

Precision Predictions: Lightcone expansion beyond leading power

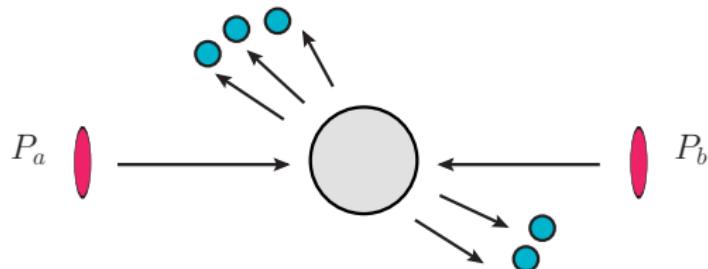
Sebastian Jaskiewicz

Matter to the Deepest 2023
University of Silesia
Ustroń
September 18th, 2023



Predictions for the LHC: Factorization

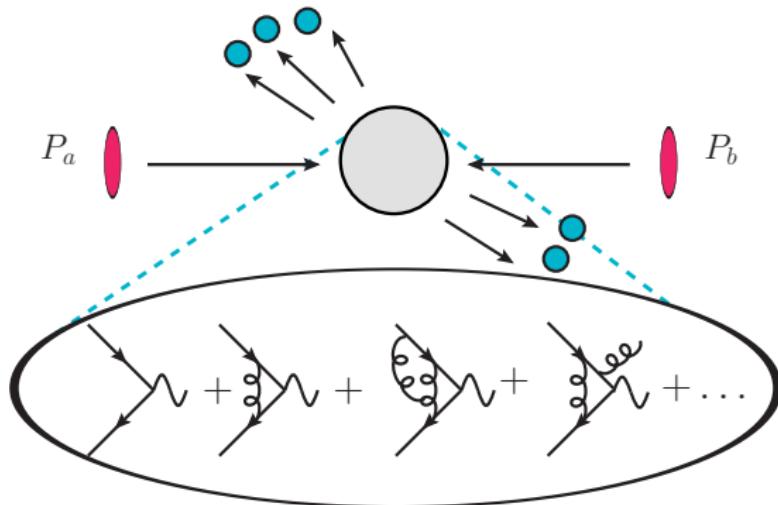
$$d\sigma \sim \sum_{a,b} \int_0^1 dx_a dx_b f_{a/A}(x_a) f_{b/B}(x_b) \hat{\sigma}_{ab} + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{Q}\right)$$



Factorization of physics at different scales is crucial in making predictions, goes back to [Collins, Soper, Sterman. Nucl. Phys. B250 (1985) 199]

Predictions for the LHC: Fixed-order

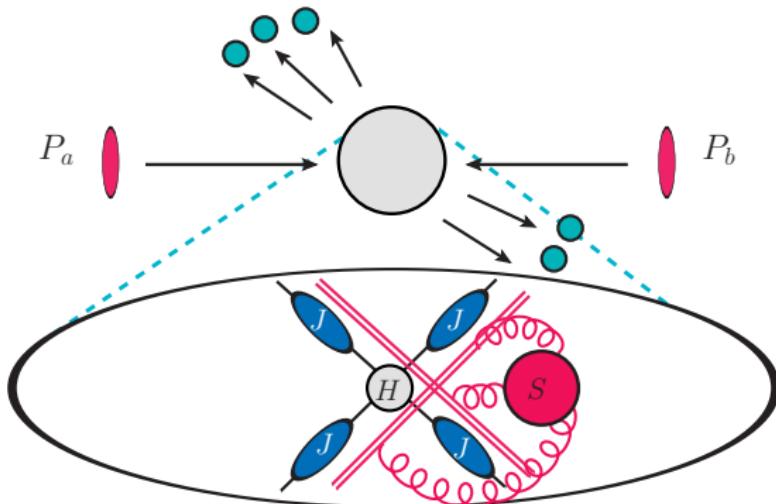
$$d\sigma \sim \sum_{a,b} \int_0^1 dx_a dx_b f_{a/A}(x_a) f_{b/B}(x_b) \hat{\sigma}_{ab} + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{Q}\right)$$



$$d\hat{\sigma}_{ab} = \underbrace{d\hat{\sigma}_{ab}^{(0)}}_{\text{LO}} + \alpha_s \underbrace{d\hat{\sigma}_{ab}^{(1)}}_{\text{NLO}} + \alpha_s^2 \underbrace{d\hat{\sigma}_{ab}^{(2)}}_{\text{NNLO}} + \alpha_s^3 \underbrace{d\hat{\sigma}_{ab}^{(3)}}_{\text{N}^3\text{LO}} + \dots$$

Predictions for the LHC: Singular limits

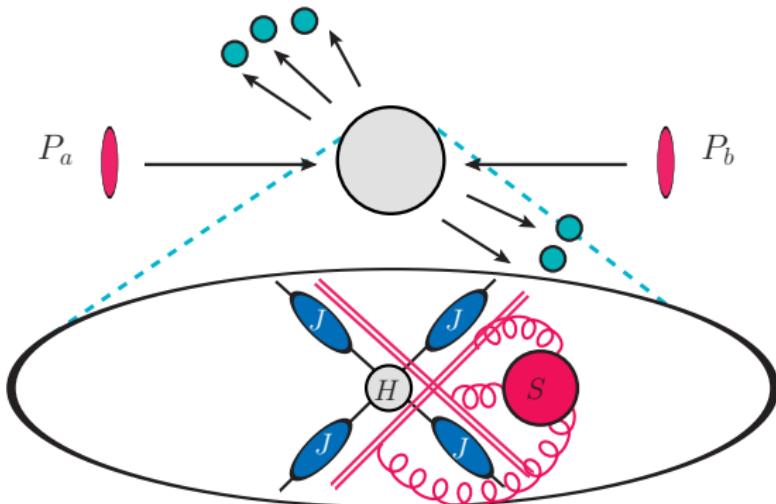
$$d\sigma \sim \sum_{a,b} \int_0^1 dx_a dx_b f_{a/A}(x_a) f_{b/B}(x_b) \left(\hat{\sigma}_{ab}^{\text{LP}} + \hat{\sigma}_{ab}^{\text{NLP}} + \dots \right) + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{Q}\right)$$



Singular limits of QCD give rise to large logarithms. Resummation of these logarithms is based on factorization of amplitudes in these limits. Expansion in Q_0/Q is necessary to resum large $\ln(Q_0/Q)$.

Predictions for the LHC: Leading power

$$d\sigma \sim \sum_{a,b} \int_0^1 dx_a dx_b f_{a/A}(x_a) f_{b/B}(x_b) \left(\hat{\sigma}_{ab}^{\text{LP}} + \hat{\sigma}_{ab}^{\text{NLP}} + \dots \right) + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{Q}\right)$$

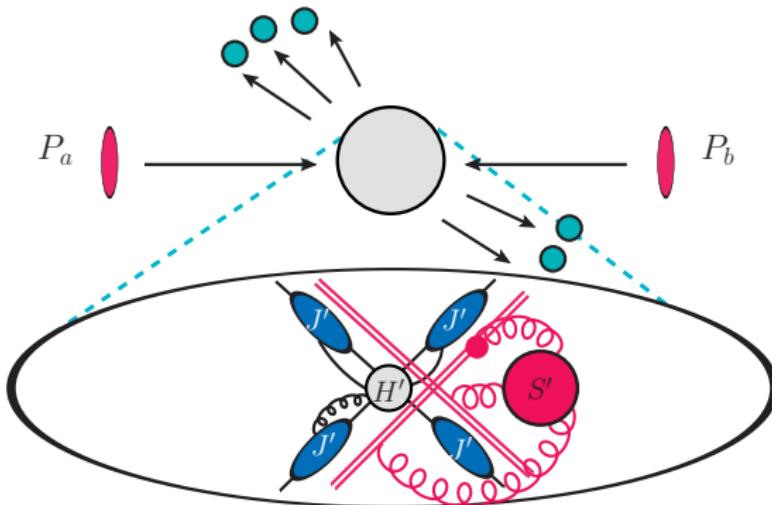


Power expansion in $\tau = (1-z), q_T^2, \dots$

$$\hat{\sigma}(\tau) = \sum_{n=0}^{\infty} \alpha_s^n \left[c_n \delta(\tau) + \sum_{m=0}^{2n-1} \left(c_{nm} \left[\frac{\ln^m \tau}{\tau} \right]_+ + d_{nm} \ln^m(\tau) \right) + \dots \right]$$

Predictions for the LHC: Power corrections

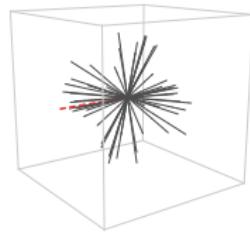
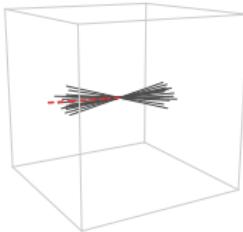
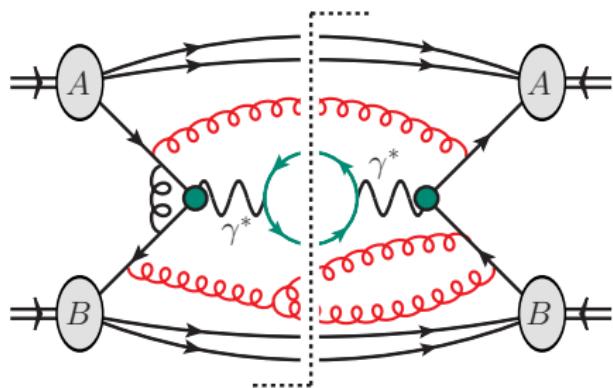
$$d\sigma \sim \sum_{a,b} \int_0^1 dx_a dx_b f_{a/A}(x_a) f_{b/B}(x_b) \left(\hat{\sigma}_{ab}^{\text{LP}} + \hat{\sigma}_{ab}^{\text{NLP}} + \dots \right) + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{Q}\right)$$



Power expansion in $\tau = (1 - z), q_T^2, \dots$

$$\hat{\sigma}(\tau) = \sum_{n=0}^{\infty} \alpha_s^n \left[c_n \delta(\tau) + \sum_{m=0}^{2n-1} \left(c_{nm} \left[\frac{\ln^m \tau}{\tau} \right]_+ + d_{nm} \ln^m(\tau) \right) + \dots \right]$$

Motivations and focus



[T. Becher, A. Broggio, A. Ferroglio, 1410.1892]

DY threshold $\tau = (1 - z)$

$$\frac{d\sigma}{dQ^2} = f_{a/A} \otimes f_{b/B} \otimes |C^{A0}|^2 \times S_{LP}$$

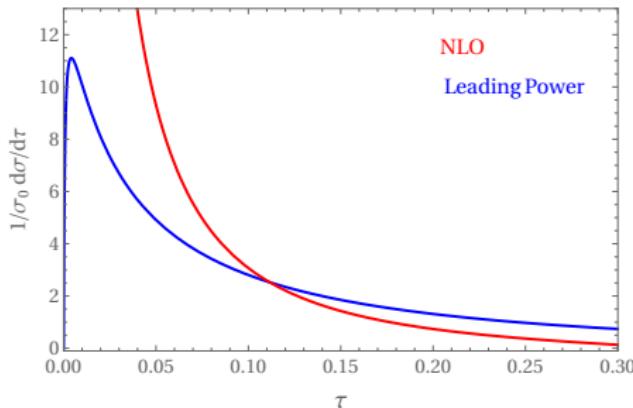
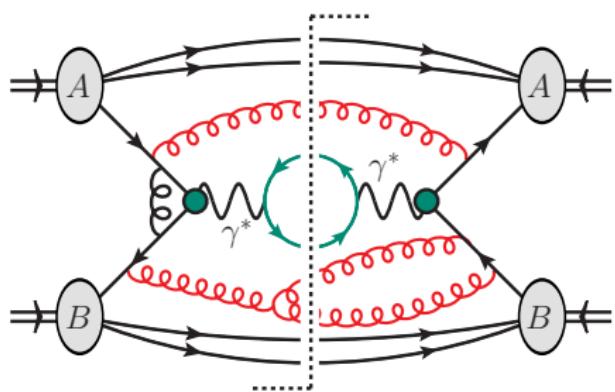
Event shapes, thrust $\tau = 1 - T$

$$\frac{d\sigma}{d\tau} = |C^{A0}|^2 \times J_c^{(q)} \otimes J_{\bar{c}}^{(\bar{q})} \otimes S_{LP}$$

LP logarithms, $\alpha_s^n \left[\frac{\ln^m \tau}{\tau} \right]_+$, $m \leq 2n - 1$ are well studied: LL, NLL, NNLL, N3LL

[T. Becher, M. Neubert, A. Xu, 0710.0680] [T. Becher, M. Schwartz, 0803.0342]

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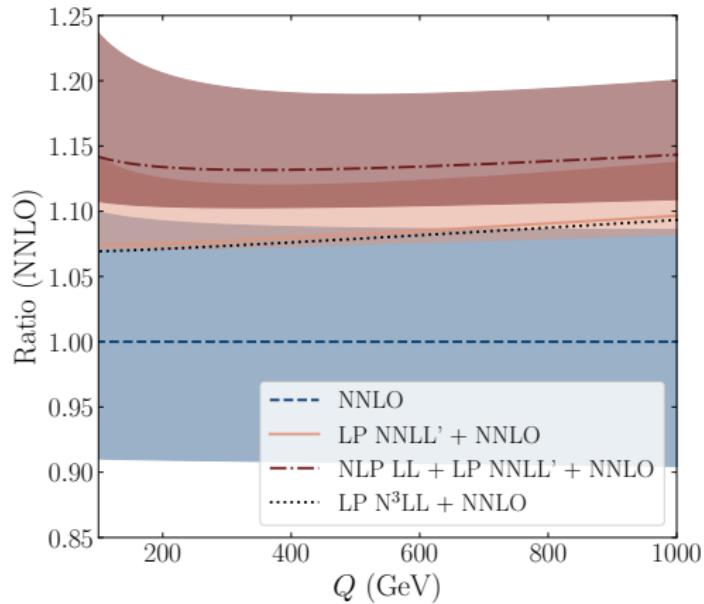
[T. Becher, M. Neubert, A. Xu, 0710.0680] [T. Becher, M. Schwartz, 0803.0342]

Why are power corrections important?

- ▶ Advancing all-order understanding of QFT structure
- ▶ Extending applicability of factorization
- ▶ Describing power-suppressed processes
- ▶ Improving matching and fixed order subtractions
- ▶ Getting a handle on theoretical errors
- ▶ With the high logarithmic resummation performed at leading power, it's natural to ask about leading logarithms at subleading power.
→ some phenomenological studies already exist

Phenomenological motivations

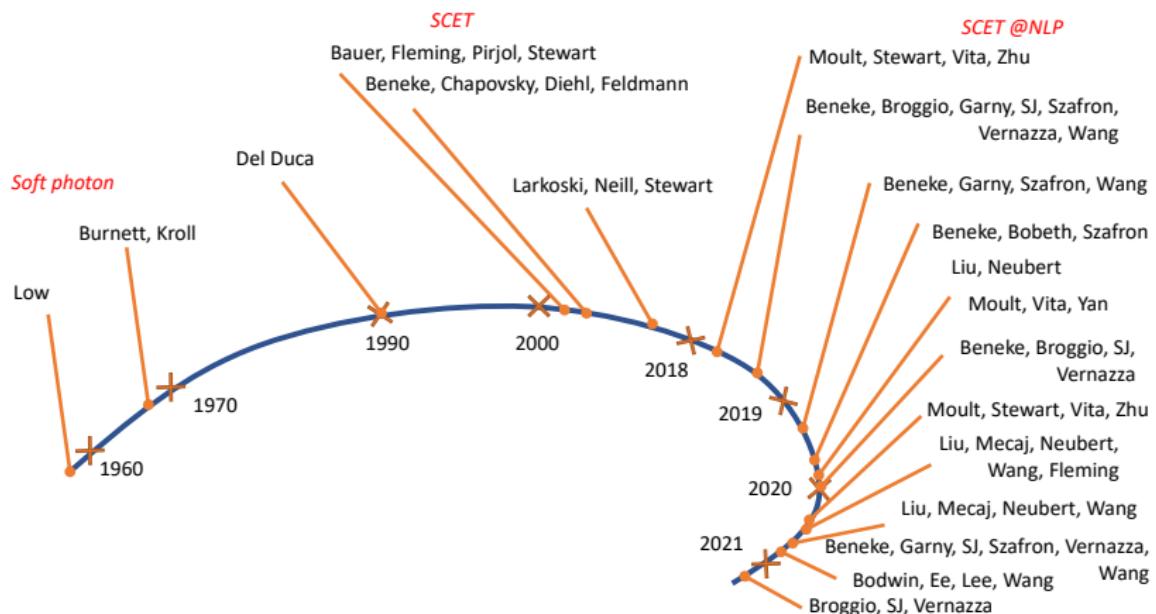
Numerical studies of inclusive Higgs and Drell-Yan production studied in
[M. van Beekveld, E. Laenen, J. Sinnenhe Damst , L. Vernazza, 2101.07270].



Studies of Leadling Power NNLL'
matched to NNLO for
DY and Higgs
concluded that **NLP LL has a comparable impact to LP N³LL.**

Towards NLP

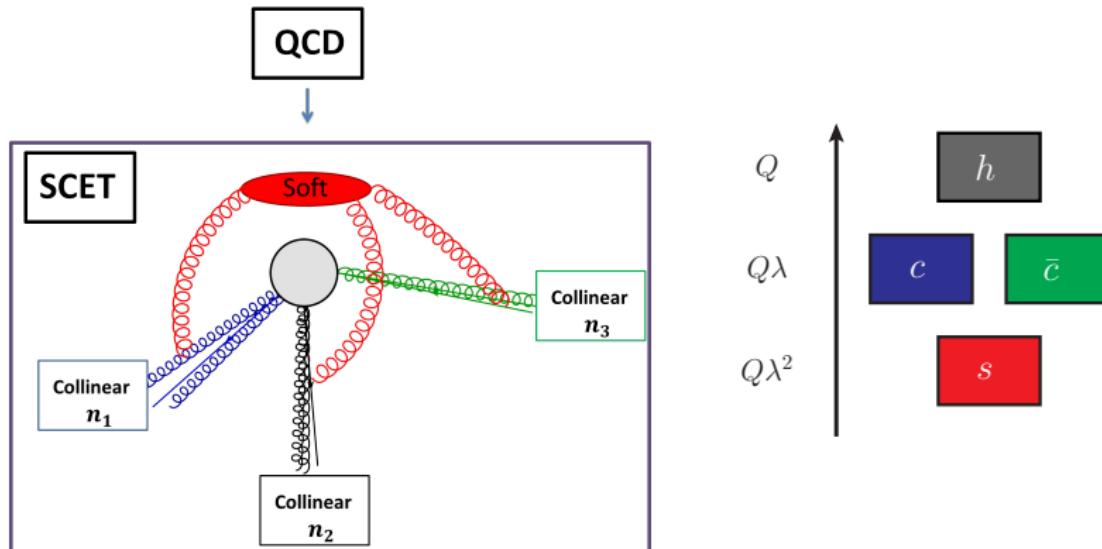
Tremendous progress in understanding of the NLP terms has been made in the last few years. See also talk by R. Balsach.



Diagrammatic approach: Bahjat-Abbas, Bonocore, Sinninghe Damst  , Laenen, Magnea, Melville, Vernazza, White; Anastasiou, Penin; Ajjath, Mukherjee, Ravindran, Sankar, Tiwari

SCET

Soft collinear effective theory is contained within QCD. It is an EFT which describes energetic particles.



- ▶ Process specific description, with collinear sectors formed by energetic particles.
- ▶ Interactions between sectors are mediated by the soft degrees of freedom.
- ▶ Every interaction is well defined in terms of power counting - this allows for systematic expansion.

Subleading power effects

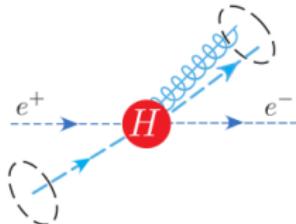
There are three sources of power suppressed corrections in SCET

- ▶ 1) Subleading power hard-scattering operators
 - ▶ More than one field present in each collinear direction
 - ▶ Soft fields present

$$\psi(x) \rightarrow \underbrace{\psi_1(x) + \dots + \psi_N(x)}_{N \text{ collinear fermion fields}} + q(x)$$

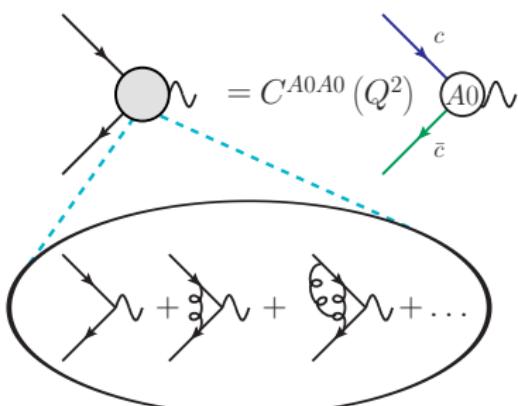
We keep the *collinear*, *anti-collinear*, and *soft* degrees of freedom. The *hard* modes are integrated out:

$$\bar{\psi} \gamma^\mu \psi = \int \prod_{i=1}^N \prod_{k=1}^{n_i} dt_{i_k} C(\{t_{i_k}\}) \prod_{i=1}^N J_i(t_{i_1}, \dots, t_{i_{n_i}})$$



(from I. Moult)

Sebastian Jaskiewicz



Power Corrections on the Lightcone

Subleading power effects

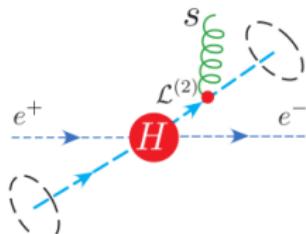
There are three sources of power suppressed corrections in SCET

- ▶ 1) Subleading power hard-scattering operators
 - ▶ More than one field present in each collinear direction
 - ▶ Soft fields present
- ▶ 2) Subleading soft-collinear interactions

$$\mathcal{L}_{\text{SCET}} = \sum_{i=1}^N \mathcal{L}_{c_i} + \mathcal{L}_{\text{soft}}$$

where each of the Lagrangians belonging to a collinear direction is expanded in powers of the small parameter $\lambda = \sqrt{1-z}$:

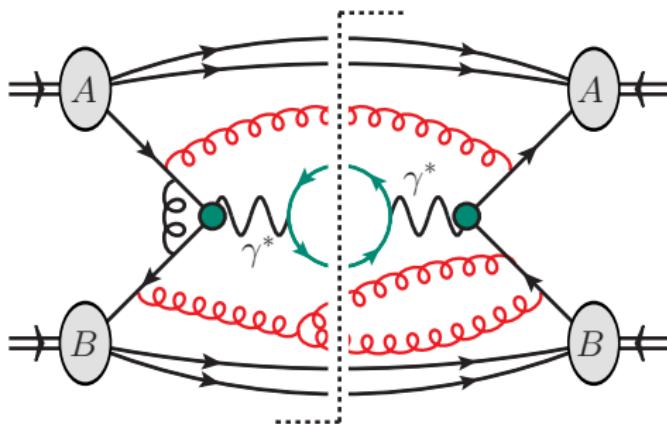
$$\mathcal{L}_{c_i} = \underbrace{\mathcal{L}_{c_i}^{(0)}}_{\text{LP}} + \underbrace{\mathcal{L}_{c_i}^{(1)}}_{\mathcal{O}(\lambda^1)} + \underbrace{\mathcal{L}_{c_i}^{(2)}}_{\mathcal{O}(\lambda^2)} + \dots \quad \text{E.g.} \quad \mathcal{L}_c^{(1)} = \bar{\chi}_c i x_\perp^\mu [in_- \partial \mathcal{B}_\mu^+] \frac{\not{n}_+}{2} \chi_c$$



Subleading power effects

There are three sources of power suppressed corrections in SCET

- ▶ 1) Subleading power hard-scattering operators
 - ▶ More than one field present in each collinear direction
 - ▶ Soft fields present
- ▶ 2) Subleading soft-collinear interactions
- ▶ 3) Kinematic corrections



The Foundations

The subleading power Lagrangians have been known for a long time [M. Beneke, A. Chapovsky, M. Diehl, T. Feldmann, hep-ph/0206152] [M. Beneke, T. Feldmann, hep-ph/0211358] and [D. Pirjol, I. Stewart, hep-ph/0211251]

Subleading operator basis for dijets [D. Kolodrubetz, I. Moult, I. Stewart, 1601.02607] [I. Feige, D. Kolodrubetz, I. Moult, I. Stewart, G. Vita, 1703.03411] [I. Moult, I. Stewart, G. Vita, 1703.03408]

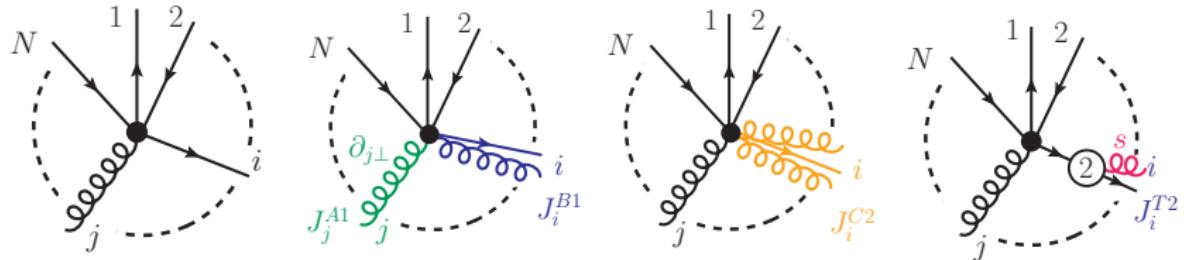
and for N-jets [M. Beneke, M. Garny, R. Szafron, J. Wang, 1712.04416, 1712.07462, 1808.04742, 1907.05463]

Building blocks (up to $\mathcal{O}(\lambda^2)$):

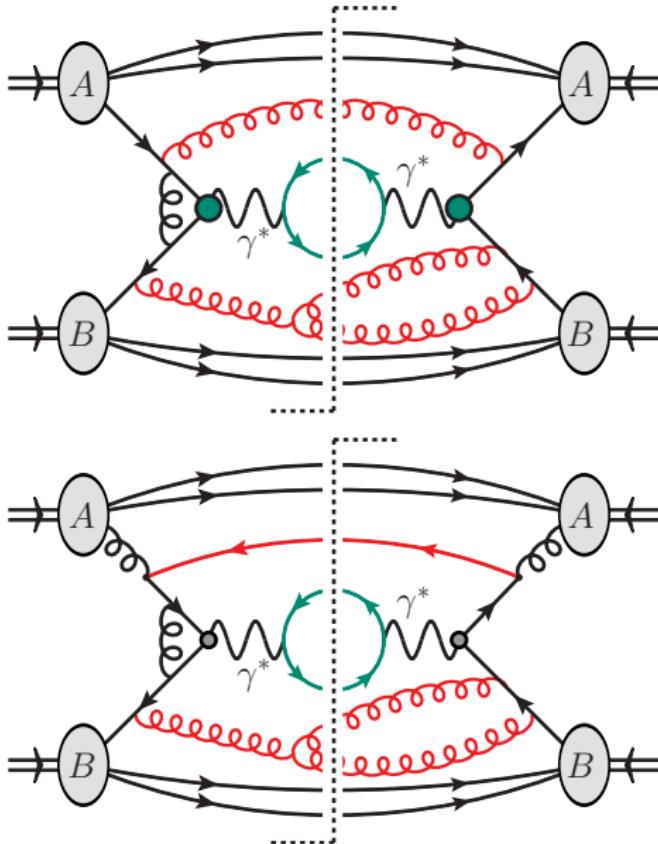
$$\chi_i(t_i n_{i+}) \equiv W_i^\dagger \xi_i$$

$$\mathcal{A}_{i\perp}^\mu(t_i n_{i+}) \equiv W_i^\dagger [i D_{\perp i}^\mu W_i]$$

Generic leading power N -jet operator:



Factorization of the Drell-Yan process

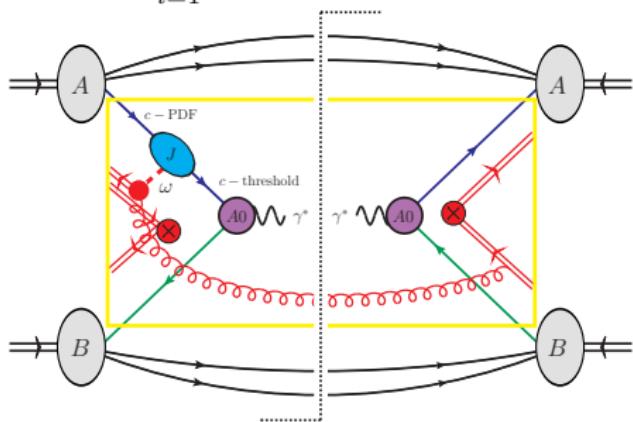


Factorisation formula at NLP

Defining $\Delta = \hat{\sigma}/z$, we arrive at a final result:

[M. Beneke, A. Broggio, SJ, L. Vernazza, 1912.01585]

$$\begin{aligned} \Delta_{\text{NLP}}^{\text{dyn}}(z) = & -2 Q \left[\left(\frac{\not{q}_-}{4} \right) \gamma_{\perp\rho} \left(\frac{\not{q}_+}{4} \right) \gamma_{\perp}^{\rho} \right]_{\beta\gamma} \int d(n+p) C^{A0}(n+p, x_b n - p_B) \\ & \times C^{*A0}(x_a n + p_A, x_b n - p_B) \sum_{i=1}^4 \int \{d\omega_j\} J_i(n+p, x_a n + p_A; \{\omega_j\}) S_i(\Omega; \{\omega_j\}) + \text{h.c.} \end{aligned}$$



For comparison, LP result is:

$$\Delta_{\text{LP}}(z) = |C(Q^2)|^2 Q S_{\text{DY}}(Q(1-z))$$

Factorisation formula at NLP

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where the *generalised soft* functions have the structure:

$$\tilde{S}_i(x; \{\omega_j\}) = \int \left\{ \frac{dz_{j-}}{2\pi} \right\} e^{-i\omega_j z_{j-}} \times \frac{1}{N_c} \text{Tr} \langle 0 | \bar{\mathbf{T}} \left([Y_+^\dagger Y_-](x) \right) \mathbf{T} \left([Y_-^\dagger Y_+](0) \mathfrak{s}_i(\{z_{j-}\}) \right) | 0 \rangle$$

with

$$\begin{aligned} \mathfrak{s}_i(\{z_{j-}\}) \in & \left\{ \frac{i\partial_\perp^\mu}{in_- \partial} \mathcal{B}_{\mu\perp}^+(z_{1-}), \frac{1}{(in_- \partial)^2} \left[\mathcal{B}^{+\mu\perp}(z_{1-}), \left[in_- \partial \mathcal{B}_{\mu\perp}^+(z_{1-}) \right] \right], \right. \\ & \left. \frac{1}{(in_- \partial)} \mathcal{B}_{\mu\perp}^+(z_{1-}) \mathcal{B}_{\nu\perp}^+(z_{2-}), \frac{1}{(in_- \partial)^2} q_{+\sigma}(z_{1-}) \bar{q}_{+\lambda}(z_{2-}) \right\} \end{aligned}$$

Results at NNLO

The S_1 soft function is the only one with C_F^2 contributions. Combine our results for two-loop soft with tree-level collinear according to factorisation theorem

$$\Delta_{\text{NLP-soft}, S_1, C_F^2}^{\text{dyn (2)2r0v}}(z) = 4 Q \Omega H^{(0)}(Q^2) \int dr J_{1,1}^{(0)}(x_a(n+p_A); \omega) S_{1,C_F^2}^{(2)2r0v}(\Omega, r)$$

$$\begin{aligned} \Delta_{\text{NLP-soft}, S_1, C_F^2}^{\text{dyn (2)2r0v}}(z) &= \frac{\alpha_s^2}{(4\pi)^2} C_F^2 \left(\frac{32}{\epsilon^3} - \frac{128}{\epsilon^2} \ln(1-z) + \frac{256}{\epsilon} \ln^2(1-z) - \frac{112\pi^2}{3\epsilon} \right. \\ &\quad \left. + \frac{32}{3} (-32 \ln^3(1-z) + 14\pi^2 \ln(1-z) - 62\zeta(3)) + \mathcal{O}(\epsilon) \right) \end{aligned}$$

The S_1 1r1v gives leading logarithms proportional to $C_F C_A$, which are cancelled exactly by the leading poles of 2r0v contribution to S_1

$$\begin{aligned} \Delta_{\text{NLP-soft}, S_1, C_F C_A}^{\text{dyn (2)2r0v}}(z) &= \frac{\alpha_s^2}{(4\pi)^2} C_F C_A \left(\frac{8}{\epsilon^3} - \frac{4}{3\epsilon^2} (24 \ln(1-z) - 11) \right. \\ &\quad - \frac{16}{9\epsilon} (-36 \ln^2(1-z) + 33 \ln(1-z) + 6\pi^2 - 16) \\ &\quad - \frac{256}{3} \ln^3(1-z) + \frac{352}{3} \ln^2(1-z) + \frac{128}{3} \pi^2 \ln(1-z) \\ &\quad \left. - \frac{1024}{9} \ln(1-z) - \frac{616\zeta(3)}{3} - \frac{154\pi^2}{9} + \frac{1484}{27} + \mathcal{O}(\epsilon) \right) \end{aligned}$$

The full NNLO cross-section can be compared with [R. Hamberg, W. van Neerven, T. Matsuura, 1991] and we find agreement.

Results at N³LO

Interestingly, we can partially validate our results at N³LO by comparing to
[\[N. Bahjat-Abbas, J. Sinninghe Damst , L. Vernazza, C. White, 1807.09246\]](#)

$$\Delta_{\text{NLP-coll}, C_F^3}^{dyn(3)}(z) = 4Q \int d\omega J_{1,1}^{(1)}(x_a n + p_A; \omega) S_{1,C_F^2}^{(2)}(\Omega; \omega)$$

Using one-loop collinear function from [\[M.Beneke, A.Broggio, SJ, L.Vernazza, 1912.01585\]](#)

$$S_{1,C_F^2}^{(2)2r0v}(\Omega, \omega) = 8 \frac{\alpha_s^2}{(4\pi)^2} C_F^2 \left(\frac{\omega(\Omega - \omega)^3}{\mu^4} \right)^{-\epsilon} \frac{1}{\omega} \frac{1}{\epsilon^2} \frac{e^{2\epsilon\gamma_E} \Gamma[1 - \epsilon]}{\Gamma[1 - 3\epsilon]} \theta(\Omega - \omega) \theta(\omega)$$

We find a closed d -dimensional result, expanding

$$\begin{aligned} \Delta_{\text{NLP-coll}, C_F^3}^{dyn(3)}(z) = & \frac{\alpha_s^3}{(4\pi)^3} C_F^3 \left(-\frac{64}{\epsilon^4} + \frac{80(4 \ln(1-z) - 1)}{\epsilon^3} + \frac{16}{\epsilon^2} \left(-50 \ln^2(1-z) \right. \right. \\ & + 25 \ln(1-z) + 7\pi^2 - 6 \Big) + \frac{1}{\epsilon} \left(\frac{4000}{3} \ln^3(1-z) - 1000 \ln^2(1-z) \right. \\ & - 560\pi^2 \ln(1-z) + 480 \ln(1-z) + 2624\zeta(3) + 140\pi^2 - 128 \Big) \\ & - \frac{5000}{3} \ln^4(1-z) + \frac{5000}{3} \ln^3(1-z) + 1400\pi^2 \ln^2(1-z) \\ & - 1200 \ln^2(1-z) - 700\pi^2 \ln(1-z) + 640 \ln(1-z) \\ & \left. \left. + \zeta(3)(3280 - 13120 \ln(1-z)) + \frac{62\pi^4}{5} + 168\pi^2 - 192 \right) \right) \end{aligned}$$

DY: LL at NLP

After derivation of factorization formulas, and solving the relevant RG equations, the final results for leading NLP logarithms are remarkably simple.

Drell-Yan at threshold:

$$\Delta_{\text{NLP}}^{\text{LL}}(z) = -\exp[4A(\mu_h, \mu) - 4A(\mu_s, \mu)] \times 4 \frac{\alpha_s C_F}{\pi} \ln \frac{\mu_s}{\mu} \theta(1-z)$$

[M. Beneke, A.Broggio, M. Garry, **SJ**, R. Szafron, L. Vernazza, J.Wang, 1809.10631]

$$\begin{aligned} \Delta_{\text{NLP}}^{\text{LL}}(z, \mu) &= -\theta(1-z) \left\{ 4C_F \frac{\alpha_s}{\pi} \left[\ln(1-z) - L_\mu \right] \right. \\ &\quad + 8C_F^2 \left(\frac{\alpha_s}{\pi} \right)^2 \left[\ln^3(1-z) - 3L_\mu \ln^2(1-z) + 2L_\mu^2 \ln(1-z) \right] \\ &\quad + 8C_F^3 \left(\frac{\alpha_s}{\pi} \right)^3 \left[\ln^5(1-z) - 5L_\mu \ln^4(1-z) + 8L_\mu^2 \ln^3(1-z) - 4L_\mu^3 \ln^2(1-z) \right] \\ &\quad + \frac{16}{3} C_F^4 \left(\frac{\alpha_s}{\pi} \right)^4 \left[\ln^7(1-z) - 7L_\mu \ln^6(1-z) + 18L_\mu^2 \ln^5(1-z) \right. \\ &\quad \left. \left. - 20L_\mu^3 \ln^4(1-z) + 8L_\mu^4 \ln^3(1-z) \right] \right. \end{aligned}$$

where we define $L_\mu = \ln(\mu/Q)$. Comparison to [R. Hamberg, W. van Neerven, T. Matsuura, 1991] and [D. de Florian, J. Mazzitelli, S. Moch, A. Vogt, 1408.6277]

Also confirmed using diagrammatic techniques [N. Bahjat-Abbas, D. Bonocore, J. Sinnenhe Damst  , E. Laenen, L. Magnea, L. Vernazza, C. White, 1905.13710]

Beyond leading logarithms: Problems at the Endpoint

$$\int_0^\Omega d\omega \underbrace{(n+p\omega)^{-\epsilon}}_{\text{collinear piece}} \underbrace{\frac{1}{\omega^{1+\epsilon}} \frac{1}{(\Omega-\omega)^\epsilon}}_{\text{soft piece}}$$

For resummation, we treat the two objects independently, and expand in ϵ prior to performing the final convolution. However, there is a problem! At two loops:

$$J_1^{(1)}(x_a n+p_A; \omega) \sim \alpha_s \log(\omega)$$

and

$$S_1(\Omega, \omega) \sim \alpha_s \delta(\omega) + \mathcal{O}(\alpha^2)$$

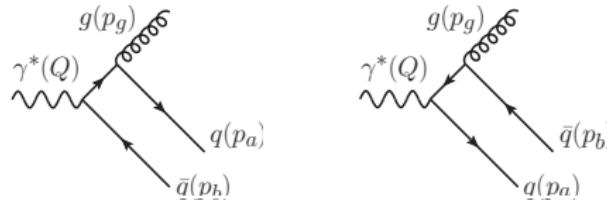
Hence, first expanding in ϵ and performing convolution after yields

$$\begin{aligned} \Delta_{\text{NLP-coll}}^{\text{dyn (2)}}(z) &= \frac{\alpha_s^2}{(4\pi)^2} \left(C_F^2 \left(-\frac{32}{\epsilon^2} - \frac{8}{\epsilon} \left[5 - 8 \ln(1-z) - 4 \int d\omega \delta(\omega) \ln \left(\frac{\omega}{Q} \right) \right] \right) \right. \\ &\quad \left. + C_A C_F \frac{40}{\epsilon} + \mathcal{O}(\epsilon^0) \right) \end{aligned}$$

A couple of issues arise. The convolution $d\omega$ integral is now divergent at the endpoint. This prohibits the application of standard RG methods.

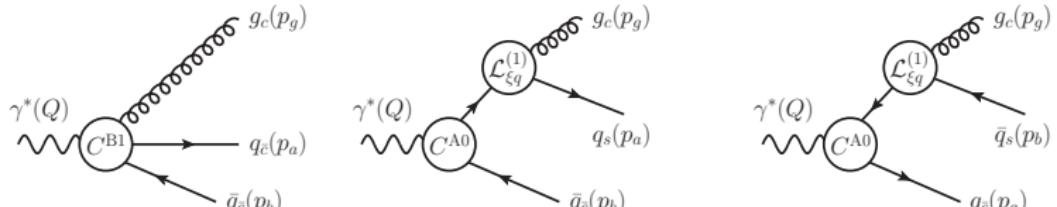
Thrust power corrections: off-diagonal channel

$$e^+ e^- \rightarrow \gamma^* \rightarrow [g]_c + [q\bar{q}]_{\bar{c}}$$



$$\begin{aligned} \bar{\psi} \gamma_\perp^\mu \psi(0) &= \int dt d\bar{t} \tilde{C}^{A0}(t, \bar{t}) \bar{\chi}_c(t n_+) \gamma_\perp^\mu \chi_{\bar{c}}(\bar{t} n_-) + (c \leftrightarrow \bar{c}) \\ &+ \sum_{i=1,2} \int dt d\bar{t}_1 d\bar{t}_2 \tilde{C}_i^{B1}(t, \bar{t}_1, \bar{t}_2) \bar{\chi}_{\bar{c}}(\bar{t}_1 n_-) \Gamma_i^{\mu\nu} \mathcal{A}_{c\perp\nu}(t n_+) \chi_{\bar{c}}(\bar{t}_2 n_-) + \dots \end{aligned}$$

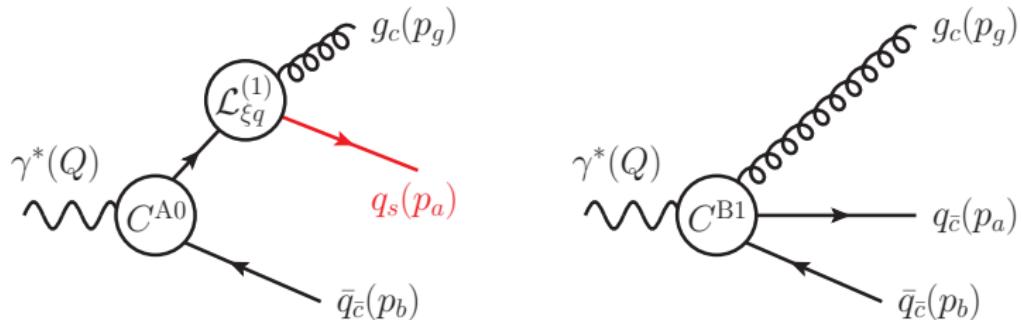
$$\mathcal{L}_{\xi q}(x) = \bar{q}_s(x_-) \mathcal{A}_{c\perp}(x) \chi_c(x) + \text{h.c.} \quad \Gamma_1^{\mu\nu} = \frac{\not{h}_-}{2} \gamma_\perp^\nu \gamma_\perp^\mu, \Gamma_2^{\mu\nu} = \frac{\not{h}_-}{2} \gamma_\perp^\mu \gamma_\perp^\nu$$



Integrand level observations

$$\frac{1}{\sigma_0} \frac{d\sigma}{dM_R dM_L} = \int d\omega d\omega' \left| C^{A0} \right|^2 \times \mathcal{J}_{\bar{c}}^{(\bar{q})} \times \mathcal{J}_c (\omega, \omega') \otimes S_{\text{NLP}} (\omega, \omega')$$

$$+ \int dr dr' C^{B1}(r) C^{B1}(r')^* \otimes \mathcal{J}_{\bar{c}}^{q\bar{q}} (r, r') \times \mathcal{J}_c^{(g)} \times S^{(g)}$$



$$\sim \int_{M_R^2/Q}^{\infty} d\omega \frac{1}{\omega^{1+\epsilon}} + \dots \quad \sim \int_0^1 dr \left[\frac{1}{r^{1+\epsilon}} + \frac{1}{(1-r)^{1+\epsilon}} \right]$$

We see that $S_{\text{NLP}} \rightarrow S^{(g)}$ and $\mathcal{J}_{\bar{c}}^{(\bar{q})} \rightarrow \mathcal{J}_{\bar{c}}^{q\bar{q}}$. The *integrands* of the two terms should become identical.

Factorization theorem rearrangement

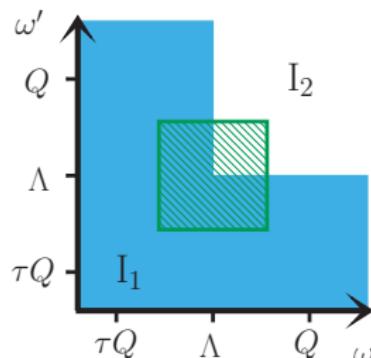
We add the following scaleless integral

$$0 = \frac{2C_F}{Q} f(\epsilon) |C^{A0}(Q^2)|^2 \tilde{\mathcal{J}}_{\bar{c}}^{(\bar{q})}(s_R) \tilde{\mathcal{J}}_c^{(g)}(s_L) \\ \times \int_0^\infty d\omega d\omega' \frac{D^{B1}(\omega Q)}{\omega} \frac{D^{B1*}(\omega' Q)}{\omega'} \left[\tilde{S}_{NLP}(s_R, s_L, \omega, \omega') \right]$$

where D^{B1} are universal objects describing the splitting of a collinear quark into a collinear gluon and a soft-collinear quark. The same object appears in the endpoint factorization theorem for $H \rightarrow gg$ [Z. Liu, M. Neubert, M. Schnubel, X. Wang, 2112.00018]

We split it by introducing a parameter Λ : $0 = I_1 + I_2$.

Then subtract I_1 from the B-type term and I_2 from the A-type term.



Resummed result for NLP-LL gluon thrust

Putting everything together [M. Beneke, M. Garny, **SJ**, J. Strohm, R. Szafron, L. Vernazza, J. Wang, 2205.04479]

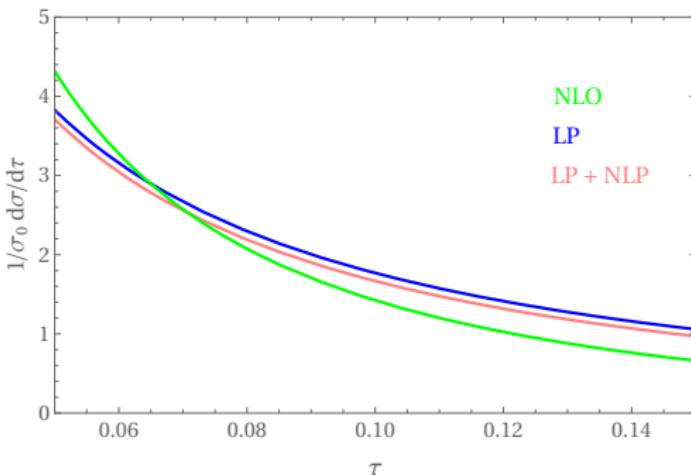
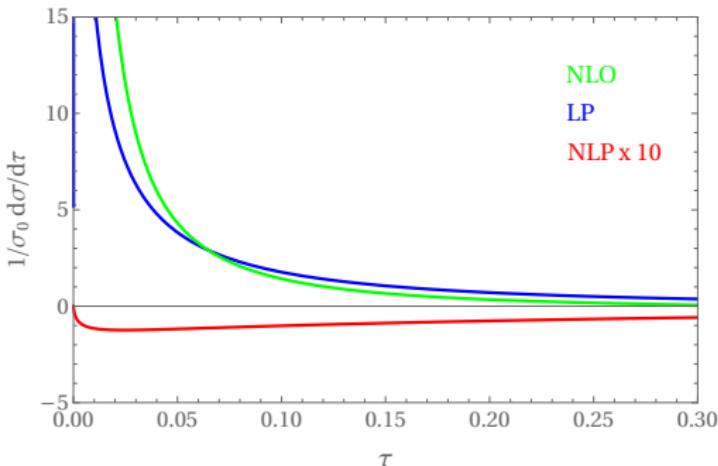
$$\begin{aligned} \frac{1}{\sigma_0} \frac{\widetilde{d\sigma}}{ds_R ds_L}|_{\text{LL}} &= 2 \cdot \frac{2C_F}{Q s_R} \frac{\alpha_s(\mu_c)}{4\pi} \exp[4C_F S(\mu_h, \mu_{\bar{c}}) + 4C_A S(\mu_s, \mu_c)] \\ &\times \left(\frac{Q^2}{\mu_h^2} \right)^{-2C_F A(\mu_h, \mu_{\bar{c}})} \left(\frac{1}{s_L s_R e^{2\gamma_E} \mu_s^2} \right)^{-2C_A A(\mu_s, \mu_c)} \\ &\times \int_\sigma^Q \frac{d\omega}{\omega} \exp[4(C_F - C_A) S(\mu_{s\Lambda}, \mu_{h\Lambda})] \left(\frac{\omega}{s_R e^{\gamma_E} \mu_{s\Lambda}^2} \right)^{-2(C_F - C_A) A(\mu_{s\Lambda}, \mu_{h\Lambda})} \\ &\times (s_R e^{\gamma_E} Q)^{2C_F A(\mu_{h\Lambda}, \mu_{\bar{c}}) + 2C_A A(\mu_c, \mu_{h\Lambda})} \end{aligned}$$

And in the double logarithmic limit, the result of previous works is recovered

$$\frac{1}{\sigma_0} \frac{d\sigma}{d\tau}|_{\text{DL}} = \frac{C_F}{C_F - C_A} \frac{1}{\ln(1/\tau)} e^{-\frac{\alpha_s C_A}{\pi} \ln^2 \tau} \left\{ 1 - e^{-\frac{\alpha_s}{\pi} (C_F - C_A) \ln^2 \tau} \right\}$$

[I. Moult, I. Stewart, G. Vita, H. X. Zhu, 1910.14038] [M. Beneke, M. Garny, **SJ**, R. Szafron, L. Vernazza, J. Wang, 2008.04943]

Numerics



Summary

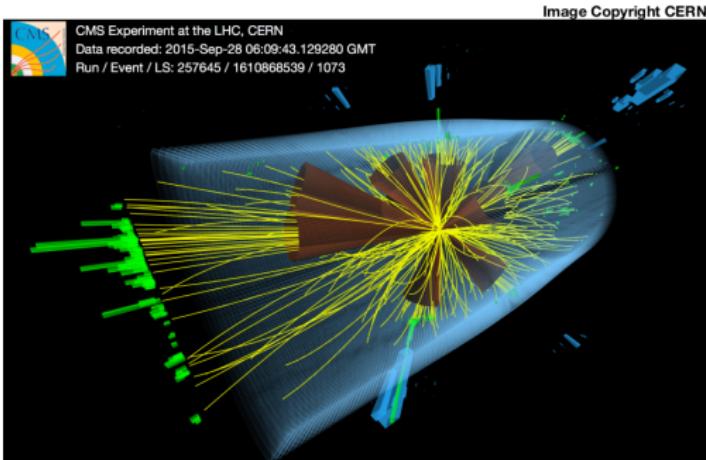
- ▶ SCET is designed to systematically include power corrections.
- ▶ There has been significant progress in understanding subleading power factorization theorems in the last years.
- ▶ The structure of the factorisation formula is more involved than the leading power counter part. New functions appear!
- ▶ We have verified the factorisation theorem at fixed orders by explicitly computing all the objects.
- ▶ We have uncovered issues preventing application of standard RG methods.
- ▶ Succeeded in refactorizing and resumming to LL accuracy the off-diagonal “gluon” thrust.
- ▶ Interesting conceptual challenges ahead. Important to understand from the point of view of gauge theories, as well as for delivering precise theoretical predictions.

Thank you!

Auxiliary slides

And now for something completely different

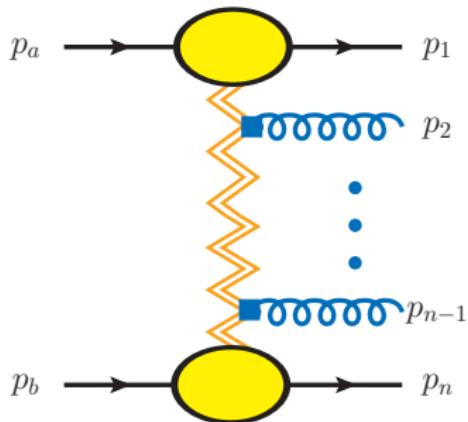
- ▶ Large high-energy logarithms are important to understand at the LHC: *High Energy Jets (HEJ)* framework
- ▶ Transverse momentum hierarchies, collinear splittings, MPI, and hadronization described by parton showers: *Pythia*.
- ▶ This talk: Merging High-Energy resummation of HEJ with Pythia parton shower → HEJ+Pythia: showered events from HEJ input. *In collaboration with Jeppe Andersen, Hitham Hassan, and Leif Lönnblad*



High Energy Jets (HEJ) framework

High Energy Jets resums large logarithms in s/t according to the BFKL equation.

[J. Andersen, J. Smillie, 0908.2786] [J. Andersen, J. Smillie, 0910.5113] [J. Andersen, J. Smillie, 1101.5394] [J. Andersen, J. Black, H. Brooks, B. Ducloué, M. Heil, A. Maier, J. Smillie, 2110.15692] [J. Andersen, B. Ducloué, H. Hassan, C. Elrick, A. Maier, G. Nail, J. Paltrinieri, A. Papaefstathiou, J. Smillie, 2303.15778].



High Energy limit for $2 \rightarrow n$ scattering

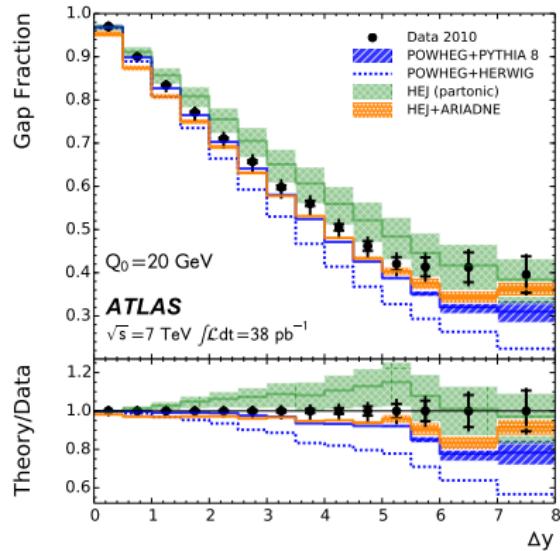
$$\hat{s} \gg \hat{s}_{ij} \gg k_\perp \quad \hat{s}_{ij} = (p_i + p_j)^2$$

$$y_1 \ll y_2 \ll \dots \ll y_{n-1} \ll y_n \quad p_{i\perp} \approx k_\perp$$

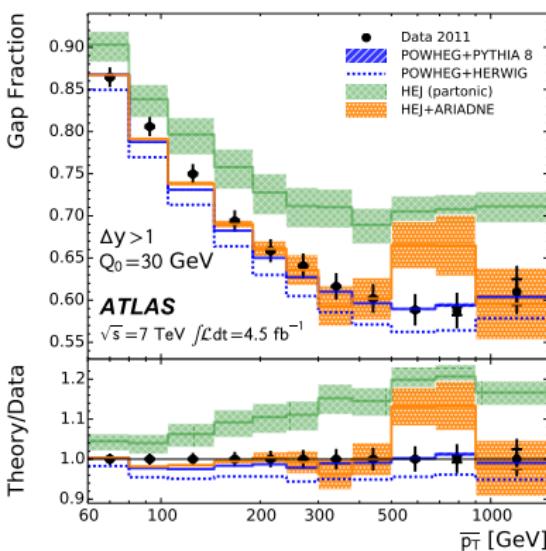
HEJ is built to describe effects from hard, wide angle emissions. Large $\ln(s/t)$ logs.

Beyond the High-Energy limit

- ▶ HEJ describes high-energy logarithms well
- ▶ Transverse momentum hierarchies are not captured → Parton Showers



[ATLAS collaboration, 1407.5756]

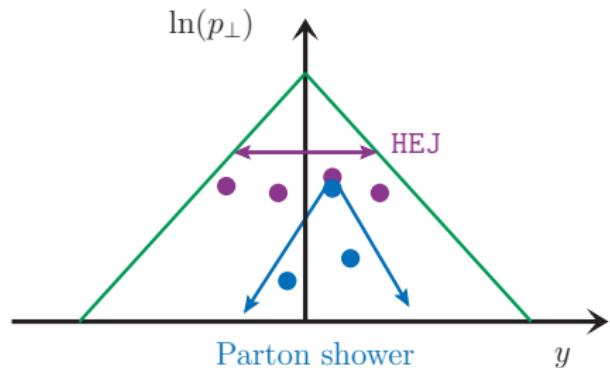


Parton Showers

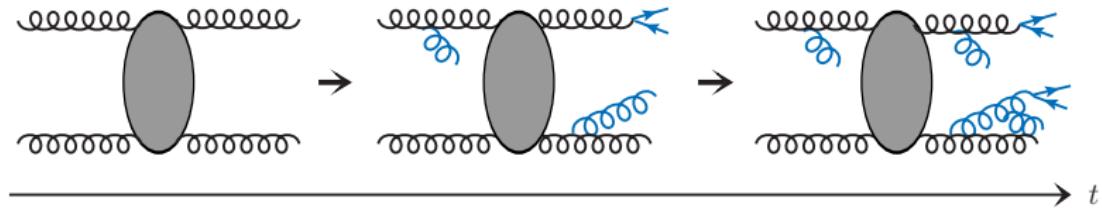
Soft and collinear splittings described by parton showers.

Our method uses **Pythia**.

[T. Sjöstrand, S. Mrenna, P. Skands, hep-ph/0603175][T. Sjöstrand, S. Ask, J. Christiansen, R. Corke, N. Desai, P. Ilten, S. Mrenna, S. Prestel, C. Rasmussen, P. Skands, 1410.3012]



$$\Delta_{ab}(t_n, t_m) = \exp \left[- \int_{t_m}^{t_n} \frac{dt}{t} \left(\frac{\alpha_s(t)}{2\pi} \int dz P_{ab}(z) \right) \right]$$



HEJ+Pythia: Merging procedure

We express HEJ in the parton shower language and feed in HEJ events.

The **HEJ+Pythia** method constructs **histories** for resummable HEJ events. → this is to ensure that the full phase space of HEJ is explored by the shower.

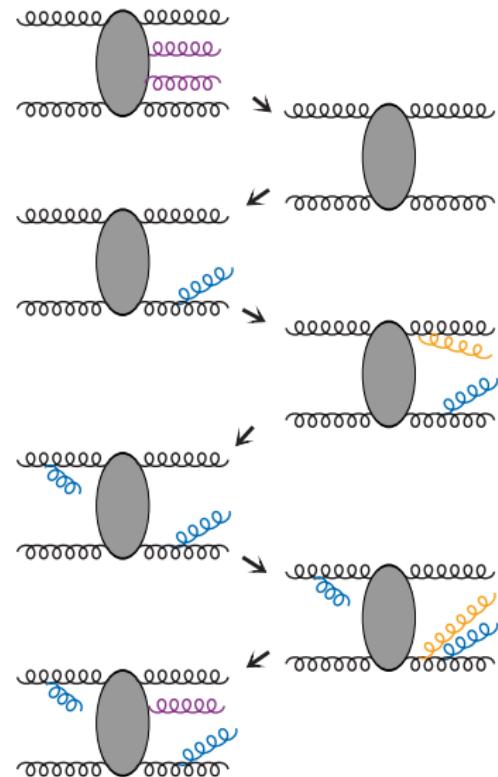
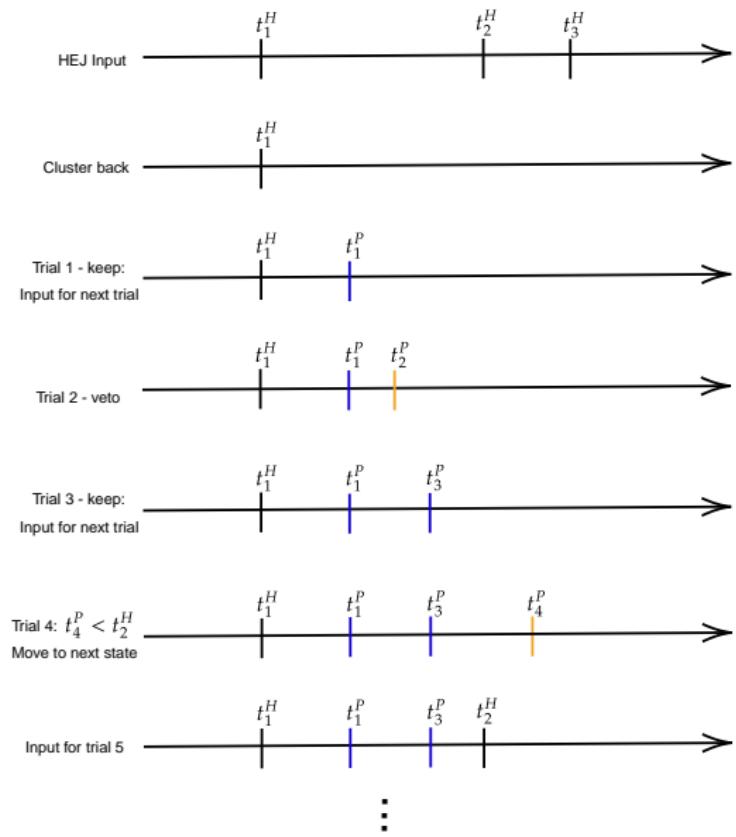
A *history* is a sequence of states $\mathcal{H} = \{S_0, \dots, S_l\}$, which charts the evolution in a parton shower from the **most clustered** state S_0 to the input event S_l .

Then, we let parton shower add **subtracted shower emissions** in between the states of the history, and **continue the Pythia shower** with a subtracted splitting kernel until hadronisation

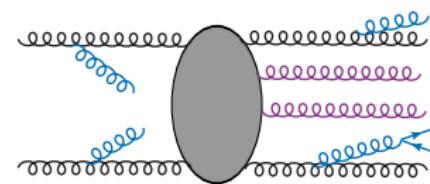
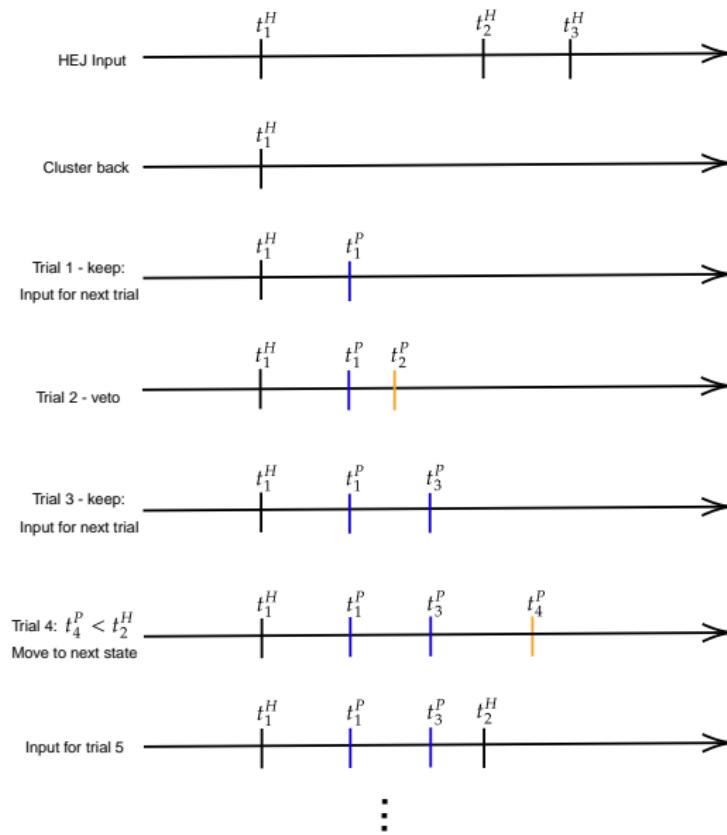
$$d\sigma_{m,n}^{\text{HEJ+Pythia}} = d\sigma_2^* \prod_{i=1}^{m-2} P_i^H \Delta_{i-1,i}^H \left[\prod_{\lambda=1}^{\lambda_i} P_{i_\lambda}^S \Delta_{i_{\lambda-1}, i_\lambda}^S \right] \prod_{j=m-2+\mathcal{N}}^n P_j^S \Delta_{j-1,j}^S$$

$$\Delta^S(t_{i-1}, t_i) = \exp \left\{ - \int_{t_i}^{t_{i-1}} dt \underbrace{\int dz \Theta(P^P(t, z) - P^H(t, z)) \left[P^P(t, z) - P^H(t, z) \right]}_{\text{Subtracted splitting probability: } P^S(t, z)} \right\}$$

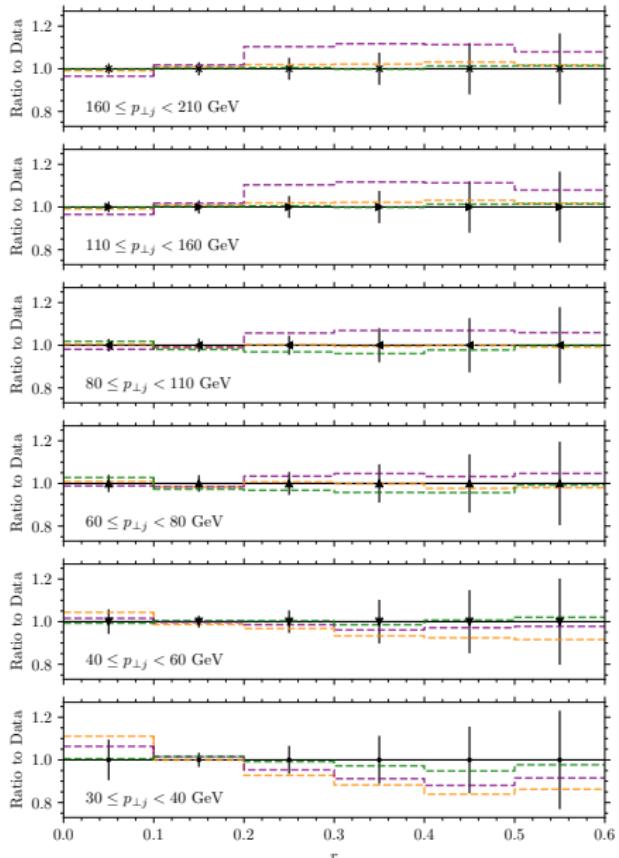
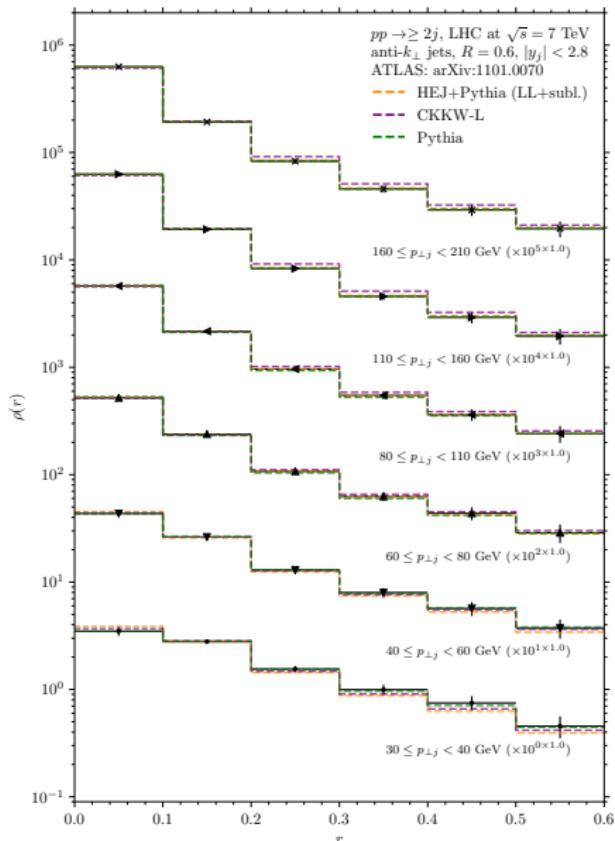
HEJ+PYTHIA: Merging



HEJ+PYTHIA: Merging



Results: Jet Shapes



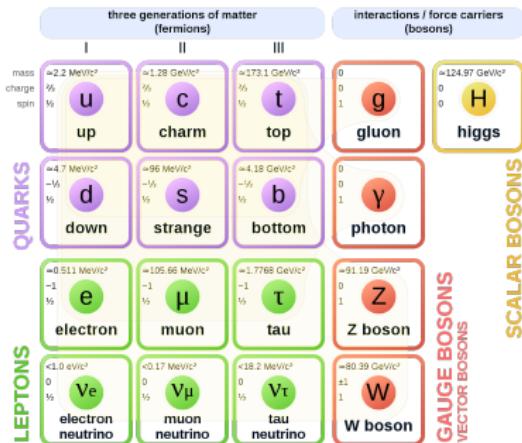
Introduction

Developing our understanding and striving for precise predictions

Standard Model

However, we know it is not the complete picture yet

Standard Model of Elementary Particles

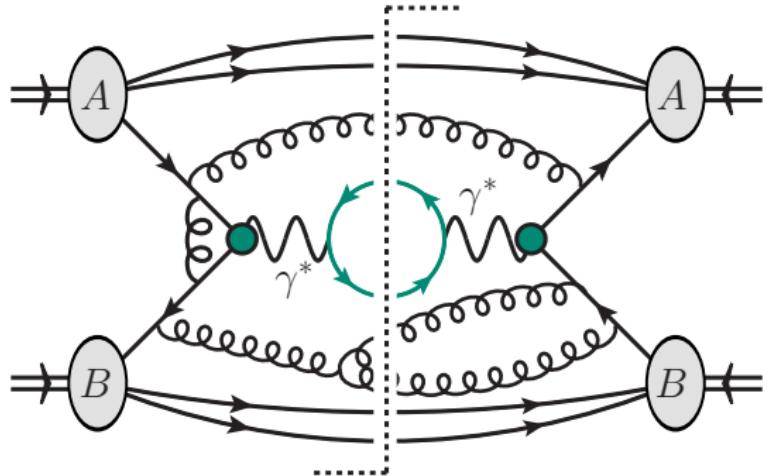


Motivations and focus: High perturbative precision

$$A(p_A) + B(p_B) \rightarrow \gamma^*(Q^2)[\rightarrow \ell(l_1)\bar{\ell}(l_2)] + X(p_X)$$

Recently calculated to N³LO

[C. Duhr, F. Dulat,
B. Mistlberger, 2001.07717]



Motivations and focus: Beyond fixed-order

$$A(p_A) + B(p_B) \rightarrow \gamma^*(Q^2)[\rightarrow \ell(l_1)\bar{\ell}(l_2)] + X(p_X)$$

Recently calculated to N³LO

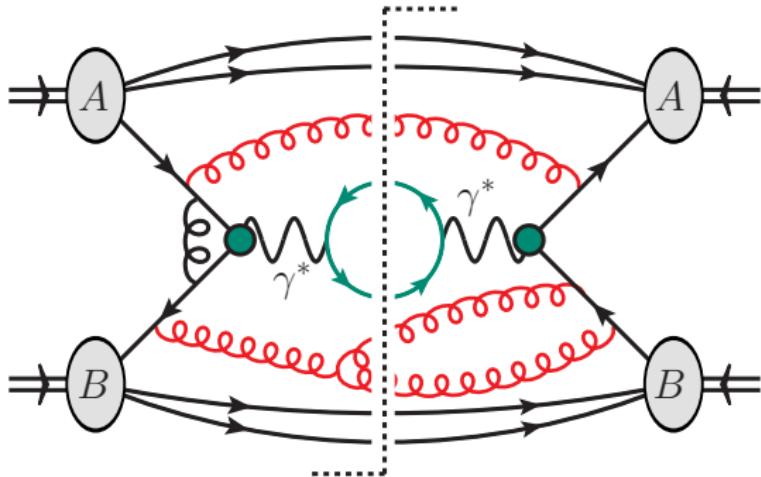
[C. Duhr, F. Dulat,
B. Mistlberger, 2001.07717]

Threshold limit:

$$z = \frac{Q^2}{s} \rightarrow 1$$

Define power counting parameter λ :

$$\lambda = \sqrt{1 - z}$$



Schematic form for production cross-sections near threshold, $z \rightarrow 1$:

$$\hat{\sigma}(z) = \sum_{n=0}^{\infty} \alpha_s^n \left[c_n \delta(1-z) + \sum_{m=0}^{2n-1} \left(c_{nm} \left[\frac{\ln^m(1-z)}{1-z} \right]_+ + d_{nm} \ln^m(1-z) \right) + \dots \right]$$

Motivations and focus: The Drell-Yan process

$$A(p_A) + B(p_B) \rightarrow \gamma^*(Q^2)[\rightarrow \ell(l_1)\bar{\ell}(l_2)] + X(p_X)$$

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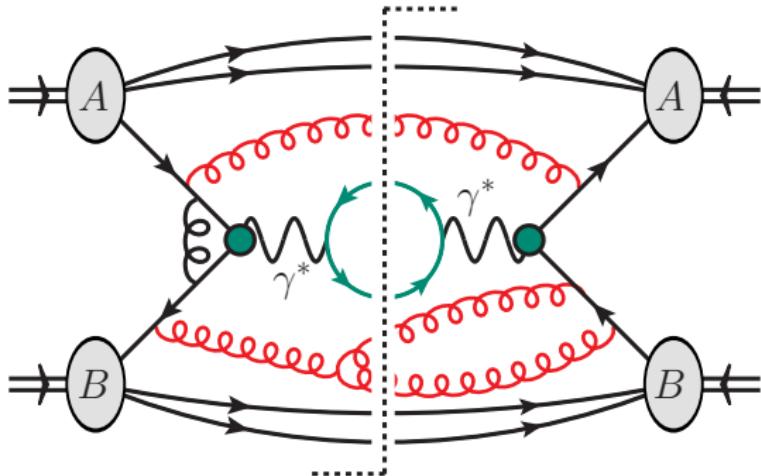
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Threshold limit:

$$z = \frac{Q^2}{\hat{s}} \rightarrow 1$$

Define power counting
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The general idea is to factorise the physics appearing at different scales:

$$\sigma \sim H \otimes J \otimes \dots \otimes J \otimes S$$

and solve RG equations for each object to sum the large logarithms → SCET

Motivations and focus: The Drell-Yan process

$$A(p_A) + B(p_B) \rightarrow \gamma^*(Q^2)[\rightarrow \ell(l_1)\bar{\ell}(l_2)] + X(p_X)$$

Recently calculated to N³LO

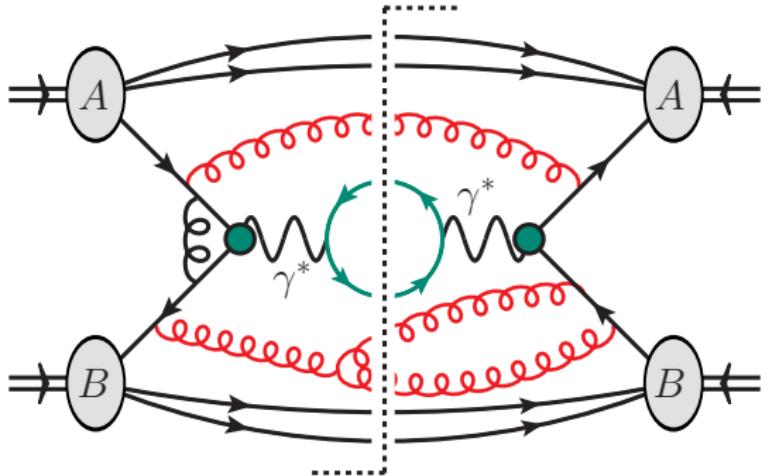
[C. Duhr, F. Dulat,
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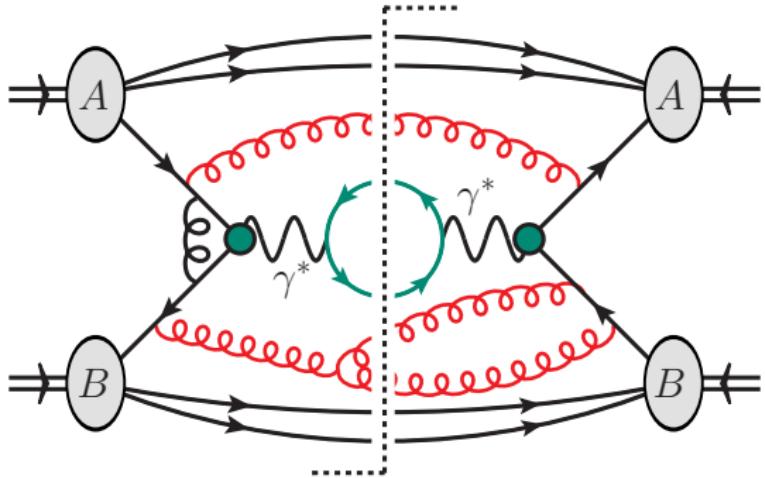
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SCET introduction: N -jet operator basis

Generic N -jet operator has the form:

[M. Beneke, M. Garny, R. Szafron, J. Wang, 1712.04416, 1712.07462, 1808.04742, 1907.05463]

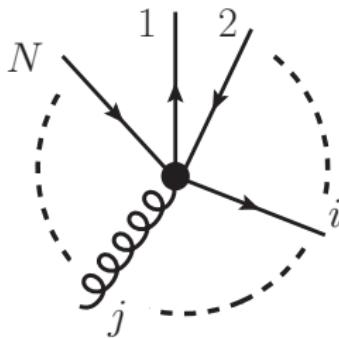
$$J = \int \prod_{i=1}^N \prod_{k=1}^{n_i} dt_{i_k} C(\{t_{i_k}\}) \prod_{i=1}^N J_i(t_{i_1}, t_{i_2}, \dots, t_{i_{n_i}})$$

where the J s are constructed by multiplying **collinear gauge invariant** building blocks in the same direction (up to $\mathcal{O}(\lambda^2)$)

$$\chi_i(t_i n_{i+}) \equiv W_i^\dagger \xi_i \quad \mathcal{A}_{i\perp}^\mu(t_i n_{i+}) \equiv W_i^\dagger [i D_{\perp i}^\mu W_i]$$

by acting on these with derivatives $i \partial_{\perp i}^\mu \sim \lambda$, and insertions of subleading SCET Lagrangian in a time-ordered product with lower power current.

Generic leading power N -jet operator:



SCET introduction: N -jet operator basis

Generic N-jet operator has the form:

[M. Beneke, M. Garny, R. Szafron, J. Wang, 1712.04416, 1712.07462, 1808.04742, 1907.05463]

$$J = \int \prod_{i=1}^N \prod_{k=1}^{n_i} dt_{i_k} C(\{t_{i_k}\}) \prod_{i=1}^N J_i(t_{i_1}, t_{i_2}, \dots, t_{i_{n_i}})$$

We adopt the notation: $J_i^{A\textcolor{teal}{n}}$, $J_i^{B\textcolor{blue}{n}}$, $J_i^{C\textcolor{orange}{n}}$, $J_i^{T\textcolor{violet}{n}}$ where:

- ▶ $A, B, C \dots$ refers to number of fields in a given collinear direction
- ▶ n is the power of λ suppression (relative to $A0$) in a given sector.

For example, up to $\mathcal{O}(\lambda^2)$ we can construct J_i^{A1} , J_i^{B1} , J_i^{C2} , J_i^{T2} respectively:

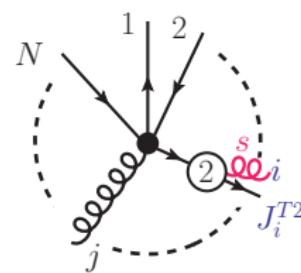
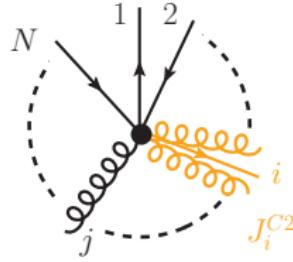
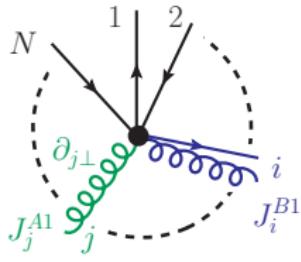
$$i\partial_{\perp i}^\mu \chi_i, \quad \chi_i(t_{i1}) \mathcal{A}_{i\perp}^\mu(t_{i2}), \quad \chi_i(t_{i1}) \mathcal{A}_{i\perp}^\nu(t_{i2}) \mathcal{A}_{i\perp}^\mu(t_{i3}), \quad i \int d^4z \mathbf{T}[\chi_i(t_{i1}) \mathcal{L}^{(2)}(z)]$$

And $\mathcal{O}(\lambda^2)$ N -jet operators are

$$J_1^{A0} \dots \textcolor{teal}{J}_i^{B1} \textcolor{teal}{J}_j^{A1} \dots J_N^{A0},$$

$$J_1^{A0} \dots \textcolor{orange}{J}_i^{C2} \dots J_N^{A0},$$

$$J_1^{A0} \dots \textcolor{violet}{J}_i^{T2} \dots J_N^{A0}, \dots$$



SCET introduction: Decoupling

In a collinear sector

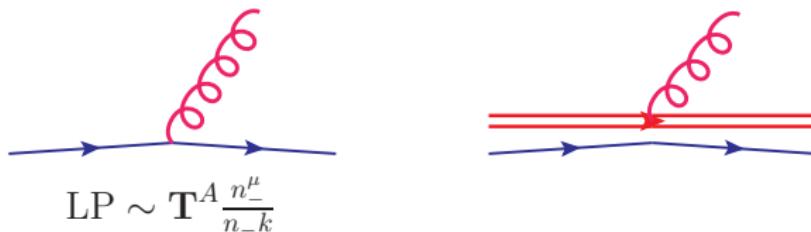
$$\mathcal{L}_{c_i} = \underbrace{\mathcal{L}_{c_i}^{(0)}}_{\text{LP}} + \underbrace{\mathcal{L}_{c_i}^{(1)}}_{\mathcal{O}(\lambda^1)} + \underbrace{\mathcal{L}_{c_i}^{(2)}}_{\mathcal{O}(\lambda^2)} + \dots$$

The decoupling transformation, $\xi_c \rightarrow Y_+ \xi_c^{(0)}$ and $A_c^\mu \rightarrow Y_+ A_c^{(0)\mu} Y_+^\dagger$, separates the soft and collinear sectors at LP
 [C. Bauer, D. Pirjol, I. Stewart, hep/0109045]

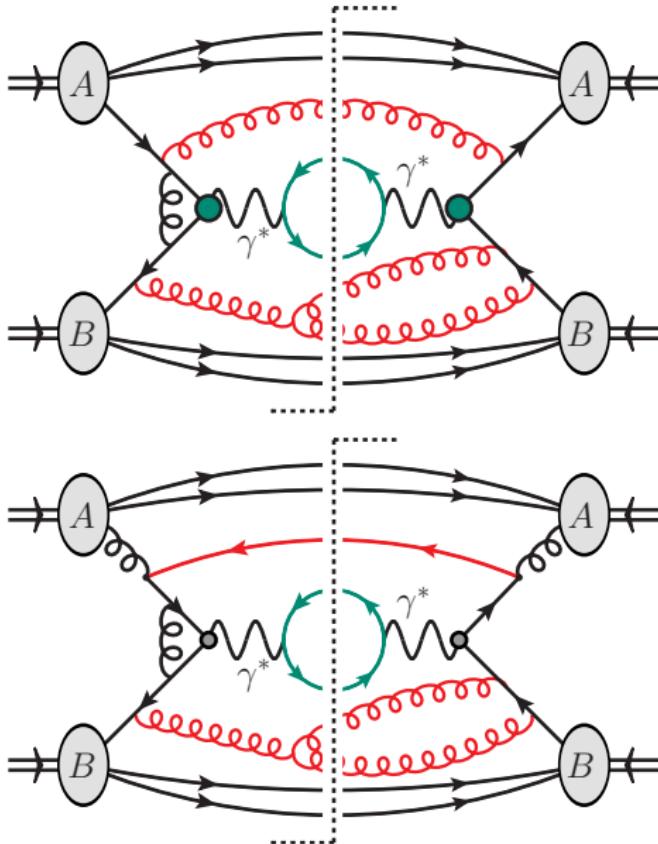
$$\mathcal{L}_c^{(0)} \ni \bar{\xi}_c (in_- D_c + g_s n_- A_s) \frac{\not{q}_+}{2} \xi_c = \bar{\xi}_c^{(0)} in_- D_c^{(0)} \frac{\not{q}_+}{2} \xi_c^{(0)}$$

where

$$Y_\pm(x) = \mathbf{P} \exp \left[ig_s \int_{-\infty}^0 ds n_\mp A_s(x + sn_\mp) \right]$$



Factorization of the Drell-Yan process



Drell-Yan Process at Threshold: Leading Power

The Drell-Yan Process

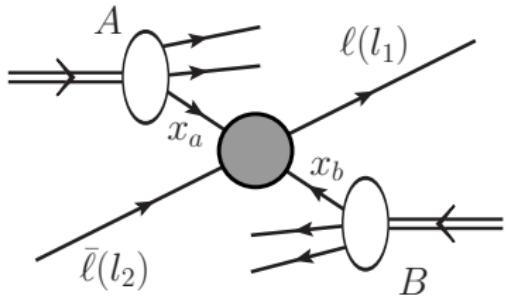
$$A(p_A) + B(p_B) \rightarrow \gamma^*(Q^2)[\rightarrow \ell(l_1)\bar{\ell}(l_2)] + X(p_X)$$

$$z = \frac{Q^2}{\hat{s}} \rightarrow 1 \quad \lambda = \sqrt{(1-z)}$$

$$p_c = (n_+ p_c, n_- p_c, p_{c\perp}) \sim Q(1, \lambda^2, \lambda)$$

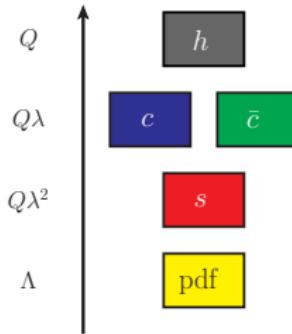
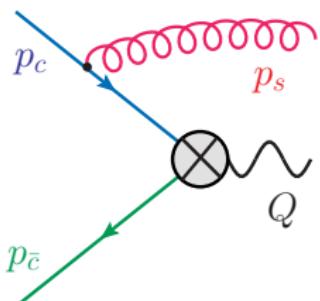
$$p_{\bar{c}} = (n_+ p_{\bar{c}}, n_- p_{\bar{c}}, p_{\bar{c}\perp}) \sim Q(\lambda^2, 1, \lambda)$$

$$\textcolor{red}{p_s} = (n_+ p_s, n_- p_s, p_{s\perp}) \sim Q(\lambda^2, \lambda^2, \lambda^2)$$



$$Q^2 \lambda^2 = Q^2(1-z) \gg \Lambda_{\text{QCD}}^2$$

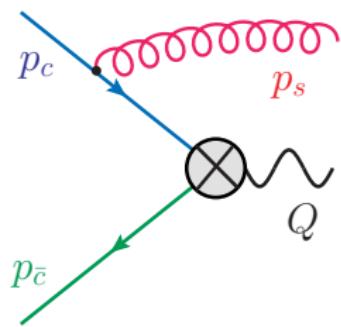
$$p_{c-\text{PDF}} \sim (Q, \Lambda^2/Q, \Lambda)$$



The Drell-Yan Process: Leading Power Amplitude

$$\bar{\psi} \gamma_\mu \psi = \int dt d\bar{t} \tilde{C}^{A0}(t, \bar{t}) J_\mu^{A0}(t, \bar{t})$$

$$J_\mu^{A0}(t, \bar{t}) = \bar{\chi}_{\bar{c}}(\bar{t} n_-) \gamma_{\perp \mu} \chi_c(t n_+)$$



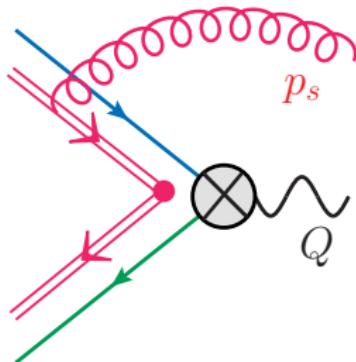
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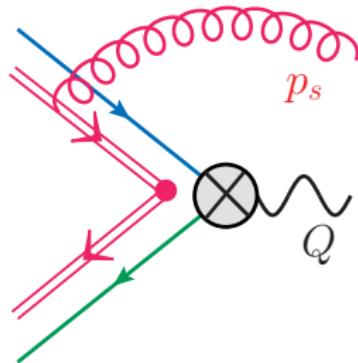
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Consider the matrix element:

$$\langle X | \bar{\psi} \gamma^\mu \psi(0) | A(p_A) B(p_B) \rangle = \int \frac{d(n+p_a)}{2\pi} \frac{d(n-p_b)}{2\pi} C^{A0}(n+p_a, -n-p_b)$$

$$\times \langle X_{\bar{c}}^{\text{PDF}} | \hat{\chi}_{\bar{c}}^{\text{PDF}}(n-p_b) | B(p_B) \rangle \gamma_\perp^\mu \langle X_c^{\text{PDF}} | \hat{\chi}_c^{\text{PDF}}(n+p_a) | A(p_A) \rangle$$

$$\times \langle X_s | \mathbf{T} [Y_-^\dagger(0) Y_+(0)] | 0 \rangle$$

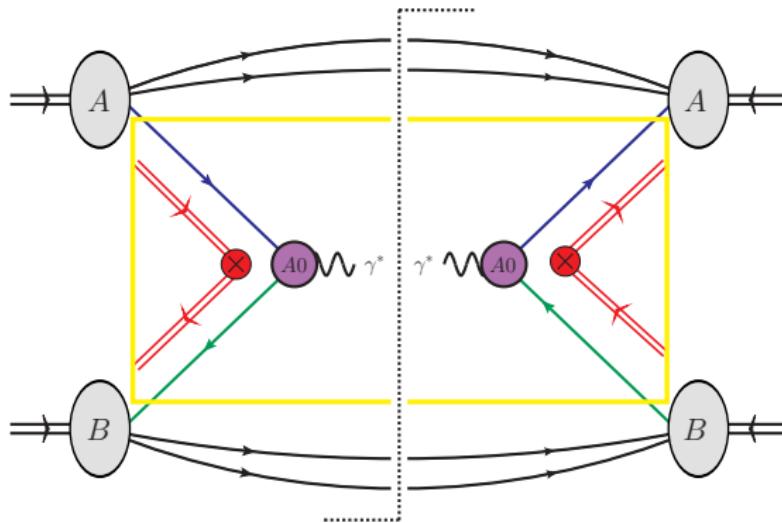
The states factorize: $\langle X | = \langle X_{\bar{c}}^{\text{PDF}} | \langle X_c^{\text{PDF}} | \langle X_s |$. The threshold collinear mode does not appear. Only the PDF collinear mode with scaling

$$p_{c-\text{PDF}} \sim (Q, \Lambda_{\text{QCD}}^2/Q, \Lambda_{\text{QCD}})$$

The Drell-Yan Process: Leading Power Cross-Section

The **leading power** factorisation formula has the following form:

$$\frac{d\sigma_{\text{DY}}}{dQ^2} \sim \sum_{a,b} \int_0^1 dx_a dx_b f_{a/A}(x_a) f_{b/B}(x_b) \left(\hat{\sigma}_{ab}^{\text{LP}} + \hat{\sigma}_{ab,\text{dyn}}^{\text{NLP}} + \hat{\sigma}_{ab,\text{kin}}^{\text{NLP}} + \dots \right) + \mathcal{O}(\Lambda/Q)$$



$$\hat{\sigma}^{\text{LP}}(z) = Q |C^{A^0}(Q^2)|^2 S_{\text{DY}}(\Omega)$$

[G. Sterman 1987] [S. Catani, L. Trentadue 1989] [G. P. Korchemsky G. Marchesini, 1993]

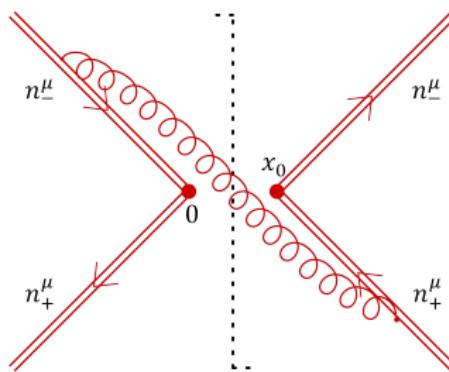
[S. Moch, A. Vogt, hep-ph/0508265] [T. Becher, M. Neubert, G. Xu, 0710.0680]

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$$S_{\text{DY}}(\Omega) = \int \frac{dx^0}{4\pi} e^{ix^0\Omega/2} \frac{1}{N_c} \text{Tr} \langle 0 | \bar{\mathbf{T}}(Y_+^\dagger(x^0) Y_-(x^0)) \mathbf{T}(Y_-^\dagger(0) Y_+(0)) | 0 \rangle$$



Note, inserting complete set of states, and performing $\int dx_0$

$$\delta(\Omega - 2E_X)$$

Drell-Yan Process at Threshold: Next-to-Leading Power

Factorisation formula at NLP

First let us compare **leading power** and **next-to-leading power** cross-sections schematically:

$$\frac{d\sigma_{\text{DY}}}{dQ^2} \sim \sum_{a,b} \int_0^1 dx_a dx_b f_{a/A}(x_a) f_{b/B}(x_b) \left(\hat{\sigma}_{ab}^{\text{LP}} + \hat{\sigma}_{ab,dyn}^{\text{NLP}} + \hat{\sigma}_{ab,kin}^{\text{NLP}} + \dots \right) + \mathcal{O}(\Lambda/Q)$$

We have discussed the **LP** piece

$$\hat{\sigma}^{\text{LP}}(z) = Q |\mathcal{C}^{A0}(Q^2)|^2 S_{\text{DY}}(\Omega)$$

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We have discussed the **LP** piece

$$\hat{\sigma}^{\text{LP}}(z) = Q |\mathcal{C}^{A0}(Q^2)|^2 S_{\text{DY}}(\Omega)$$

and as will be shown the **NLP** is given by

$$\hat{\sigma}^{\text{NLP}}(z) = \sum_{\text{terms}} [C \otimes \mathcal{J} \otimes \bar{\mathcal{J}}]^2 \otimes \mathcal{S}$$

- ▶ C is the hard Wilson matching coefficient
- ▶ \mathcal{S} is the **generalised** soft function
- ▶ \mathcal{J} s are the collinear functions

Let us now motivate the emergence of this structure at next-to-leading power.

Collinear functions at NLP

Beyond LP, the decoupling transformation does not remove soft-collinear interactions in the Lagrangian.

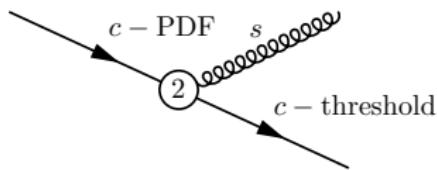
Consider an example of subleading SCET Lagrangian:

[M. Beneke and Th. Feldmann, hep-ph/0211358]

$$\mathcal{L}_{2\xi}^{(2)} = \frac{1}{2} \bar{\chi}_c^{(0)} z_\perp^\mu z_\perp^\rho [i\partial_\rho \text{in}_- \partial \mathcal{B}_\mu^+ (z_-)] \frac{\not{q}_+}{2} \chi_c^{(0)} \quad \text{where} \quad \mathcal{B}_\pm^\mu = Y_\pm^\dagger [iD_s^\mu Y_\pm]$$

The subleading power Lagrangian terms enter the basis through time-ordered product operators

$$\left(J_{A0,2\xi}^{T2}(t) \right)^\mu = i \int d^4 z \mathbf{T} \left[J_{A0}^\mu(t) \mathcal{L}_{2\xi}^{(2)}(z) \right]$$



Collinear functions at NLP

PDF collinear modes can be radiated into the final state:

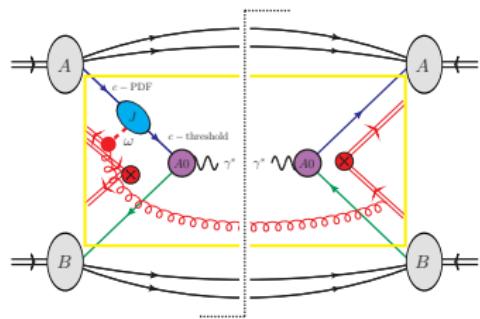
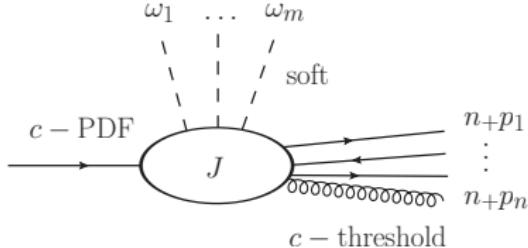
$$\textcolor{blue}{p_c} \sim Q(1, \lambda^2, \lambda) \text{ and } p_{c-\text{PDF}} \sim (Q, \Lambda^2/Q, \Lambda)$$

$$i \int d^4z \mathbf{T} \left[\chi_c(tn_+) \mathcal{L}^{(2)}(z) \right] = 2\pi \sum_i \int du \int dz_- \tilde{J}_i(t, u; z_-) \chi_c^{\text{PDF}}(un_+) \mathfrak{s}_i(z_-)$$

$$\mathfrak{s}_i(z_-) \in \left\{ \frac{i\partial_\perp^\mu}{in_- \partial} \mathcal{B}_{\mu\perp}^+(z_-), \frac{i\partial_{[\mu\perp}}}{in_- \partial} \mathcal{B}_{\nu\perp}^+(z_-), \frac{1}{(in_- \partial)} [\mathcal{B}_{\mu\perp}^+(z_-), \mathcal{B}_{\nu\perp}^+(z_-)], \dots \right\}$$

[M. Beneke, A. Broggio, M. Garry, **SJ**, R. Szafron, L. Vernazza, J. Wang, 1809.10631]

[M. Beneke, A. Broggio, **SJ**, L. Vernazza, 1912.01585]



Collinear functions at NLP

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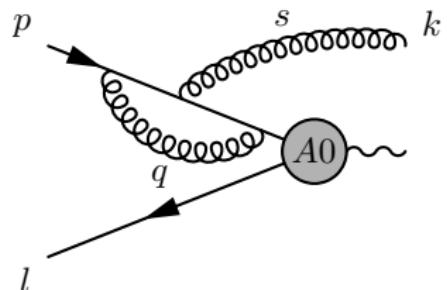
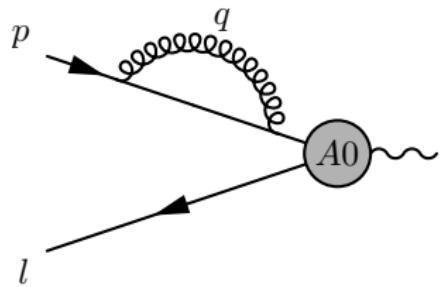
$$\mathfrak{s}_i(z_-) \in \left\{ \frac{i\partial_\perp^\mu}{in_- \partial} \mathcal{B}_{\mu\perp}^+(z_-), \frac{i\partial_{[\mu\perp}}}{in_- \partial} \mathcal{B}_{\nu\perp}^+(z_-), \frac{1}{(in_- \partial)} [\mathcal{B}_{\mu\perp}^+(z_-), \mathcal{B}_{\nu\perp}^+(z_-)], \dots \right\}$$

[M. Beneke, A. Broggio, M. Garry, **SJ**, R. Szafron, L. Vernazza, J. Wang, 1809.10631]

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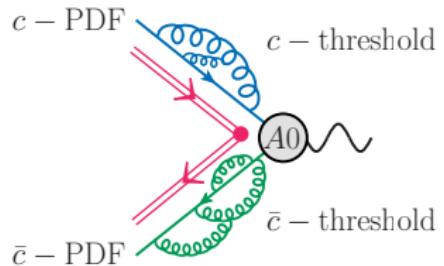
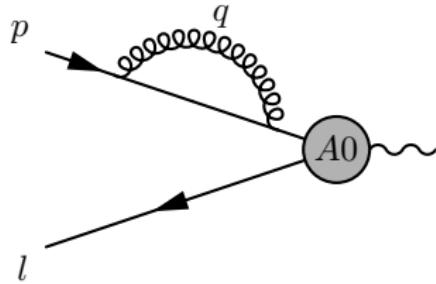
In the diagrammatic approach, soft emissions from jets are captured by the *radiative jet function* [V. Del Duca, 1990], [D. Bonocore, E. Laenen, L. Magnea, S. Melville, L. Vernazza, C.D. White, 1503.05156], [D. Bonocore, E. Laenen, L. Magnea, L. Vernazza, C.D. White, 1610.06842]

Absence of collinear functions at LP



The purely threshold-collinear loops are scaleless and vanish.

Absence of collinear functions at LP



The purely threshold-collinear loops are scaleless and vanish.

At **leading power**, we can apply the **decoupling transformation** $\chi_c^{(0)} = Y_+^\dagger(0)\chi_c$ which removed the soft-collinear interactions from the Lagrangian.

[C. Bauer, D. Pirjol, and I. Stewart, hep-ph/0109045]

The threshold-collinear modes can be identified with c -PDF modes, $\chi_c \rightarrow \chi_c^{\text{PDF}}$

$$\chi_c(t n_+) = \int du \tilde{J}(t, u) \chi_c^{\text{PDF}}(u n_+) \quad \tilde{J}(t, u) = \delta(t - u)$$

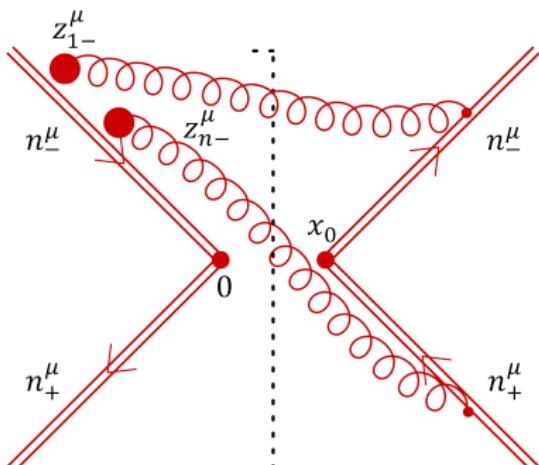
Generalised soft functions

The necessary presence of the time-ordered products at NLP gives rise to the concept of generalized soft functions with explicit gauge fields. Schematically we have

$$S(\Omega, \{\omega\}) = \int \frac{dx^0}{4\pi} e^{ix^0 \Omega/2} \left(\prod_{j=1}^n \int \frac{d(n_+ z_j)}{4\pi} e^{-i\omega_j (n_+ z_j)/2} \right) \\ \times \text{Tr} \langle 0 | \bar{\mathbf{T}} \left[Y_+^\dagger(x^0) Y_-(x^0) \right] \mathbf{T} \left[Y_-^\dagger(0) Y_+(0) \times \mathcal{L}_s^{(n)}(z_{1-}) \times \dots \times \mathcal{L}_s^{(n)}(z_{n-}) \right] | 0 \rangle$$

$\mathcal{L}_s^{(n)}(z_{j-})$ contains $\mathcal{B}_{\perp\nu}^+(z_{j-})$ fields, not only Wilson lines.

[M. Beneke , F. Campanario, T. Mannel, B.D. Pecjak, hep/0411395]

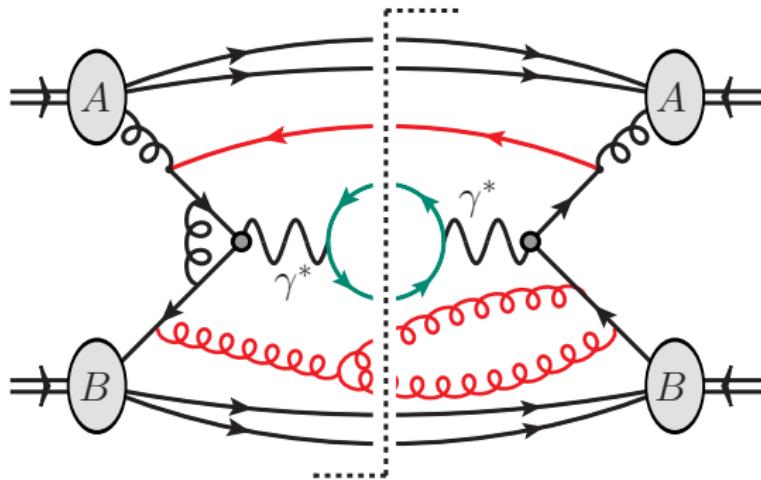


Power Corrections on the Lightcone

Factorisation formula at NLP

The off-diagonal DY process

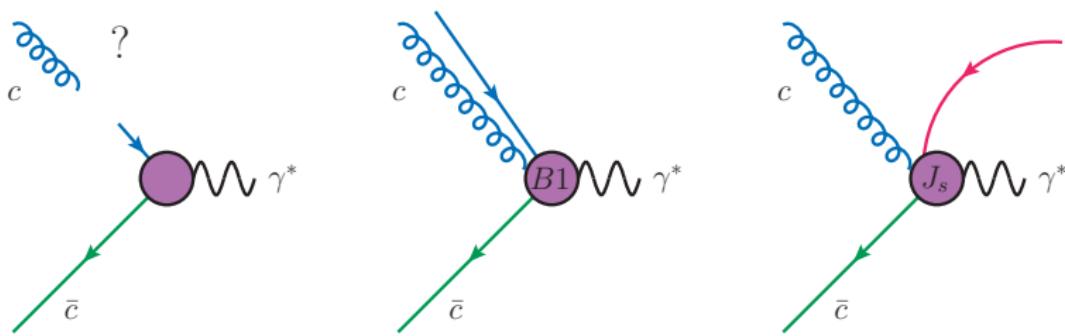
Now we consider initial state gluon from a proton A, which must be converted into a collinear quark via an emission of a soft antiquark.



In the EFT picture

This is inherently a subleading power channel.

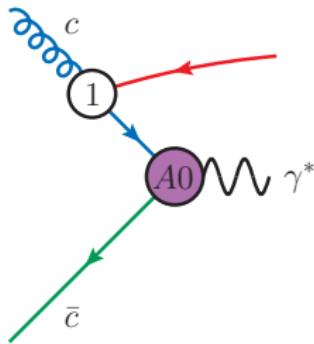
$$\bar{\psi} \gamma_\rho \psi(0) = \sum_{m_1, m_2} \int \{dt_k\} \{d\bar{t}_k\} \tilde{C}^{m_1, m_2} (\{t_k\}, \{\bar{t}_k\}) J_s(0) J_\rho^{m_1, m_2} (\{t_k\}, \{\bar{t}_k\})$$



In the EFT picture

This is inherently a subleading power channel.

$$\bar{\psi} \gamma_\mu \psi = \int dt d\bar{t} \tilde{C}^{A0}(t, \bar{t}) \underbrace{\bar{\chi}_{\bar{c}}(\bar{t} n_-) \gamma_{\perp \mu} \chi_c(t n_+)}_{J_\mu^{A0}(t, \bar{t})}$$



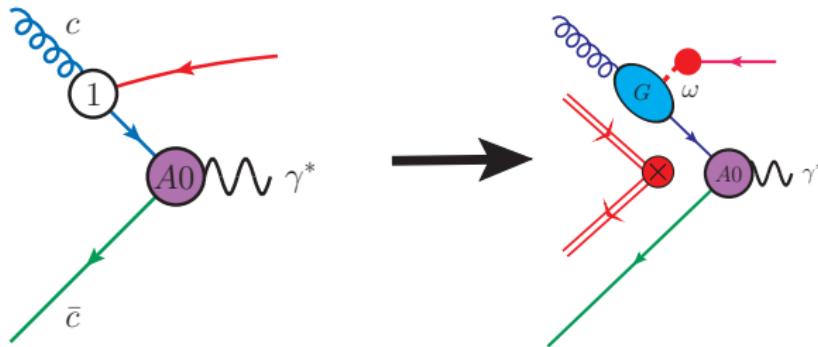
Use an insertion of a subleading power Lagrangian term with a soft quark, starting at $\mathcal{O}(\lambda)$

$$\left(J_{A0, \xi q}^{T2}(t) \right)^\mu = i \int d^d z \mathbf{T} \left[\chi_c(t n_+) \mathcal{L}_{\xi q}^{(1)}(z) \right] \quad \mathcal{L}_{\xi q}^{(1)} = \bar{q}_+ \mathcal{A}_{c\perp} \chi_c + \text{h.c.}$$

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NLP factorization formula for Drell-Yan: $g\bar{q}$

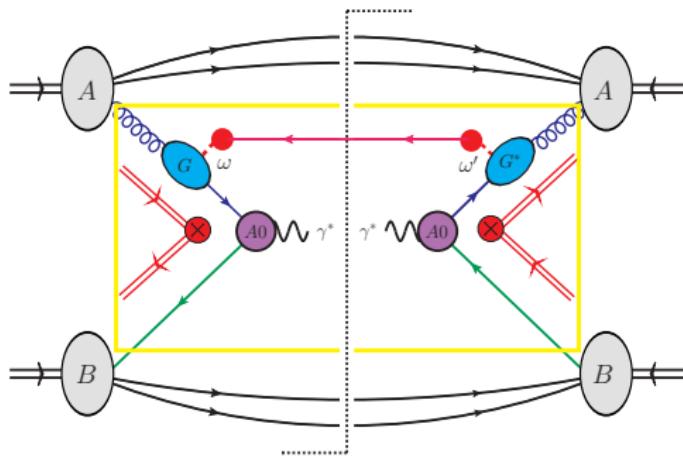
The partonic cross-section is

$$\Delta(z) = \frac{1}{(1-\epsilon)} \frac{\hat{\sigma}(z)}{z} \quad \Delta_{g\bar{q},\text{NLP}}(z) = \Delta_{g\bar{q},\text{NLP}}^{\text{dyn}}(z)$$

where

[[SJ, PhD thesis](#)][A. Broggio, [SJ](#), L. Vernazza, 2306.06037]

$$\Delta_{g\bar{q},\text{NLP}}^{\text{dyn}}(z) = 8H(Q^2) \int d\omega d\omega' G_{\xi q}^*(x_a n + p_A; \omega') G_{\xi q}(x_a n + p_A; \omega) S(\Omega, \omega, \omega')$$

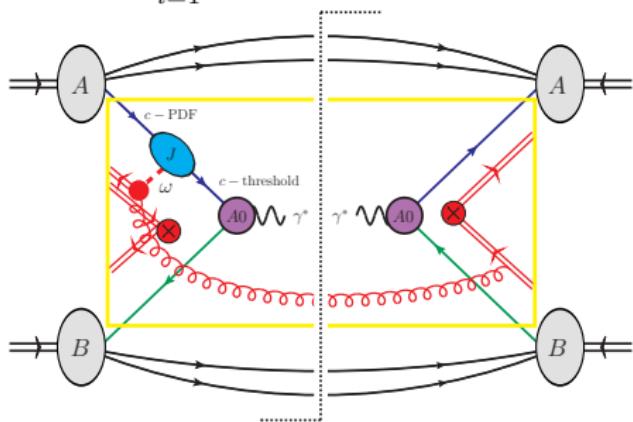


Factorisation formula at NLP

Defining $\Delta = \hat{\sigma}/z$, we arrive at a final result:

[M. Beneke, A. Broggio, SJ, L. Vernazza, 1912.01585]

$$\begin{aligned} \Delta_{\text{NLP}}^{\text{dyn}}(z) = & -2 Q \left[\left(\frac{\not{q}_-}{4} \right) \gamma_{\perp\rho} \left(\frac{\not{q}_+}{4} \right) \gamma_{\perp}^{\rho} \right]_{\beta\gamma} \int d(n+p) C^{A0}(n+p, x_b n - p_B) \\ & \times C^{*A0}(x_a n + p_A, x_b n - p_B) \sum_{i=1}^4 \int \{d\omega_j\} J_i(n+p, x_a n + p_A; \{\omega_j\}) S_i(\Omega; \{\omega_j\}) + \text{h.c.} \end{aligned}$$



For comparison, LP result is:

$$\Delta_{\text{LP}}(z) = |C(Q^2)|^2 Q S_{\text{DY}}(Q(1-z))$$

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where the *generalised soft* functions have the structure:

$$\tilde{S}_i(x; \{\omega_j\}) = \int \left\{ \frac{dz_{j-}}{2\pi} \right\} e^{-i\omega_j z_{j-}} \times \frac{1}{N_c} \text{Tr} \langle 0 | \bar{\mathbf{T}} \left([Y_+^\dagger Y_-](x) \right) \mathbf{T} \left([Y_-^\dagger Y_+](0) \mathfrak{s}_i(\{z_{j-}\}) \right) | 0 \rangle$$

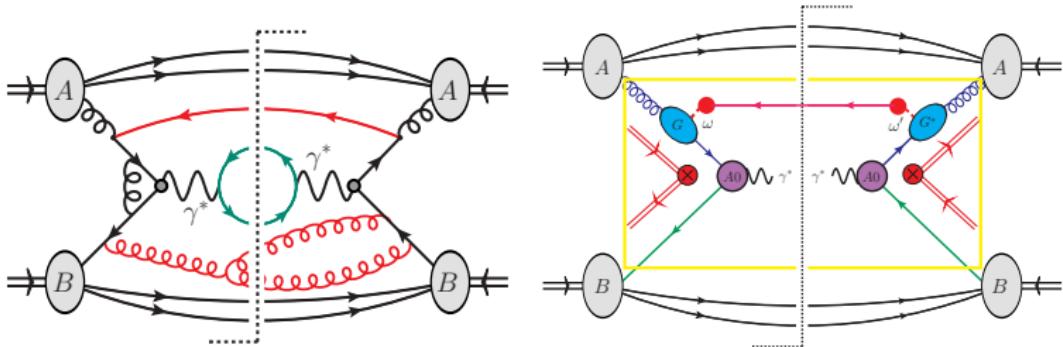
with

$$\begin{aligned} \mathfrak{s}_i(\{z_{j-}\}) \in & \left\{ \frac{i\partial_\perp^\mu}{in_- \partial} \mathcal{B}_{\mu\perp}^+(z_{1-}), \frac{1}{(in_- \partial)^2} \left[\mathcal{B}^{+\mu\perp}(z_{1-}), \left[in_- \partial \mathcal{B}_{\mu\perp}^+(z_{1-}) \right] \right], \right. \\ & \left. \frac{1}{(in_- \partial)} \mathcal{B}_{\mu\perp}^+(z_{1-}) \mathcal{B}_{\nu\perp}^+(z_{2-}), \frac{1}{(in_- \partial)^2} q_{+\sigma}(z_{1-}) \bar{q}_{+\lambda}(z_{2-}) \right\} \end{aligned}$$

Validation

The full NNLO cross-section can be compared with [R. Hamberg, W. van Neerven, T. Matsuura, 1991]

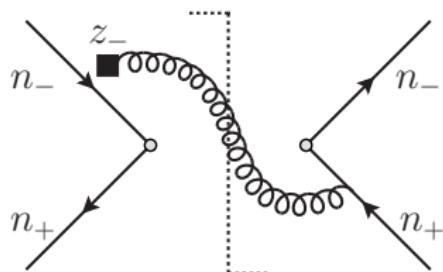
$$\Delta_{g\bar{q}, \text{NLP}}^{dyn}(z) = 8H(Q^2) \int d\omega d\omega' G_{\xi q}^*(x_a n + p_A; \omega') G_{\xi q}(x_a n + p_A; \omega) S(\Omega, \omega, \omega')$$



Soft function at NLO

The only generalised soft function with non-vanishing contributions at $\mathcal{O}(\alpha_s)$ is

$$S_1(\Omega, \omega) = \int \frac{dx^0}{4\pi} \int \frac{d(n_+ z)}{4\pi} e^{ix^0 \Omega/2 - i\omega(n_+ z)/2} \\ \times \frac{1}{N_c} \text{Tr} \langle 0 | \bar{\mathbf{T}} \left[Y_+^\dagger(x^0) Y_-(x^0) \right] \mathbf{T} \left[Y_-^\dagger(0) Y_+(0) \frac{i\partial_\perp^\nu}{in_- \partial} \mathcal{B}_{\nu\perp}^+(z_-) \right] |0\rangle$$

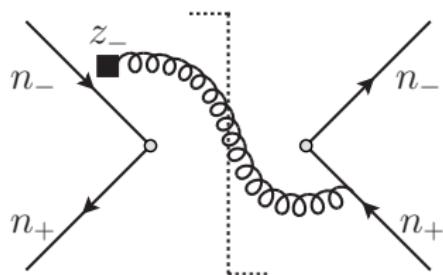


$$\langle g^A(k) | \mathbf{T} \left(Y_-^\dagger(0) Y_+(0) \frac{i\partial_\perp^\nu}{in_- \partial} \mathcal{B}_{\nu\perp}^+(z_-) \right) |0\rangle = \mathbf{T}^A \frac{g_s}{(n_- k)} \left[k_\perp^\eta - \frac{k_\perp^2}{(n_- k)} n_-^\eta \right] \epsilon_\eta^*(k) e^{iz_- k} \\ \rightarrow \delta(n_- k - \omega)$$

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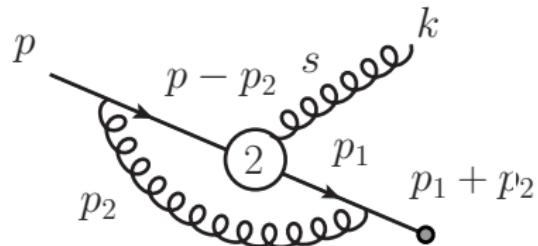
$$S_1(\Omega, \omega) = \int \frac{dx^0}{4\pi} \int \frac{d(n+z)}{4\pi} e^{ix^0\Omega/2 - i\omega(n+z)/2} \\ \times \frac{1}{N_c} \text{Tr} \langle 0 | \bar{\mathbf{T}} \left[Y_+^\dagger(x^0) Y_-(x^0) \right] \mathbf{T} \left[Y_-^\dagger(0) Y_+(0) \frac{i\partial_\perp^\nu}{in_- \partial} \mathcal{B}_{\nu\perp}^+(z_-) \right] |0\rangle$$



$$S_1^{(1)}(\Omega, \omega) = \frac{\alpha_s C_F}{2\pi} \frac{\mu^{2\epsilon} e^{\epsilon\gamma_E}}{\Gamma[1-\epsilon]} \frac{1}{\omega^{1+\epsilon}} \frac{1}{(\Omega-\omega)^\epsilon} \theta(\omega)\theta(\Omega-\omega) .$$

After $\int d\omega$ the NLO cross section $\Delta_{\text{NLP}}^{\text{dyn}(1)}$ is correctly reproduced.

Collinear functions at one-loop



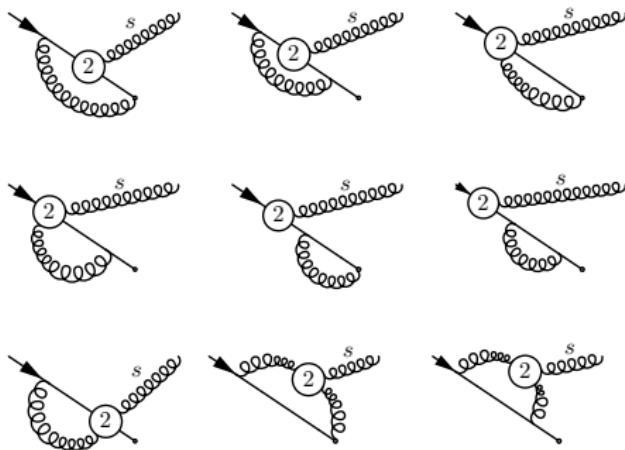
Use NLP Feynman rules to calculate these new objects up to $\mathcal{O}(\alpha_s)$. [[M. Beneke, M. Garny, R. Szafron, J. Wang, 1808.04742](#)]
Some non-standard features present!

$$X^\mu = \partial^\mu \left[(2\pi)^d \delta^{(d)} \left(\sum p_{\text{in}} - \sum p_{\text{out}} \right) \right]$$

$$\mathcal{O}(\lambda^2) : \quad ig_s \mathbf{T}^A A^{\rho\nu}(k, p, p') \frac{\not{p}_+}{2} (k_\rho g_{\nu\mu} - k_\nu g_{\rho\mu})$$

$$A^{\rho\nu}(k, p, p') \equiv \frac{1}{2} \left[(n_- X) n_+^\rho n_-^\nu + (k X_\perp) X_\perp^\rho n_-^\nu + X_\perp^\rho \left(\frac{\not{p}'_\perp}{n_+ p'} \gamma_\perp^\nu + \gamma_\perp^\nu \frac{\not{p}_\perp}{n_+ p} \right) \right]$$

Collinear functions at one-loop



$$\begin{aligned}
 J_1(n+q, n+p; \omega) = & -\frac{1}{n+p} \delta(n+q - n+p) + 2 \frac{\partial}{\partial n+q} \delta(n+q - n+p) \\
 & + \frac{\alpha_s}{4\pi} \frac{1}{(n+p)} \left(\frac{n+p \omega}{\mu^2} \right)^{-\epsilon} \frac{e^{\epsilon \gamma_E} \Gamma[1+\epsilon] \Gamma[1-\epsilon]^2}{(-1+\epsilon)(1+\epsilon) \Gamma[2-2\epsilon]} \\
 & \times \left(C_F \left(-\frac{4}{\epsilon} + 3 + 8\epsilon + \epsilon^2 \right) - C_A (-5 + 8\epsilon + \epsilon^2) \right) \delta(n+q - n+p)
 \end{aligned}$$

[M. Beneke, A. Broggio, SJ, L. Vernazza, 1912.01585]

Validation at Fixed Order: $\Delta_{\text{NLP-coll}}^{\text{dyn}(2)}$ piece

Focus on one piece of the factorization formula

$$\Delta_{\text{NLP-coll}}^{\text{dyn}(2)}(z) = 4Q \int d\omega J_{1,1}^{(1)}(x_a n_+ p_A; \omega) S_1^{(1)}(\Omega; \omega).$$

The factorization formula is valid for unrenormalized objects. Performing the convolution in d -dimensions reproduces fixed NNLO result:

$$\begin{aligned} \Delta_{\text{NLP-coll}}^{(2)} &= \frac{\alpha_s^2}{(4\pi)^2} \left(C_A C_F \left(\frac{20}{\epsilon} - 60 \log(1-z) + 8 + \mathcal{O}(\epsilon^1) \right) \right. \\ &\quad \left. + C_F^2 \left(\frac{-16}{\epsilon^2} - \frac{20}{\epsilon} + \frac{48}{\epsilon} \log(1-z) + 60 \log(1-z) - 72 \log^2(1-z) + \mathcal{O}(\epsilon^1) \right) \right) \end{aligned}$$

after we set the scale to hard. In agreement with equation (4.22) of [D. Bonocore, E. Laenen, L. Magnea, S. Melville, L. Vernazza, C. White, 1503.05156]

We have also computed and checked the remaining contributions

- ▶ Hard: 1-loop hard and NLO soft functions, $\Delta_{\text{NLP-hard}}^{\text{dyn}(2)}$.
[M. Beneke, A. Broggio, SJ, L. Vernazza, 1912.01585]
- ▶ Soft: NNLO soft functions, $\Delta_{\text{NLP-soft}}^{\text{dyn}(2)} \rightarrow$ focus on next.
[A. Broggio, SJ, L. Vernazza, 2107.07353]

NLP factorization formula for Drell-Yan

The partonic cross-section is

$$\Delta(z) = \frac{1}{(1-\epsilon)} \frac{\hat{\sigma}(z)}{z} \quad \Delta_{\text{NLP}}(z) = \Delta_{\text{NLP}}^{\text{dyn}}(z) + \Delta_{\text{NLP}}^{\text{kin}}(z)$$

where

[M. Beneke, A.Broggio, SJ, L. Vernazza, 1912.01585]

$$\begin{aligned} \Delta_{\text{NLP}}^{\text{dyn}}(z) &= -\frac{2}{(1-\epsilon)} Q \left[\left(\frac{\not{q}_-}{4} \right) \gamma_{\perp\rho} \left(\frac{\not{q}_+}{4} \right) \gamma_{\perp}^{\rho} \right]_{\beta\gamma} \\ &\times \int d(n+p) C^{A0,A0}(n+p, x_b n - p_B) C^{*A0A0}(x_a n + p_A, x_b n - p_B) \\ &\times \sum_{i=1}^4 \int \{d\omega_j\} J_{i,\gamma\beta}(n+p, x_a n + p_A; \{\omega_j\}) S_i(\Omega; \{\omega_j\}) + \text{h.c.} \end{aligned}$$

We now focus on $\mathcal{O}(\alpha_s^2)$ calculation of:

$$S_i(\Omega; \{\omega_j\}) = \int \frac{dx^0}{4\pi} e^{i\Omega x^0/2} \int \left\{ \frac{dz_{j-}}{2\pi} \right\} e^{-i\omega_j z_{j-}} S_i(x_0; \{z_{j-}\})$$

Generalised Soft Functions

We make use of the soft building blocks

$$\mathcal{B}_\pm^\mu = Y_\pm^\dagger [i D_s^\mu Y_\pm] , \quad q^\pm = Y_\pm^\dagger q_s$$

The relevant soft functions are

$$S_1(x^0; z_-) = \frac{1}{N_c} \text{Tr} \langle 0 | \bar{\mathbf{T}} \left[Y_+^\dagger(x^0) Y_-(x^0) \right] \mathbf{T} \left(\left[Y_-^\dagger(0) Y_+(0) \right] \frac{i \partial_\perp^\nu}{in_- \partial} \mathcal{B}_{\nu \perp}^+(z_-) \right) |0\rangle$$

$$\begin{aligned} S_3(x^0; z_-) &= \frac{1}{N_c} \text{Tr} \langle 0 | \bar{\mathbf{T}} \left[Y_+^\dagger(x^0) Y_-(x^0) \right] \\ &\times \mathbf{T} \left(\left[Y_-^\dagger(0) Y_+(0) \right] \frac{1}{(in_- \partial)^2} \left[\mathcal{B}^{+\mu \perp}(z_-), [in_- \partial \mathcal{B}_{\mu \perp}^+(z_-)] \right] \right) |0\rangle \end{aligned}$$

$$\begin{aligned} S_{4;\mu\nu,bf}^{AB}(x^0; z_{1-}, z_{2-}) &= \frac{1}{N_c} \text{Tr} \langle 0 | \bar{\mathbf{T}} \left[Y_+^\dagger(x^0) Y_-(x^0) \right]_{ba} \\ &\times \mathbf{T} \left(\left[Y_-^\dagger(0) Y_+(0) \right]_{af} \mathcal{B}_{\mu \perp}^{+A}(z_{1-}) \mathcal{B}_{\nu \perp}^{+B}(z_{2-}) \right) |0\rangle \\ S_{5;bfgh,\sigma\lambda}(x^0; z_{1-}, z_{2-}) &= \frac{1}{N_c} \langle 0 | \bar{\mathbf{T}} \left[Y_+^\dagger(x^0) Y_-(x^0) \right]_{ba} \\ &\times \mathbf{T} \left(\left[Y_-^\dagger(0) Y_+(0) \right]_{af} \frac{g_s^2}{(in_- \partial z_1)(in_- \partial z_2)} q_{+\sigma g}(z_{1-}) \bar{q}_{+\lambda h}(z_{2-}) \right) |0\rangle \end{aligned}$$

Not functions of Wilson lines only!

Generalised Soft Functions: Matrix Elements

Single emission:

$$\langle g^A(k) | \mathbf{T} \left(Y_-^\dagger(0) Y_+(0) \frac{i\partial_\perp^\nu}{in_- \partial} \mathcal{B}_{\nu\perp}^+ (z_-) \right) | 0 \rangle = \mathbf{T}^A \frac{g_s}{(n_- - k)} \left[k_\perp^\eta - \frac{k_\perp^2}{(n_- - k)} n_-^\eta \right] \epsilon_\eta^*(k) e^{iz_- k}$$

Generalised Soft Functions: Matrix Elements

Single emission:

$$\langle g^A(k) | \mathbf{T} \begin{pmatrix} Y_-^\dagger(0) Y_+(0) & \frac{i\partial_\perp^\nu}{in_- \partial} \mathcal{B}_{\nu\perp}^+ (z_-) \end{pmatrix} | 0 \rangle = \mathbf{T}^A \frac{g_s}{(n_- - k)} \left[k_\perp^\eta - \frac{k_\perp^2}{(n_- - k)} n_-^\eta \right] \epsilon_{\eta}^*(k) e^{iz_- k}$$

Double emission:

$$\begin{aligned} \langle g^{K_1}(k_1) g^{K_2}(k_2) | \mathbf{T} \begin{bmatrix} Y_-^\dagger(0) Y_+(0) & \frac{i\partial_\perp^\mu}{in_- \partial} \mathcal{B}_{\mu\perp}^+ (z_-) \end{bmatrix} | 0 \rangle &= \\ g_s^2 \mathbf{T}^{K_2} \mathbf{T}^{K_1} \frac{1}{(n_- - k_1)} \frac{n_-^{\eta_2}}{(n_- - k_2)} \left[k_{1\perp}^{\eta_1} - \frac{k_{1\perp}^2}{(n_- - k_1)} n_-^{\eta_1} \right] \epsilon_{\eta_1}^*(k_1) \epsilon_{\eta_2}^*(k_2) e^{iz_- k_1} \\ + \dots \\ + g_s^2 i f^{K_1 K_2 K} \mathbf{T}^K \frac{1}{n_-(k_1 + k_2)} &\left(- \frac{(k_{1\perp}^{\eta_2} + k_{2\perp}^{\eta_2}) n_-^{\eta_1}}{(n_- - k_1)} + \frac{(k_{1\perp}^{\eta_1} + k_{2\perp}^{\eta_1}) n_-^{\eta_2}}{(n_- - k_2)} \right. \\ - \frac{n_-^{\eta_1} n_-^{\eta_2}}{n_-(k_1 + k_2)(n_- - k_1)(n_- - k_2)} &\left[(n_- - k_1) \left(k_{1\perp}^2 + k_{1\perp} \cdot k_{2\perp} \right) \right. \\ \left. - (n_- - k_2) \left(k_{2\perp} \cdot k_{1\perp} + k_{2\perp}^2 \right) \right] \epsilon_{\eta_1}^*(k_1) \epsilon_{\eta_2}^*(k_2) e^{iz_- (k_1 + k_2)} \\ + \dots \end{aligned}$$

\rightarrow $\delta(\omega - n_- k_1)$ constraint in the integrand.

Generalised Soft Functions: Matrix Elements

Single emission:

$$\langle g^A(k) | \mathbf{T} \begin{pmatrix} Y_-^\dagger(0) Y_+(0) & \frac{i\partial_\perp^\nu}{in_- \partial} \mathcal{B}_{\nu\perp}^+ (z_-) \end{pmatrix} | 0 \rangle = \mathbf{T}^A \frac{g_s}{(n_- - k)} \left[k_\perp^\eta - \frac{k_\perp^2}{(n_- - k)} n_-^\eta \right] \epsilon_\eta^*(k) e^{iz_- k}$$

Double emission:

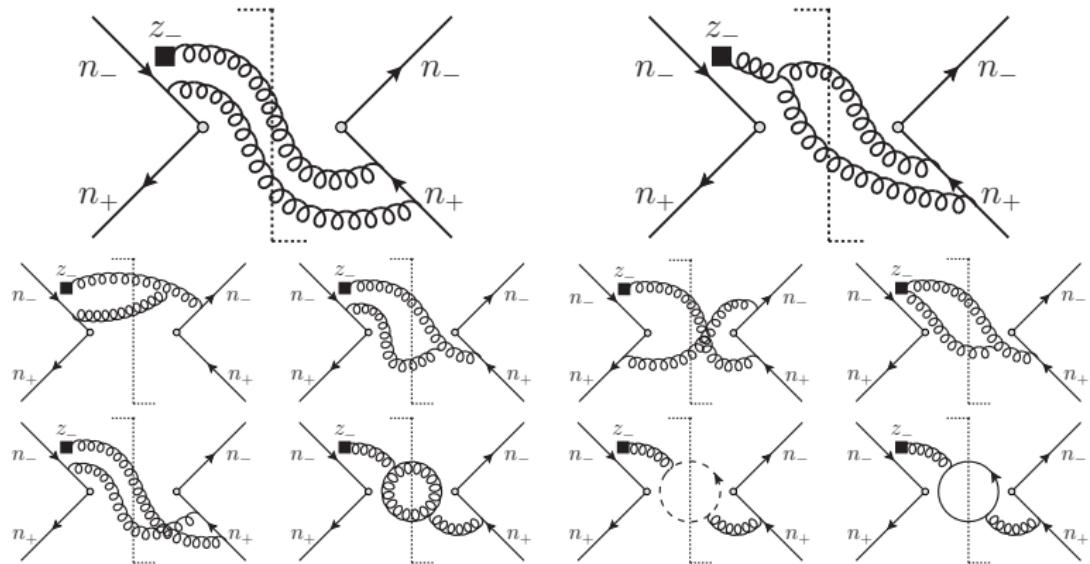
$$\begin{aligned} \langle g^{K_1}(k_1) g^{K_2}(k_2) | \mathbf{T} \begin{pmatrix} Y_-^\dagger(0) Y_+(0) & \frac{i\partial_\perp^\mu}{in_- \partial} \mathcal{B}_{\mu\perp}^+ (z_-) \end{pmatrix} | 0 \rangle &= \\ g_s^2 \mathbf{T}^{K_2} \mathbf{T}^{K_1} \frac{1}{(n_- - k_1)} \frac{n_-^{\eta_2}}{(n_- - k_2)} \left[k_{1\perp}^{\eta_1} - \frac{k_{1\perp}^2}{(n_- - k_1)} n_-^{\eta_1} \right] \epsilon_{\eta_1}^*(k_1) \epsilon_{\eta_2}^*(k_2) e^{iz_- k_1} \\ + \dots \\ + g_s^2 i f^{K_1 K_2 K} \mathbf{T}^K \frac{1}{n_-(k_1 + k_2)} &\left(- \frac{(k_{1\perp}^{\eta_2} + k_{2\perp}^{\eta_2}) n_-^{\eta_1}}{(n_- - k_1)} + \frac{(k_{1\perp}^{\eta_1} + k_{2\perp}^{\eta_1}) n_-^{\eta_2}}{(n_- - k_2)} \right. \\ - \frac{n_-^{\eta_1} n_-^{\eta_2}}{n_-(k_1 + k_2)(n_- - k_1)(n_- - k_2)} \left[(n_- - k_1) \left(k_{1\perp}^2 + k_{1\perp} \cdot k_{2\perp} \right) \right. \\ \left. \left. - (n_- - k_2) \left(k_{2\perp} \cdot k_{1\perp} + k_{2\perp}^2 \right) \right] \right) \epsilon_{\eta_1}^*(k_1) \epsilon_{\eta_2}^*(k_2) e^{iz_- (k_1 + k_2)} \\ + \dots \end{aligned}$$

$\rightarrow \delta(\omega - n_- k_1 - n_- k_2)$ constraint in the integrand.

Some sample diagrams

For the S_1 soft function:

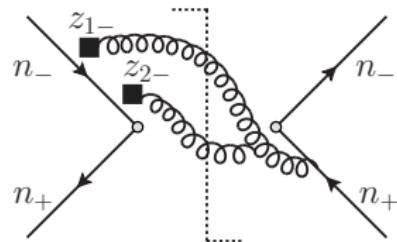
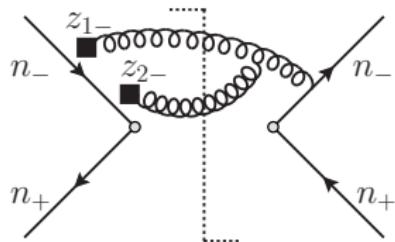
$$S_1(x^0; z_-) = \frac{1}{N_c} \text{Tr} \langle 0 | \bar{\mathbf{T}} \left[Y_+^\dagger(x^0) Y_-(x^0) \right] \mathbf{T} \left(\left[Y_-^\dagger(0) Y_+(0) \right] \frac{i\partial_\perp^\nu}{in_- \partial} \mathcal{B}_{\nu\perp}^+ (z_-) \right) | 0 \rangle$$



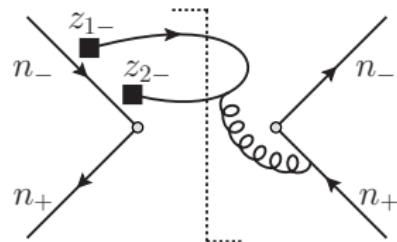
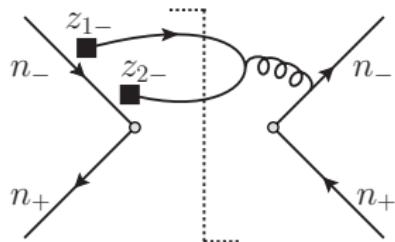
...

Some sample diagrams

For the S_4 soft function:

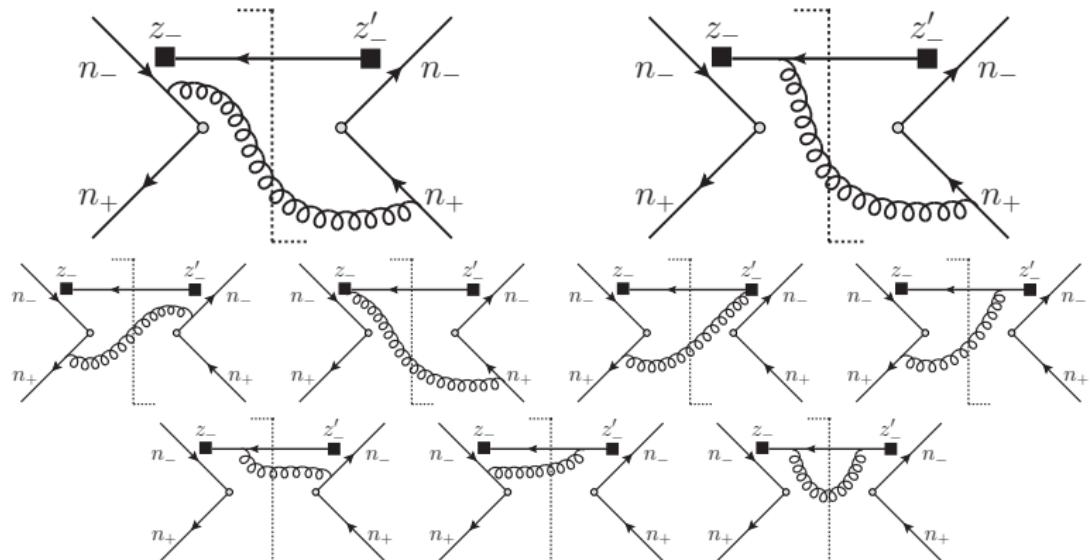


For the S_5 soft function:



Some sample two-loop diagrams: double real

$$\begin{aligned}
 S(\Omega, \omega, \omega') = & \sum_{s, \lambda} \int \frac{d^d k_1}{(2\pi)^{d-1}} \delta^+(k_1^2) \int \frac{d^d k_2}{(2\pi)^{d-1}} \delta^+(k_2^2) \delta(\omega - n_- k_1 - n_- k_2) \delta(\omega' - n_- k_1) \\
 & \times \frac{1}{C_F C_A} \langle 0 | \bar{\mathbf{T}} \left(\frac{g_s}{in_- \partial_z}, \bar{q}_{+\sigma} k(z'_-) \right) \mathbf{T}^D \{ Y_+^\dagger(0) Y_-(0) \} \Big) | X_s \rangle \\
 & \times \frac{\not{n}_{-\sigma\beta}}{4} \langle X_s | \mathbf{T} \left(\{ Y_-^\dagger(0) Y_+(0) \} \mathbf{T}^D \frac{g_s}{in_- \partial_z} q_{+\beta}(z_-) \right) | 0 \rangle \delta(\Omega - 2k_1^0 - 2k_2^0)
 \end{aligned}$$



The calculation

Methods developed for calculations of two-loop soft functions at *leading power*.

[Y. Li, S. Mantry, F. Petriello, 1105.5171] [T. Becher, G. Bell, S. Marti, 1201.5572]

[A. Ferroglio, B. Pecjak, L.L. Yang, 1207.4798]

First, we find the relevant topologies for, and perform, the reduction. For example:

$$P_1 = (k_1 + k_2)^2, \quad P_2 = n_+ k_2, \quad P_3 = n_- (k_1 + k_2),$$

$$P_4 = k_1^2, \quad P_5 = k_2^2, \quad P_6 = (\Omega - n_- k_1 - n_- k_2 - n_+ k_1 - n_+ k_2), \quad P_7 = (\omega - n_- k_1)$$

$$\hat{I}_{\mathfrak{T}}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = (4\pi)^4 \left(\frac{e^{\gamma_E} \mu^2}{4\pi} \right)^{2\epsilon} \int \frac{d^d k_1}{(2\pi)^{d-1}} \frac{d^d k_2}{(2\pi)^{d-1}} \prod_{i=1}^7 \frac{1}{P_i^{\alpha_i}}$$

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$$\delta(k_1^2) = \frac{1}{2\pi i} \left[\frac{1}{k_1^2 + i0^+} - \frac{1}{k_1^2 - i0^+} \right]$$

[C. Anastasiou, K. Melnikov, hep-ph/0207004]

The calculation

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- ▶ The reduction is implemented in **LiteRed**
- ▶ 9 topologies are needed to reduce the soft functions
- ▶ We have 8 Master Integrals (MIs)
 - ▶ 5 MIs implementing the $\delta(\omega - n_- k_1)$ constraint: $\hat{I}_1 - \hat{I}_5$
 - ▶ 2 MIs with $\delta(\omega - n_- k_1 - n_- k_2)$: \hat{I}_6 and \hat{I}_7
 - ▶ 1 MI with $\delta(\omega_1 - n_- k_1) \delta(\omega_2 - n_- k_2)$: \hat{I}_8

The calculation

Methods developed for calculations of two-loop soft functions at *leading power*.

[Y. Li, S. Mantry, F. Petriello, 1105.5171] [T. Becher, G. Bell, S. Marti, 1201.5572]

[A. Ferroglio, B. Pecjak, L.L. Yang, 1207.4798]

First, we find the relevant topologies for, and perform, the reduction. For example:

$$P_1 = (k_1 + k_2)^2, \quad P_2 = n_+ k_2, \quad P_3 = k_1^2, \quad P_4 = k_2^2, \quad P_5 = (\omega - n_- k_1)$$

$$P_6 = (\Omega - n_- k_1 - n_- k_2 - n_+ k_1 - n_+ k_2), \quad P_7 = (\omega' - n_- k_1 - n_- k_2)$$

$$\hat{I}_{\mathfrak{T}}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = (4\pi)^4 \left(\frac{e^{\gamma_E} \mu^2}{4\pi} \right)^{2\epsilon} \int \frac{d^d k_1}{(2\pi)^{d-1}} \frac{d^d k_2}{(2\pi)^{d-1}} \prod_{i=1}^7 \frac{1}{P_i^{\alpha_i}}$$

- ▶ The reduction is implemented in **LiteRed**
- ▶ $4 + 1$ topologies are needed to reduce the soft functions
- ▶ We have $6 + 2$ Master Integrals (MIs)
 - ▶ 4 MIs implementing the $\delta(\omega - n_- k_1)\delta(\omega' - n_- k_1 - n_- k_2)$ and $\omega \leftrightarrow \omega'$ constraint: $\hat{I}_1 - \hat{I}_4$
 - ▶ 1 MI with $\delta(\omega - n_- k_1)\delta(\omega - \omega')$: \hat{I}_5
 - ▶ 1 MI with $\delta(\omega - n_- k_1 - n_- k_2)\delta(\omega - \omega')$: \hat{I}_6

Reduced expressions

Double real

$$\begin{aligned}
 S_{g\bar{q}}^{(2)2r0v}(\Omega, \omega, \omega') &= \frac{\alpha_s^2}{(4\pi)^2} T_F \left[C_F \left(\frac{(4-\epsilon)(1-\epsilon)(1-2\epsilon)}{\epsilon^2 \omega^2 (\Omega - \omega)} \hat{I}_6 + \frac{4(2-3\epsilon)(1-3\epsilon)}{\epsilon^2 \omega (\Omega - \omega)^2} \hat{I}_5 \right) \delta(\omega - \omega') \right. \\
 &\quad + (C_A - 2C_F) \left(\frac{(1-2\epsilon)(\omega + \omega')}{\epsilon \omega \omega' (\Omega - \omega)(\omega - \omega')} \hat{I}_3 - \frac{(1-2\epsilon)(\omega + \omega')}{\epsilon \omega \omega' (\Omega - \omega')(\omega - \omega')} \hat{I}_1 \right. \\
 &\quad \left. \left. + \frac{(\Omega - \omega)(\omega + \omega')}{2\omega \omega'} \hat{I}_4 + \frac{(\Omega - \omega')(\omega + \omega')}{2\omega \omega'} \hat{I}_2 \right) \right]
 \end{aligned}$$

where for example

$$\hat{I}_1(\Omega, \omega, \omega') \equiv \hat{I}_{\mathcal{A}}(0, 0, 1, 1, 1, 1, 1), \quad \hat{I}_2(\Omega, \omega, \omega') \equiv \hat{I}_{\mathcal{A}}(1, 1, 1, 1, 1, 1, 1)$$

and family \mathcal{A} propagators are

$$\begin{aligned}
 P_1 &= (k_1 + k_2)^2, & P_2 &= n_+ k_2, & P_3 &= k_1^2, & P_4 &= k_2^2, \\
 P_5 &= (\Omega - n_- k_1 - n_- k_2 - n_+ k_1 - n_+ k_2), & P_6 &= (\omega - n_- k_1), \\
 P_7 &= (\omega' - n_- k_1 - n_- k_2)
 \end{aligned}$$

The integrals are calculated using the differential equations method.

DE method for MIs

Convenient to change to dimensionless variable $\omega \rightarrow r \Omega$

$$\begin{aligned} I'_1(r) &= \frac{1}{\Omega^2} \left(\frac{\Omega}{\mu} \right)^{4\epsilon} \hat{I}_1(\Omega, r), & I'_2(r) &= \frac{1}{\Omega} \left(\frac{\Omega}{\mu} \right)^{4\epsilon} \hat{I}_2(\Omega, r), \\ I'_3(r) &= \left(\frac{\Omega}{\mu} \right)^{4\epsilon} \hat{I}_3(\Omega, r), & I'_4(r) &= \Omega \left(\frac{\Omega}{\mu} \right)^{4\epsilon} \hat{I}_4(\Omega, r), \\ I'_5(r) &= \Omega^2 \left(\frac{\Omega}{\mu} \right)^{4\epsilon} \hat{I}_5(\Omega, r) \end{aligned}$$

System of DEs can be put into canonical form [J. Henn, 1304.1806]

$$\begin{aligned} \frac{d\vec{I}(r)}{dr} &= \epsilon A(r) \cdot \vec{I}(r) \\ I'_1(r) &= \frac{2(1-r)^2}{2 - 9\epsilon + 9\epsilon^2} I_1(r), \\ I'_3(r) &= \frac{1}{\epsilon^2} I_3(r), \\ I'_4(r) &= -\frac{1}{\epsilon^2(1-r)} I_4(r), \\ I'_5(r) &= -\frac{1+r}{2\epsilon^2(1-r)r} I_4(r) + \frac{1}{\epsilon^2 r} I_5(r) \end{aligned} \quad A(r) = \begin{bmatrix} -\frac{1}{r} + \frac{3}{1-r} & 0 & 0 & 0 \\ \frac{2}{r} & -\frac{2}{r} & 0 & 0 \\ \frac{2}{r} & \frac{2}{r} & \frac{4}{1-r} & 0 \\ \frac{1}{r} & \frac{1}{r} & \frac{1}{r} & -\frac{2}{r} \end{bmatrix}$$

Virtual real result

This contribution is given by

[A. Broggio, SJ, L. Vernazza, 2306.06037]

$$\begin{aligned}
 S_{g\bar{q}}^{(2)1r1v}(\Omega, \omega, \omega') &= \frac{\alpha_s^2 T_F}{(4\pi)^2} (2C_F - C_A) \frac{e^{2\epsilon\gamma_E} \Gamma[1+\epsilon]}{\epsilon \Gamma[1-\epsilon]} \\
 &\cdot \text{Re} \left\{ \left[\frac{\omega' - \omega}{\omega(\omega')^2} {}_2F_1 \left(1, 1 + \epsilon, 1 - \epsilon, -\frac{\omega}{\omega'} \right) - \frac{1}{\omega\omega'} \right] \left(\frac{\mu^4}{\omega\omega'(\Omega - \omega')^2} \right)^\epsilon \theta(\omega) \right. \\
 &+ \left. \frac{2(\omega + \omega')}{\omega\omega'(\omega' - \omega)} \left(\frac{\mu^4}{(\omega' - \omega)^2(\Omega - \omega')^2} \right)^\epsilon \frac{\Gamma[1-\epsilon]^2}{\Gamma[1-2\epsilon]} \theta(\omega' - \omega) \right\} \theta(\omega') \theta(\Omega - \omega')
 \end{aligned}$$

Integrating over ω, ω'

$$\begin{aligned}
 \Delta_{g\bar{q}}^{(2)}(z)|_{\text{NLP,s,1r1v}} &= - \left(\frac{\alpha_s}{4\pi} \right)^2 T_F (C_A - 2C_F) \left(\frac{\mu^2}{\Omega^2} \right)^{2\epsilon} \\
 &\times \frac{2\text{Re}[e^{-i\epsilon\pi}] e^{2\epsilon\gamma_E} \Gamma[1-2\epsilon] \Gamma[1-\epsilon]^2 \Gamma[1+\epsilon]^2}{\epsilon^3 \Gamma[1-4\epsilon]}
 \end{aligned}$$

Results at NNLO

The S_1 soft function is the only one with C_F^2 contributions. Combine our results for two-loop soft with tree-level collinear according to factorisation theorem

$$\Delta_{\text{NLP-soft}, S_1, C_F^2}^{\text{dyn (2)2r0v}}(z) = 4 Q \Omega H^{(0)}(Q^2) \int dr J_{1,1}^{(0)}(x_a(n+p_A); \omega) S_{1,C_F^2}^{(2)2r0v}(\Omega, r)$$

$$\begin{aligned} \Delta_{\text{NLP-soft}, S_1, C_F^2}^{\text{dyn (2)2r0v}}(z) &= \frac{\alpha_s^2}{(4\pi)^2} C_F^2 \left(\frac{32}{\epsilon^3} - \frac{128}{\epsilon^2} \ln(1-z) + \frac{256}{\epsilon} \ln^2(1-z) - \frac{112\pi^2}{3\epsilon} \right. \\ &\quad \left. + \frac{32}{3} (-32 \ln^3(1-z) + 14\pi^2 \ln(1-z) - 62\zeta(3)) + \mathcal{O}(\epsilon) \right) \end{aligned}$$

The S_1 1r1v gives leading logarithms proportional to $C_F C_A$, which are cancelled exactly by the leading poles of 2r0v contribution to S_1

$$\begin{aligned} \Delta_{\text{NLP-soft}, S_1, C_F C_A}^{\text{dyn (2)2r0v}}(z) &= \frac{\alpha_s^2}{(4\pi)^2} C_F C_A \left(\frac{8}{\epsilon^3} - \frac{4}{3\epsilon^2} (24 \ln(1-z) - 11) \right. \\ &\quad - \frac{16}{9\epsilon} (-36 \ln^2(1-z) + 33 \ln(1-z) + 6\pi^2 - 16) \\ &\quad - \frac{256}{3} \ln^3(1-z) + \frac{352}{3} \ln^2(1-z) + \frac{128}{3} \pi^2 \ln(1-z) \\ &\quad \left. - \frac{1024}{9} \ln(1-z) - \frac{616\zeta(3)}{3} - \frac{154\pi^2}{9} + \frac{1484}{27} + \mathcal{O}(\epsilon) \right) \end{aligned}$$

The full NNLO cross-section can be compared with [R. Hamberg, W. van Neerven, T. Matsuura, 1991] and we find agreement.

Results at N³LO

Interestingly, we can partially validate our results at N³LO by comparing to
[\[N. Bahjat-Abbas, J. Sinnenhe Damst , L. Vernazza, C. White, 1807.09246\]](#)

$$\Delta_{\text{NLP-coll}, C_F^3}^{dyn(3)}(z) = 4Q \int d\omega J_{1,1}^{(1)}(x_a n + p_A; \omega) S_{1,C_F^2}^{(2)}(\Omega; \omega)$$

Using one-loop collinear function from [\[M.Beneke, A.Broggio, SJ, L.Vernazza, 1912.01585\]](#)

$$S_{1,C_F^2}^{(2)2r0v}(\Omega, \omega) = 8 \frac{\alpha_s^2}{(4\pi)^2} C_F^2 \left(\frac{\omega(\Omega - \omega)^3}{\mu^4} \right)^{-\epsilon} \frac{1}{\omega} \frac{1}{\epsilon^2} \frac{e^{2\epsilon\gamma_E} \Gamma[1 - \epsilon]}{\Gamma[1 - 3\epsilon]} \theta(\Omega - \omega) \theta(\omega)$$

We find a closed d -dimensional result, expanding

$$\begin{aligned} \Delta_{\text{NLP-coll}, C_F^3}^{dyn(3)}(z) = & \frac{\alpha_s^3}{(4\pi)^3} C_F^3 \left(-\frac{64}{\epsilon^4} + \frac{80(4 \ln(1-z) - 1)}{\epsilon^3} + \frac{16}{\epsilon^2} \left(-50 \ln^2(1-z) \right. \right. \\ & + 25 \ln(1-z) + 7\pi^2 - 6 \Big) + \frac{1}{\epsilon} \left(\frac{4000}{3} \ln^3(1-z) - 1000 \ln^2(1-z) \right. \\ & - 560\pi^2 \ln(1-z) + 480 \ln(1-z) + 2624\zeta(3) + 140\pi^2 - 128 \Big) \\ & - \frac{5000}{3} \ln^4(1-z) + \frac{5000}{3} \ln^3(1-z) + 1400\pi^2 \ln^2(1-z) \\ & - 1200 \ln^2(1-z) - 700\pi^2 \ln(1-z) + 640 \ln(1-z) \\ & \left. \left. + \zeta(3)(3280 - 13120 \ln(1-z)) + \frac{62\pi^4}{5} + 168\pi^2 - 192 \right) \right) \end{aligned}$$

Recent progress

- ▶ First NLP resummations
 - ▶ [I. Moult, I. Stewart, G. Vita, X. Zhu, 1804.04665]
 - ▶ [M. Beneke, A. Broggio, M. Garny, SJ, R. Szafron, L. Vernazza, J. Wang, 1809.10631]
 - ▶ [N. Bahjat-Abbas, D. Bonocore, J. Sinnenhe Damst , E. Laenen, L. Magnea, L. Vernazza, C. White, 1905.13710]
- ▶ End point divergences: thrust and $gg \rightarrow h$
 - ▶ [M. Beneke, M. Garny, SJ, J. Strohm, R. Szafron, L. Vernazza, J. Wang, 2205.04479]
 - ▶ [Z. Liu, M. Neubert, M. Schnubel, X. Wang, 2112.00018]
 - ▶ [C. Anastasiou, A. Penin, 2004.03602] [T. Liu, S. Modi, A. Penin, 2111.01820]
- ▶ Fixed order subtractions
- ▶ TMDs
- ▶ Subleading Effects in Soft-Gluon Emission at One-Loop in Massless QCD
 - ▶ → see talk by T. Schellenberger
- ▶ Flavour physics and QED corrections: $B \rightarrow \mu^+ \mu^-$
 - ▶ [M. Beneke, C. Bobeth, R. Szafron, 1908.07011]
 - ▶ [T. Hurth, R. Szafron, 2301.01739]
 - ▶ [C. Cornell, M. K nig, M. Neubert, 2212.14430]