TOWARDS DEVELOPING QUANTUM CORRECTION TO A NEW WILSON-LINE BASED ACTION FOR GLUODYNAMICS

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Scattering amplitudes in Yang-Mills action on the light-cone

The Yang-Mills action on the constant light-cone time x^+ in the light-cone gauge $\hat{A}^+ = 0$, integrating out the $\hat{A}^-$ fields, has the following form:

$$S_{\mathrm{YM}}^{\mathrm{LC}}\left[A^{\bullet}, A^{\star}\right] = \int dx^+ \left(- \int d^3\mathbf{x}\, \mathrm{Tr}\, \hat{A}^{\bullet}\Box\hat{A}^{\star} + \mathcal{L}_{-++}^{(\mathrm{LC})} + \mathcal{L}_{--+}^{(\mathrm{LC})} + \mathcal{L}_{--++}^{(\mathrm{LC})} \right), \quad (1)$$

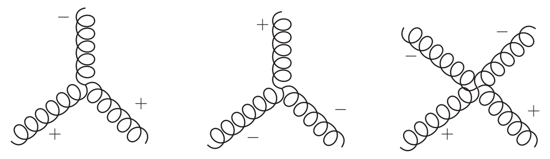

Fig. 1: The Yang-Mills action (1) has two 3-gluon vertices $(++-)$, $(--+)$ and a 4-gluon vertex $(++--)$.

Pure gluonic scattering amplitudes in the Yang-Mills theory:

- Draw all Feynman diagrams using the vertices shown in Fig. 1. Loop diagrams involve integration over the loop momenta.
- The number of diagrams grows factorially, already at the tree level, with the number of external legs making it a cumbersome technique. The final result, on the other hand, can sometimes be expressed with a single term:

$$A_n^{tree}(1^{\pm}, 2^+, \ldots, n^+) = 0; \qquad A_n^{tree}(1^+ \ldots, j^-, \ldots, l^-, \ldots n^+) = \frac{\langle jl\rangle^4}{\langle 12\rangle \cdots \langle n1\rangle}, \quad \langle ij\rangle \equiv \epsilon^{\alpha\beta}(\lambda_i)_\alpha(\lambda_j)_\beta\,. \quad (2)$$

The above amplitudes are tree level color-ordered. The latter are known as *Maximally Helicity Violating* (MHV) [1].

Cachazo-Svrcek-Witten (CSW) Rules

In the Cachazo-Svrcek-Witten (CSW) [2] method the amplitudes are computed via the following rules: (Consider example in Fig. 2.)

- The MHV amplitudes (2) can be used as vertices with the spinors continued off-shell.
- These MHV vertices are glued together using scalar propagator.

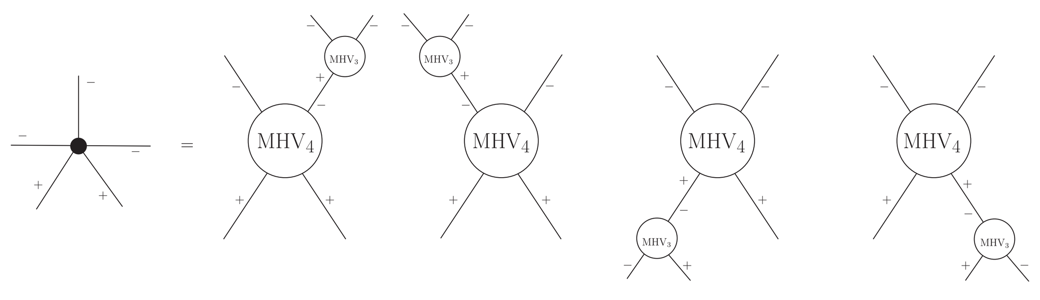

Fig. 2: The contributions to the color-ordered 5-point $\overline{\mathrm{MHV}}$ amplitude, with helicity $(---++)$.

The MHV Lagrangian

Later, an action dubbed as the "MHV action" was derived [3] with all the necessary ingredients to implement the CSW rules:

- The MHV action is obtained from the Yang-Mills action by canonically transforming both the fields to a new pair $(\hat{B}^{\bullet}, \hat{B}^{\star})$:

$$\mathcal{L}_{-+}[A^{\bullet}, A^{\star}] + \mathcal{L}_{-++}[A^{\bullet}, A^{\star}] \longrightarrow \mathcal{L}_{-+}[B^{\bullet}, B^{\star}], \quad (3)$$

- The MHV action consists of an infinite set of MHV vertices $\sum \mathcal{L}_{--+\cdots+}^{(\mathrm{LC})}$ and a scalar propagator:

$$S_{\mathrm{YM}}^{\mathrm{LC}}\left[B^{\bullet}, B^{\star}\right] = \int dx^+ \left(- \int d^3\mathbf{x}\, \mathrm{Tr}\, \hat{B}^{\bullet}\Box\hat{B}^{\star} + \mathcal{L}_{--+}^{(\mathrm{LC})} + \ldots + \mathcal{L}_{--+\cdots+}^{(\mathrm{LC})} + \ldots \right), \quad (4)$$

- The physical meaning of the new fields in the MHV action was first discussed in [4, 5].

In the original work [3] the MHV action was constructed using only analytic properties of the transformations and equivalence theorem for the S-matrix.

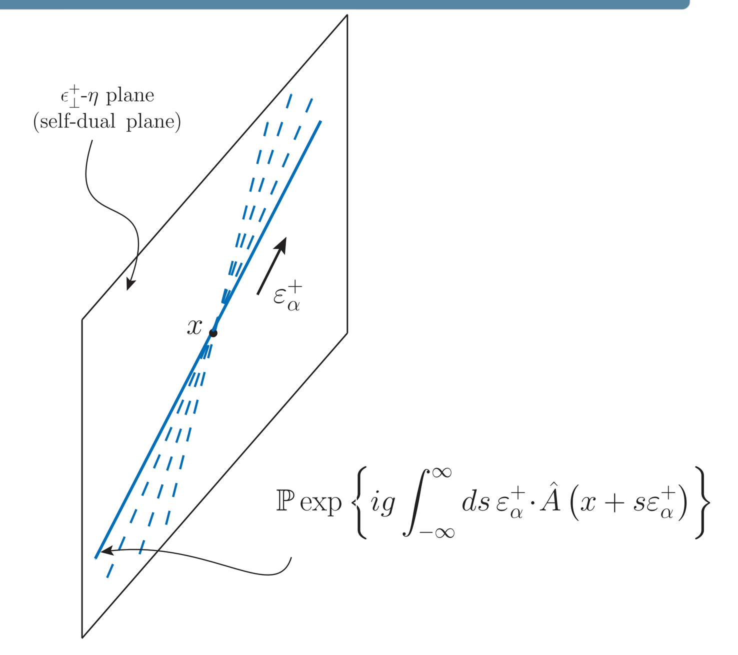

Fig. 3: $B_a^{\bullet}(x)$ is given by the straight infinite Wilson line lying on the plane spanned by $\varepsilon_\alpha^+ = \varepsilon_\perp^+ - \alpha\eta$ (with $\varepsilon_\perp^+ = (0,1,i,0)/\sqrt{2}$, $\eta = (1,0,0,-1)/\sqrt{2}$) and integrated over all α (the dashed lines represent tilted Wilson lines due to the change of α).

Straight infinite Wilson-lines in MHV action

The physical interpretation of $\hat{B}^{\bullet}$, $\hat{B}^{\star}$ as Wilson lines:

- In [4] the $B^{\bullet}$ field in the MHV action was shown to be the straight infinite Wilson line $B_a^{\bullet}[A^{\bullet}](x) = \mathcal{W}_{(+)}^a[A](x)$. Where, for a K^μ, the straight infinite Wilson line functional $\mathcal{W}_{(\pm)}[K]$ reads:

$$\mathcal{W}_{(\pm)}^a[K](x) = \int_{-\infty}^{\infty} d\alpha\, \mathrm{Tr}\left\{\frac{1}{2\pi g}t^a\, \partial_-\, \mathbb{P}\exp\left[ig\int_{-\infty}^{\infty} ds\, \varepsilon_\alpha^{\pm}\cdot\hat{K}\left(x + s\varepsilon_\alpha^{\pm}\right)\right]\right\}, \quad (5)$$

with $\varepsilon_\alpha^{\pm\,\mu} = \varepsilon_\perp^{\pm\,\mu} - \alpha\eta^\mu$.

- The Wilson line $B_a^{\bullet}(x)$ is on the so-called self-dual plane (the plane on which the tensors are self dual) spanned by $\varepsilon_\perp^+$ and η (see Fig. 3). However, on this plane, the Wilson line is not along a fixed direction. It is, rather, integrated over α.
- The $B_a^{\star}(x)$ was shown in [5] to be a similar Wilson line, but with an insertion of the negative helicity gluon field at certain point on the line (see Fig. 4).

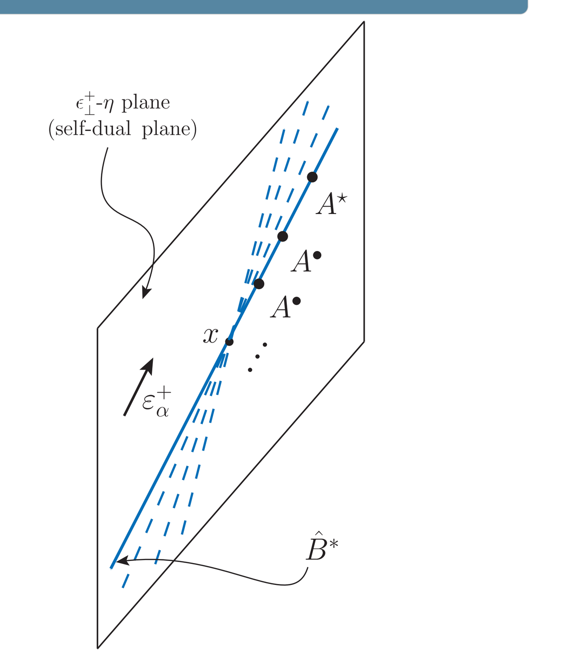

Fig. 4: The $B^{\star}$ field can be represented as the straight infinite Wilson line similar to the one from Fig. 3, but where one $A^{\bullet}$ field has been replaced by the $A^{\star}$ field.

$$B_a^{\star}[A^{\bullet}, A^{\star}](x) = \int d^3\mathbf{y}\left[\frac{\partial_-^2(y)}{\partial_-^2(x)}\frac{\delta\mathcal{W}_{(+)}^a[A](x^+;\mathbf{x})}{\delta A_c^{\bullet}(x^+;\mathbf{y})}\right]A_c^{\star}(x^+;\mathbf{y})\,; \qquad \text{where } \partial_-(x) = \partial/\partial x^- \quad (6)$$

A new Wilson Line based action for gluodynamics

Motivation:

- Eq. (3), eliminates one of the triple gloun vertices $(++-)$. The other triple gloun vertex $(+--)$ still exists in the MHV action. The triple point vertices are not very effective building blocks for calculating amplitudes. Moreover, in the on-shell limit they are zero (for real momenta). The smallest amplitude which is finite in the on-shell limit is the four-point MHV.
- It is natural to think that the $A^{\star}$ fields, in $B^{\star}$, belong to a Wilson line living within the anti-self-dual plane spanned by ε_α^- (recall that the $B^{\bullet}$ lives on the plane spanned by ε_α^+). Therefore, in [5], we conjectured that the solution (6) should be a cut through a bigger structure, spanning over both the planes.

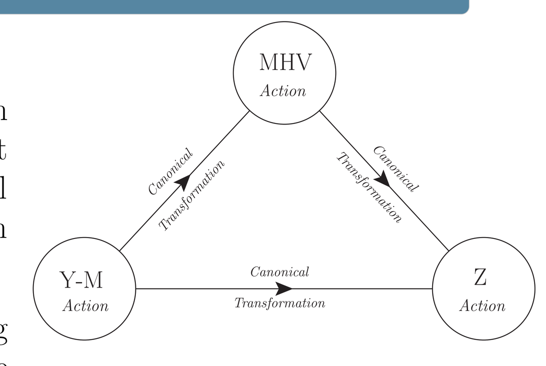

Fig. 5: Two ways to derive the new action. First is the direct method which involves the generating functional (8). Second involves two consecutive canonical field transformation.

In [6], motivated by the arguments above:

- We found a more general canonical transformation based on path ordered exponential of the gauge fields, extending over both the self-dual and the anti-self-dual planes:

$$\left\{\hat{A}^{\bullet}, \hat{A}^{\star}\right\} \to \left\{\hat{Z}^{\bullet}[A^{\bullet}, A^{\star}], \hat{Z}^{\star}[A^{\bullet}, A^{\star}]\right\}, \quad (7)$$

We schematically depict the structure of the $Z^{\star}$ field in Fig. 6.

- The transformation maps the kinetic term and both the triple-gluon vertices of the Yang-Mills action into a free term in the new action. The generating functional $\mathcal{G}[A^{\bullet}, Z^{\star}]$ reads:

$$\mathcal{G}[A^{\bullet}, Z^{\star}](x^+) = -\int d^3\mathbf{x}\;\; \hat{\mathcal{W}}_{(-)}^{-1}[Z](x)\; \partial_-\hat{\mathcal{W}}_{(+)}[A](x)\,, \quad (8)$$

- The transformation from the Yang-Mills action into the new action generated by the functional (8) is equivalent to two canonical transformations: first transforming the self-dual part of the Yang-Mills action to the kinetic term in MHV action, and then transforming the anti-self-dual part in the latter to kinetic term in the new action (see Fig. 5).

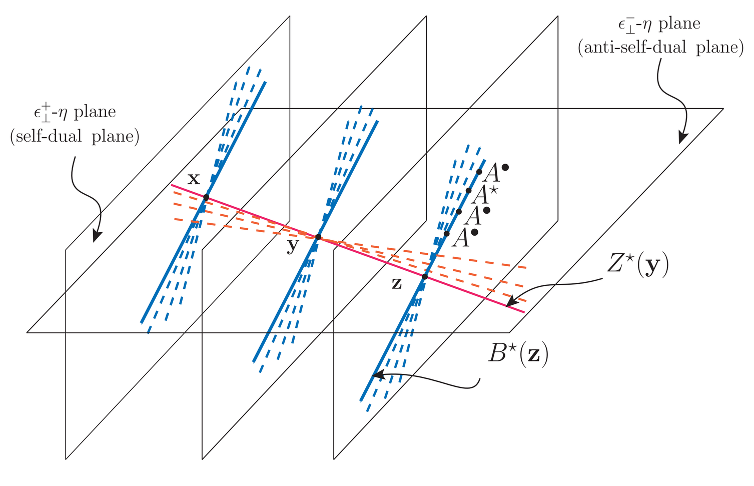

Fig. 6: Schematic presentation of the geometric structure of the $Z^{\star}$ field (the structure of $Z^{\bullet}$ is quite similar). $Z^{\star}$ field is a Wilson line (with exactly the same analytic form as $B^{\bullet}$) of only $B^{\star}$ fields on anti-self-dual plane. Notice, each vertical plane is self-dual plane with $B^{\star}$ embedded in it as was shown in Fig. 4 .

The new action, dubbed as *Z-field action* hereafter, has the following generic structure:

$$S_{\mathrm{YM}}^{\mathrm{LC}}\left[Z^{\bullet}, Z^{\star}\right] = \int dx^+ \Big\{ - \int d^3\mathbf{x}\, \mathrm{Tr}\, \hat{Z}^{\bullet}\Box\hat{Z}^{\star}$$
$$+ \mathcal{L}_{--++}^{(\mathrm{LC})} + \mathcal{L}_{--+++}^{(\mathrm{LC})} + \mathcal{L}_{--++++}^{(\mathrm{LC})} + \ldots$$
$$+ \mathcal{L}_{---++}^{(\mathrm{LC})} + \mathcal{L}_{---+++}^{(\mathrm{LC})} + \mathcal{L}_{---++++}^{(\mathrm{LC})} + \ldots$$
$$\vdots$$
$$+ \mathcal{L}_{---\cdots-++}^{(\mathrm{LC})} + \mathcal{L}_{---\cdots-+++}^{(\mathrm{LC})} + \mathcal{L}_{---\cdots-++++}^{(\mathrm{LC})} + \ldots \Big\}, \quad (9)$$

where the n-point color ordered, split helicity interaction vertex, $n \geq 4$, that couples m negative helicity fields, $m \geq 2$, and $n - m$ positive helicity fields, has a compact analytic form [6] (shown diagrammatically in Fig. 7). A similar compact version for the vertex can be derived for any helicity configuration.

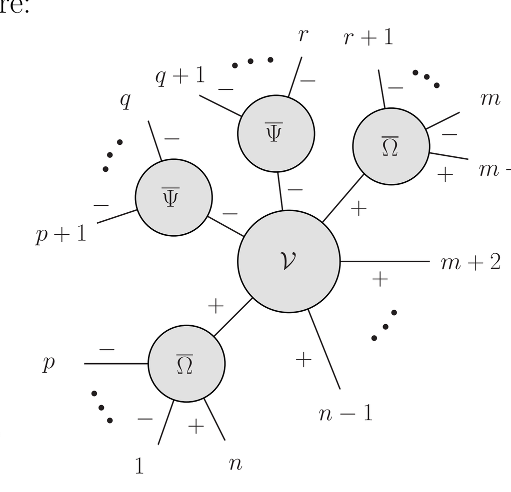

Fig. 7: A general contribution to the Z-theory vertex obtained by canonically transforming the MHV action. The central blob is the MHV vertex. $\overline{\Omega}$ and $\overline{\Psi}$ are the expansion kernels of B fields in terms of Z fields [6].

The Z-field action has the following properties:

i) There are no three point interaction vertices.

ii) At the classical level there are no all-plus, all-minus, as well as $(-+\cdots+)$, $(-\cdots-+)$ vertices.

iii) There are MHV vertices, $(--+\cdots+)$, corresponding to MHV amplitudes in the on-shell limit.

iv) There are $\overline{\mathrm{MHV}}$ vertices, $(-\cdots-++)$, corresponding to $\overline{\mathrm{MHV}}$ amplitudes in the on-shell limit.

v) All vertices have the form which can be easily calculated.

Pure gluonic tree amplitudes using the new action

Using this new action we computed several tree amplitudes and found them to be in agreement with the standard methods.

- The MHV and $\overline{\mathrm{MHV}}$ vertices alone give the corresponding on-shell amplitudes. It is easily obtained from Fig. 7.

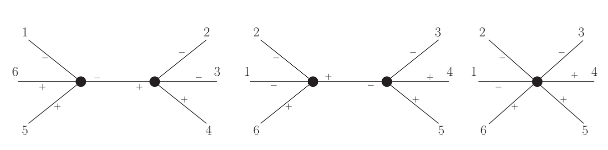

Fig. 8: Diagrams contributing to 6-point NMHV amplitude $(---+++)$.

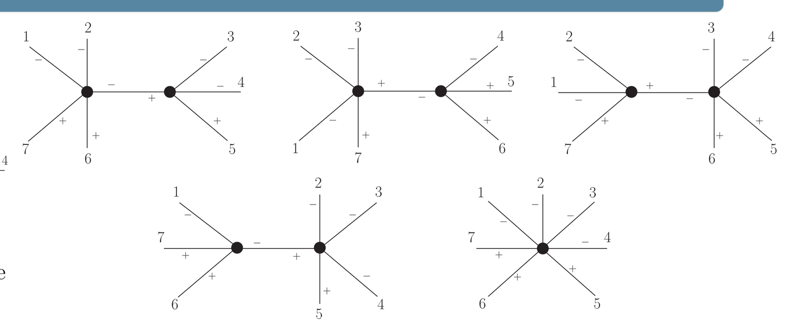

Fig. 9: Diagrams contributing to 7-point NNMHV amplitude $(----+++)$.

- For 6-point Next-To-MHV (NMHV) amplitude $(---+++)$ we have three contributions shown in Fig. 8.
- For 7-point NNMHV amplitude $(----+++)$ we had five contributing diagrams depicted in Fig. 9.
- For 8-point NNMHV $(----++++)$, we had 13 contributing diagrams. The absence of triple-gluon vertices resulted in fewer diagrams to compute amplitudes, when compared to the CSW method.
- The total number of contributions for split helicity tree amplitudes follow Delannoy numbers.

Delannoy numbers in the new action

The number of diagrams obtained for the case of split helicity tree amplitudes using the Z-field action appear random. However, when tabulated, we discovered that the total number of diagrams matched the Delannoy numbers [7] (see Fig. 10). Delannoy numbers describe the total number of paths from the southwest corner $(0,0)$ of a 2D lattice to the northeast corner (m,n), using only single steps north, northeast, or east. The appearance of these numbers, in association with the Z-field action, could be understood as follows:

n \ m	0	1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1	1
1	1	3	5	7	9	11	13	15	17
2	1	5	13	25	41	61	85	113	145
3	1	7	25	63	129	231	377	575	833
4	1	9	41	129	321	681	1289	2241	3649
5	1	11	61	231	681	1683	3653	7183	13073
6	1	13	85	377	1289	3653	8989	19825	40081
7	1	15	113	575	2241	7183	19825	48639	108545
8	1	17	145	833	3649	13073	40081	108545	265729
9	1	19	181	1159	5641	22363	75517	224143	598417

Fig. 10: The Delannoy numbers. $m+2$ and $n+2$ represents the total number of positive and negative helicities in a $m+n+4$ point split helicity tree amplitude.

- The vertices of the Z-field action can be arranged into a 2D lattice keeping the 4-point MHV at origin and increasing the positive helicity step wise along the horizontal and negative helicity along the vertical.
- The contributions to the tree amplitudes include three types: the vertex itself; and combining vertices for which we have two ways $(+-)$ (positive helicity of one vertex combined with negative of other through a propagator) and $(-+)$. These correspond to the tree allowed moves in Delannoy numbers.

Quantum corrections to the new action

With the success of the Z-field action generalizing the MHV method in computing tree amplitudes for various pure gluonic scattering processes, the next natural step is to understand the *quantum corrections*. There is an immediate difficulty:

- Missing terms: Consider one-loop $(++\cdots+)$ amplitudes. This amplitude is non-zero [8, 9]. Such amplitudes cannot be calculated within the Z-field action, since every vertex has at least two and two $-$ helicity fields.

A similar problem appears in the MHV action and a couple of different methods have been used in the literature to recover the all-plus helicity loop amplitudes [10, 11].We, however, will use the technique of *effective action* [12]. To derive this [13],

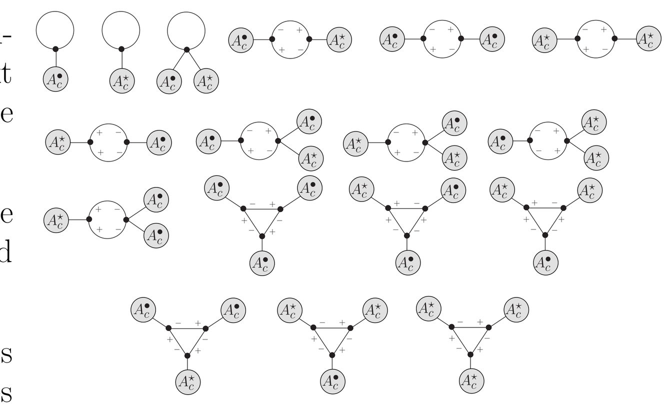

Fig. 11: One loop terms, up to 3-point, generated by the trace of the log term in the one-loop effective Yang-Mills action.

- We start with the generating functional for the Green functions for the full Yang-Mills action:

$$Z_{\mathrm{YM}}[J] = \int [dA]\, e^{i\left(S_{\mathrm{YM}}^{\mathrm{LC}}[A] + \int d^4x\, \mathrm{Tr}\hat{J}_j(x)\hat{A}^j(x)\right)}; \quad \text{where } j = \bullet, \star\,. \quad (10)$$

- Expand the action, up to second order in fields, around the classical solution $A_c[J] = (A_c^{\bullet}[J], A_c^{\star}[J])$. The higher order terms are necessary for corrections beyond one loop. The linear term vanishes due to the classical equations of motion, whereas the integration over the quadratic term gives

$$Z_{\mathrm{YM}}[J] \approx \exp\left\{iS_{\mathrm{YM}}^{\mathrm{LC}}[A_c] + i\int d^4x\, \mathrm{Tr}\hat{J}_i(x)\hat{A}_c^i(x) - \frac{1}{2}\mathrm{Tr}\ln\left(\frac{\delta^2 S_{\mathrm{YM}}^{\mathrm{LC}}[A_c]}{\delta\hat{A}^i(x)\delta\hat{A}^j(y)}\right)\right\} \quad (11)$$

where the second trace is over color and position. The logarithm term above generates the one loop contributions (diagrammatically represented in Fig. 11).

Applying Mansfield transformation to (11) we get:

$$Z_{\mathrm{MHV}}[J] \approx \exp\left\{iS_{\mathrm{MHV}}[B] + i\int d^4x\, \hat{J}_i(x)\hat{A}_c^i[B(x)] - \frac{1}{2}\ln\left(\frac{\delta^2 S_{\mathrm{YM}}[A_c[B]]}{\delta\hat{A}^i(x)\delta\hat{A}^j(y)}\right)\right\} \quad (12)$$

The generic form of the loop contributions arising from (12) are shown in Fig. 12. Notice, the pure self-dual contributions and the ones obtained by the mixing of MHV and self-dual vertex are no longer missing. This is because the cancellations that result in the Mansfield's transformation does not take place in the log term. In fact we computed one-loop $(++++)$ $(+++-)$ amplitudes and found agreement with known results.

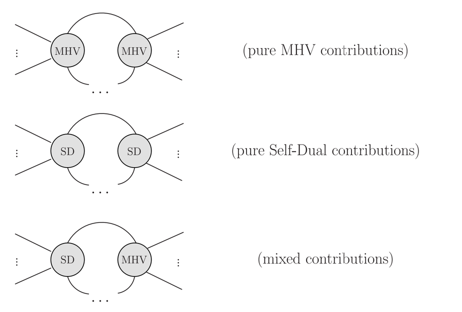

Fig. 12: One loop terms generated by the one-loop effective MHV action.

- The effective action approach separates the classical action from the loop contributions thereby making it ideal to develop quantum correction to such formulations of Yang-Mills theory.
- Having one-loop effective action for the Yang-Mills/MHV action, the one-loop effective Z-field action can be derived following either of the two ways shown in Fig. 5.

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