mathcal{H}$, as long as the exact same amplitude is used consistently everywhere.

Shifted kinematics

Shifted kinematics

Idea: Evaluate $\mathcal{H}$ using a different set of conserved momenta so that $\mathcal{H}$ is uniquely defined.

[T.H. Burnett, N.M. Kroll - *Phys.Rev.Lett.* (1967)] [V. Del Duca, E. Laenen, L. Magnea, L. Vernazza, C.D. White - *JHEP* (2017)] [D. Bonocore, A. Kulesza - *Phys.Rev.B* (2021)]

The expression for LBK theorem looks like a first order expansion:

$$\overline{|\mathcal{A}|}^2_{\mathrm{LP+NLP}} = -\sum_{i,j} \frac{(\eta_i Q_i p_i)\cdot(\eta_j Q_j p_j)}{(p_i\cdot k)(p_j\cdot k)} \left[1 + \frac{(p_j\cdot k)p_{i\mu}}{p_i\cdot p_j} G_j^{\mu\nu} \frac{\partial}{\partial p_j^\nu}\right] \overline{|\mathcal{H}|}^2$$

$$\overline{|\mathcal{A}|}^2_{\mathrm{LP+NLP}} = -\left(\sum_{i,j} \frac{(\eta_i Q_i p_i)\cdot(\eta_j Q_j p_j)}{(p_i\cdot k)(p_j\cdot k)}\right) \overline{|\mathcal{H}(p+\delta p)|}^2$$

$$= -C\overline{|\mathcal{H}(p+\delta p)|}^2$$

$$\delta p_j^\nu = \eta_j Q_j C^{-1} \sum_i \left(\frac{\eta_i Q_i p_{i\mu}}{p_i\cdot k}\right) G_j^{\mu\nu}$$

$p_j + \delta p_j$ fulfil the conservation of momentum for $\mathcal{H}$;

$$\sum_j \delta p_j = -k \Longrightarrow \sum_j (p_j + \delta p_j) = 0$$

On-shell shifted kinematics

$$\overline{|\mathcal{A}|}^2_{\mathrm{LP+NLP}} = -C\overline{|\mathcal{H}(p+\delta p)|}^2$$

$$\delta p_j^\nu = \eta_j Q_j C^{-1} \sum_i \left(\frac{\eta_i Q_i p_{i\mu}}{p_i \cdot k} \right) G_j^{\mu\nu} = \mathcal{O}(k)$$

The shifts modify the mass of the particles by NNLP terms.

$$p_j \cdot \delta p_j = 0 \Longrightarrow (p_j + \delta p_j)^2 = m_j^2 + \mathcal{O}\big(k^2\big)$$

This is consistent with the approximation, but not ideal for numerical implementations.

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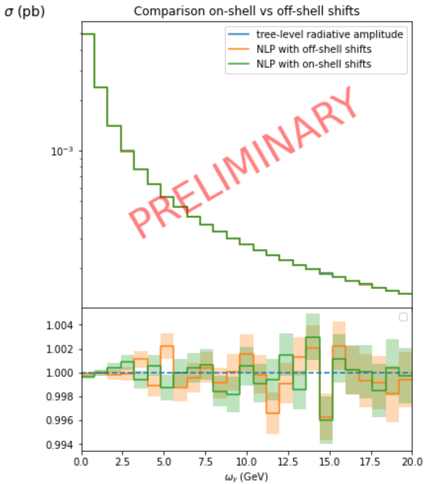
We found an alternative way to do the shifts that:

- is consistent with LBK theorem at NLP,
- satisfies four-momentum conservation,
- keeps the particles on-shell to all orders in the expansion of k.

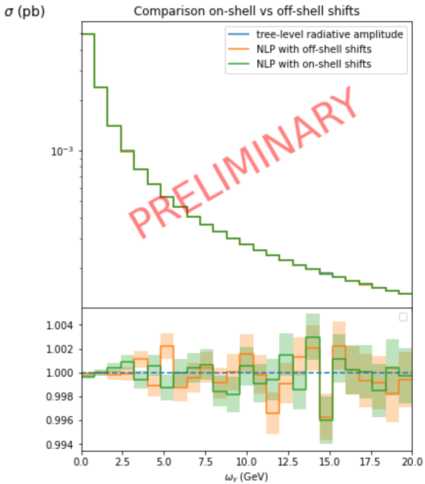
Results



Radiative amplitude calculated with MadGraph5,
non-radiative amplitude obtained analytically.

$e^- e^+ \to \mu^- \mu^+ \gamma, \qquad \sqrt{S} = 1\text{TeV}$

$p_{T,\gamma} > 0.1\text{GeV}, \quad p_{T,\mu} > 10\text{GeV}, \quad |\eta| < 2.5, \quad \Delta R > 0.4$



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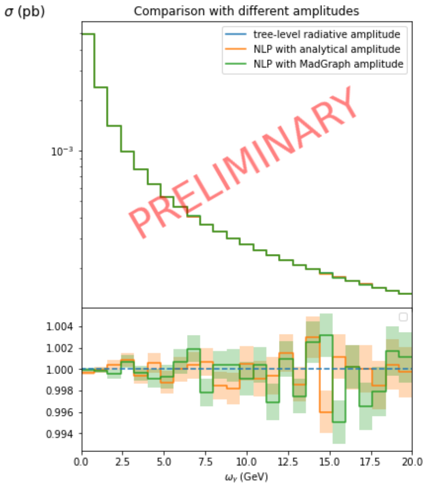
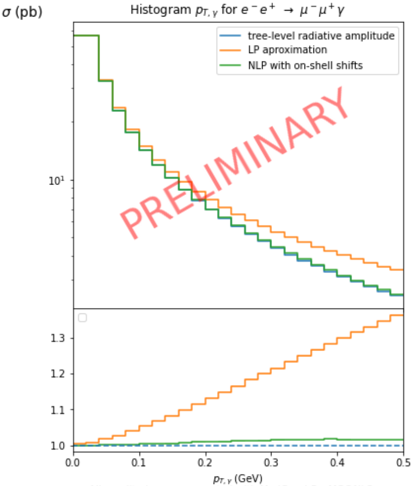


Radiative amplitude calculated with MadGraph5_aMC@NLO.
NLP amplitudes are obtained using the new on-shell shifts.

$e^-e^+ \to \mu^-\mu^+\gamma, \quad \sqrt{S} = 1\text{TeV}$

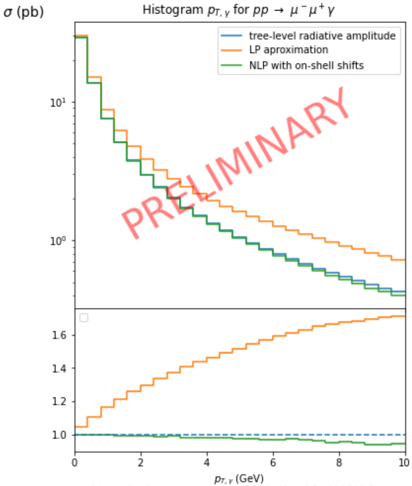
$p_{T,\gamma} > 0.1\text{GeV}, \quad p_{T,\mu} > 10\text{GeV}, \quad |\eta| < 2.5, \quad \Delta R > 0.4$



All amplitudes are generated with MadGraph5_aMC@NLO.

$e^-e^+ \to \mu^-\mu^+\gamma, \quad \sqrt{S} = 91.1\text{GeV}$

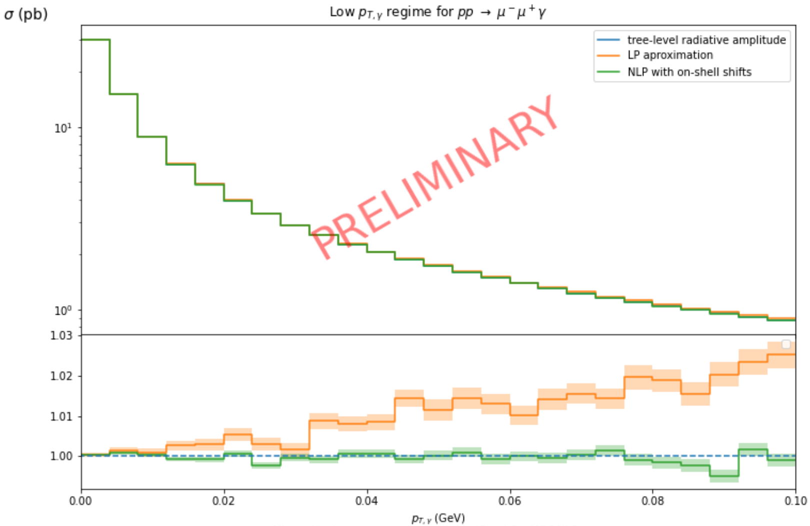
$10MeV < p_{T,\gamma} < 5\text{GeV}, \quad p_{T,\mu} > 10\text{GeV}, \quad |\eta| < 2.5, \quad \Delta R > 0.4$



All amplitudes are generated with MadGraph5_aMC@NLO.

$pp \to \mu^-\mu^+\gamma, \quad \sqrt{S} = 13\text{TeV}$

$100MeV < p_{T,\gamma} < 50\text{GeV}, \quad p_{T,\mu} > 10\text{GeV}, \quad |\eta| < 2.5, \quad \Delta R > 0.4$



All amplitudes are generated with MadGraph5_aMC@NLO.

$pp \to \mu^-\mu^+\gamma, \qquad \sqrt{S} = 13\text{TeV}$

$1MeV < p_{T,\gamma} < 1\text{GeV}, \quad p_{T,\mu} > 10\text{GeV}, \quad |\eta| < 2.5, \quad \Delta R > 0.4$

Conclusions

- Precision predictions call for understanding the NLP terms.
- LBK theorem is free of inconsistencies and can be used safely for calculating soft photon spectra.
- Reformulation of LBK theorem using on-shell shifted kinematics opens the door to an efficient implementation for the NLP approximation for the emission of (ultra-)soft photons (e.g. as measured in the future by ALICE3 detector).
- More work has to be done in order to understand the origin of the soft photon anomaly observed at LEP.