t-channel singularity in the early Universe studies
the problem and its solution

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Motivation: how much dark matter is there?
cosmological Boltzmann equation

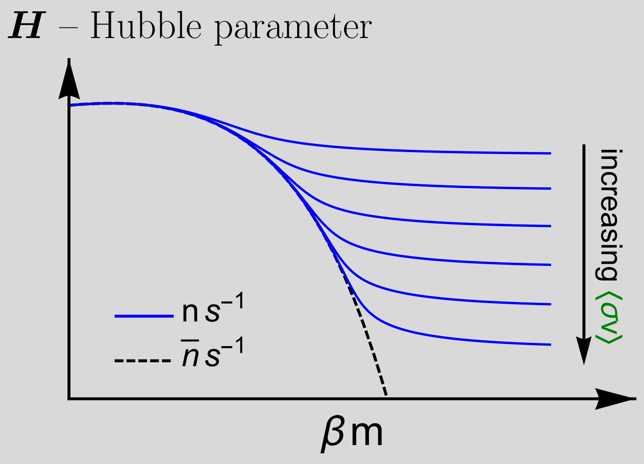
goal: connect **amount of particles** x to **reaction rate**

$$\dot{n}_x + 3Hn_x = -\sum_{a,i,j} \langle\sigma v\rangle_{xa\to ij}\left(n_x n_a - \bar{n}_x\bar{n}_a\frac{n_i n_j}{\bar{n}_i\bar{n}_j}\right) + \text{decay terms}$$

$$\langle\sigma v\rangle_{xa\to ij} \equiv \int \underbrace{d\Phi_x\, d\Phi_a\, d\Phi_i\, d\Phi_j}_{\text{contains } ds\,dt} |\mathcal{M}|^2_{xa\to ij}(2\pi)^4\delta^{(4)}(p_x+p_a-p_i-p_j)\frac{\bar{f}_x\bar{f}_a}{\bar{n}_x\bar{n}_a}$$

is it always well-defined?

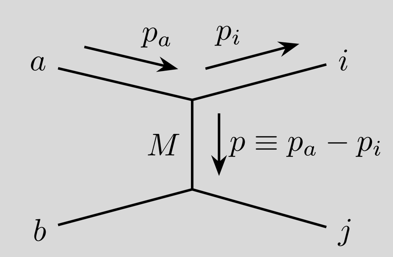
$\langle\sigma v\rangle_{xa\to ij}$ – reaction rate (thermally averaged cross section)
n_k – number densities ($k = x, a, i, j$)
$\bar{n}_k$ – equilibrium number densities ($k = x, a, i, j$)
$\bar{f}_k$ – equilibrium distribution functions ($k = x, a$)
H – Hubble parameter



Exemplary **solution of the Boltzmann equation** for a simple one-component dark matter case. The plot uses inverse temperature of the Universe (β) as a time scale and s denotes entropy density. The stronger **interactions** are, the smaller final **density** is.

What if the matrix element is singular?
t-channel singularity

the singularity



$$t \equiv p^2$$

$$|\mathcal{M}|^2_{ab\to ij} = \frac{|A|^2(s,t)}{(t-M^2)^2} \xrightarrow{\int dt} \infty$$

if $t_{\min} < M^2 < t_{\max}$
integration over dt produces **infinity**

conditions for occurrence

$$t_{\min} < M^2 < t_{\max} \quad\Leftrightarrow\quad s_1 < s < s_2$$

to get $\langle\sigma v\rangle_{ab\to ij}$ we integrate over $s \in (s_{\min}, \infty)$

so $\quad t_{\min} < M^2 < t_{\max} \quad\Leftrightarrow\quad s_{\min} < s_2$

(s_1, s_2, $s_{\min}$ – functions of m_a, m_b, m_i, m_j, M)

$\Rightarrow$ **condition** for singular $\langle\sigma v\rangle_{ab\to ij}$:

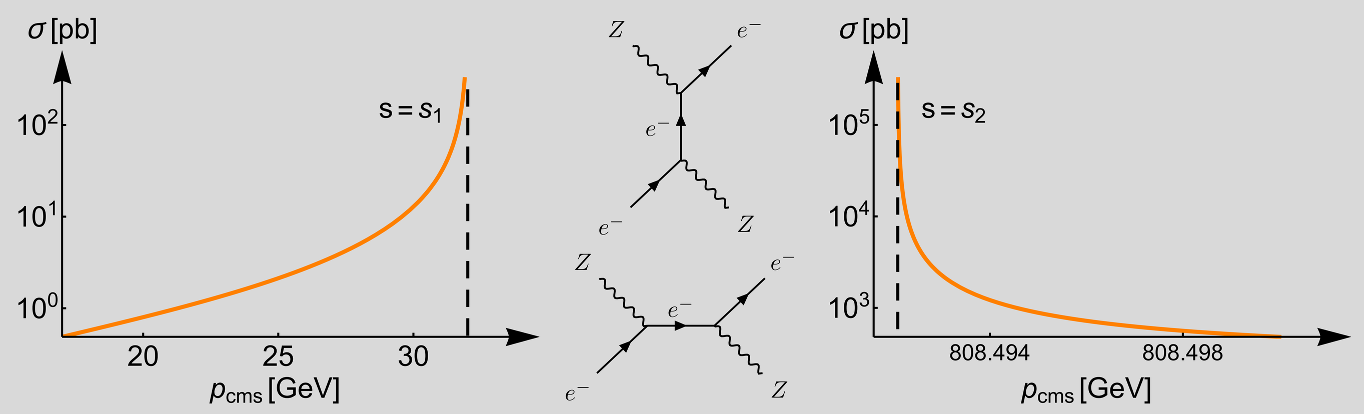
either

$m_a > m_i + M$ and $m_j > m_b + M$

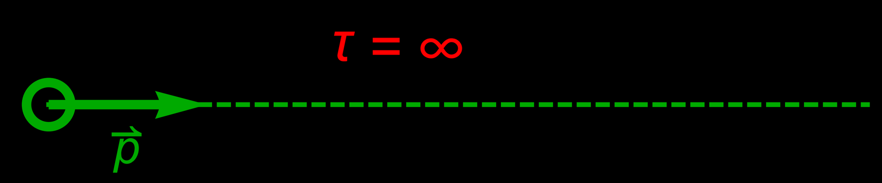
or

$m_i > m_a + M$ and $m_b > m_j + M$

example in the Standard Model: weak Compton scattering ($Ze^- \to e^- Z$)



in vacuum

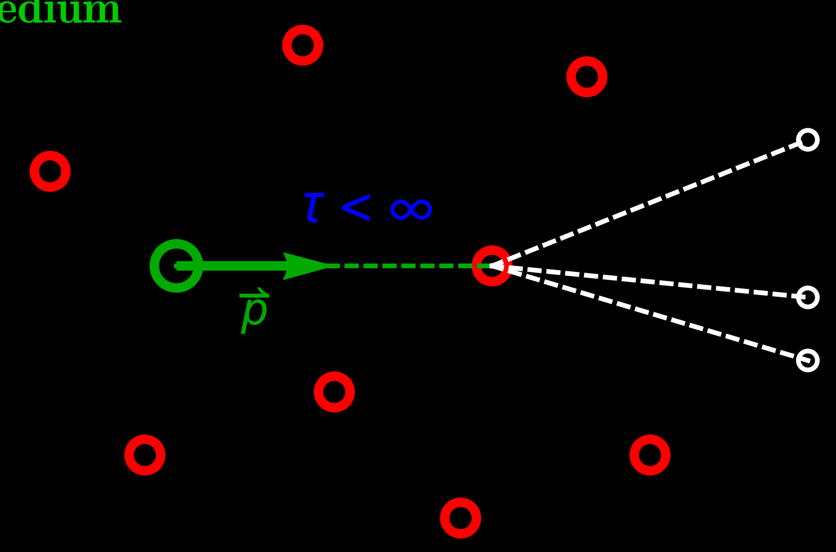


Regularization: idea

mediator interacts with **the medium**
⇓
finite lifetime
⇓
effective **thermal decay width**
⇓
finite resummed propagator

$$\frac{A}{t-M^2} \to \frac{A}{t-M^2+i\,M\Gamma(E,T)}$$

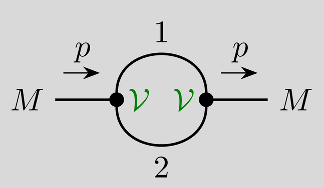
in a medium



Regularization: details

method: calculate **1-loop corrections** within **thermal field theory** (real-time formalism used)

$$M\Gamma(E_p,T) = \Im\left[\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\Big[\mathcal{V}\,\Delta_1^+(k+p)\,\mathcal{V}\,\Delta_2^{\text{sym}}(k,T) + \mathcal{V}\,\Delta_1^{\text{sym}}(k,T)\,\mathcal{V}\,\Delta_2^-(k-p)\Big]\right]$$



note: form of the propagators $\Delta_j^\pm$, Δ_j^{sym} ($j = 1, 2$) depends on the **spin:**

scalar propagators

$$\Delta_j^\pm(p) = \Delta^\pm(p)$$
$$\Delta_j^{\text{sym}}(p,T) = \Delta_B^{\text{sym}}(p,T)$$

fermion propagators

$$G_j^\pm(p) = (\not{p}+m)\,\Delta^\pm(p)$$
$$G_j^{\text{sym}}(p,T) = (\not{p}+m)\,\Delta_F^{\text{sym}}(p,T)$$

vector propagators

$$D_j^\pm(p)_{\mu\nu} = \left[-g_{\mu\nu}+\frac{p_\mu p_\nu}{m^2}\right]\Delta^\pm(p)$$
$$D_j^{\text{sym}}(p,T)_{\mu\nu} = \left[-g_{\mu\nu}+\frac{p_\mu p_\nu}{m^2}\right]\Delta_B^{\text{sym}}(p,T)$$

where

$$\Delta^\pm(p) \equiv \frac{1}{p^2-m^2\pm i\,\text{sgn}\,p_0\,0^+}$$
$$\Delta_{\text{F,B}}^{\text{sym}}(p,T) \equiv -\frac{i\pi}{E_p}\Big(\delta(E_p-p_0)+\delta(E_p+p_0)\Big)\,f_{\text{F,B}}(\beta E_p)$$

$\mathcal{V}$ – vertex factor

$$f_{\text{F,B}}(x) \equiv \begin{cases}\frac{e^x-1}{e^x+1} & \text{for fermions}\\ \frac{e^x+1}{e^x-1} & \text{for bosons}\end{cases}$$

After tedious calculations:

(1: boson, 2: boson) $\quad M\Gamma(E_p,T) = \frac{1}{16\pi}\frac{X_0}{\beta|p|}\left[\ln\frac{e^{\beta(b+a)}-1}{e^{\beta(b-a)}-1} - \ln\frac{e^{\beta(b+a)}e^{-\beta E_p}-1}{e^{\beta(b-a)}e^{-\beta E_p}-1}\right]$

(1: boson, 2: fermion) $\quad M\Gamma(E_p,T) = \frac{1}{16\pi}\frac{X_0}{\beta|p|}\left[\ln\frac{e^{\beta(b+a)}-1}{e^{\beta(b-a)}-1} - \ln\frac{e^{\beta(b+a)}e^{-\beta E_p}+1}{e^{\beta(b-a)}e^{-\beta E_p}+1}\right]$

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where

$$\beta \equiv \frac{1}{T} \qquad |p| \equiv \sqrt{E_p^2-M^2}$$

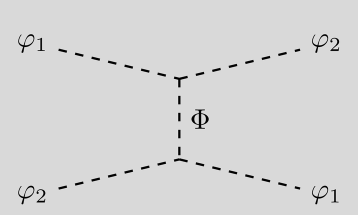
$$a \equiv \frac{|p|}{2M^2}\sqrt{m_1^2-(m_2-M)^2}\sqrt{m_1^2-(m_2+M)^2} \qquad b \equiv \frac{m_1^2-m_2^2+M^2}{2M^2}E_p$$

X_0 – vertex- and spin-dependent factor, calculated in a given model

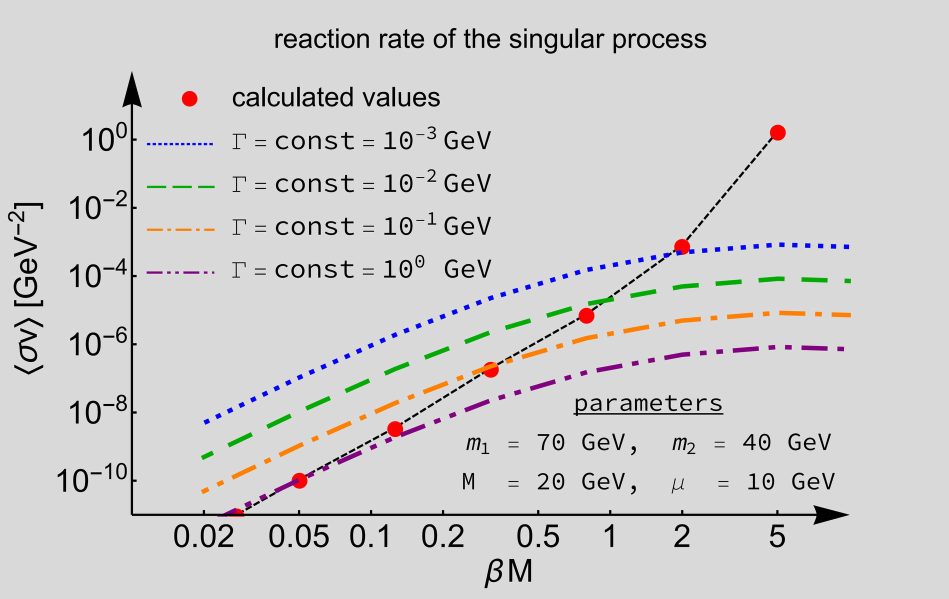
Example: 3-scalar toy model

particles: φ_1, φ_2, Φ **parameters:** μ (coupling constant), masses

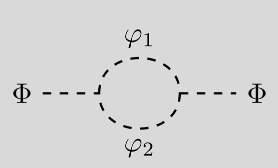
singular process



singular if
$\mathbf{m_1 > m_2 + M}$
or $\mathbf{m_2 > m_1 + M}$



regularizing loop

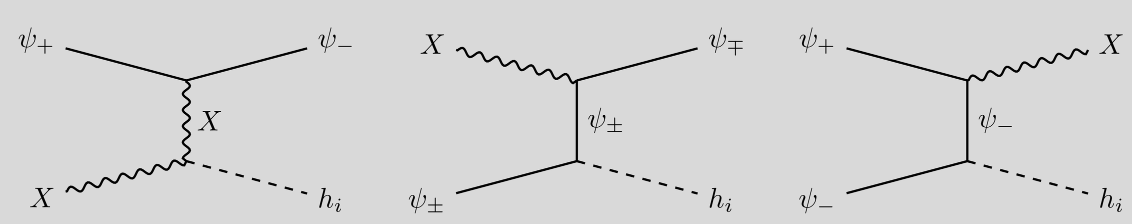


B. Grządkowski, M. Iglicki, S. Mrówczyński
doi: 10.1016/j.nuclphysb.2022.115967

Example: vector-fermion dark matter

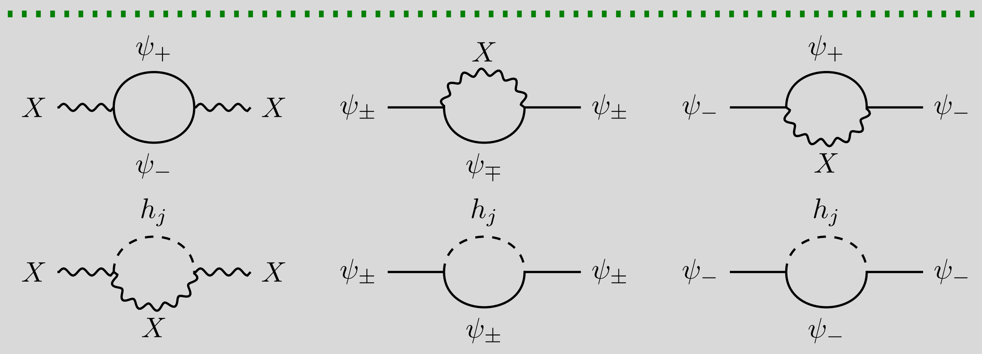
particles: X (gauge vector), ψ_+, ψ_- (Majorana fermions), h_1, h_2 (scalar states, $m_{h_1} = 125$ GeV), SM **parameters:** g_x (coupling constant), α (scalar sector mixing angle), masses

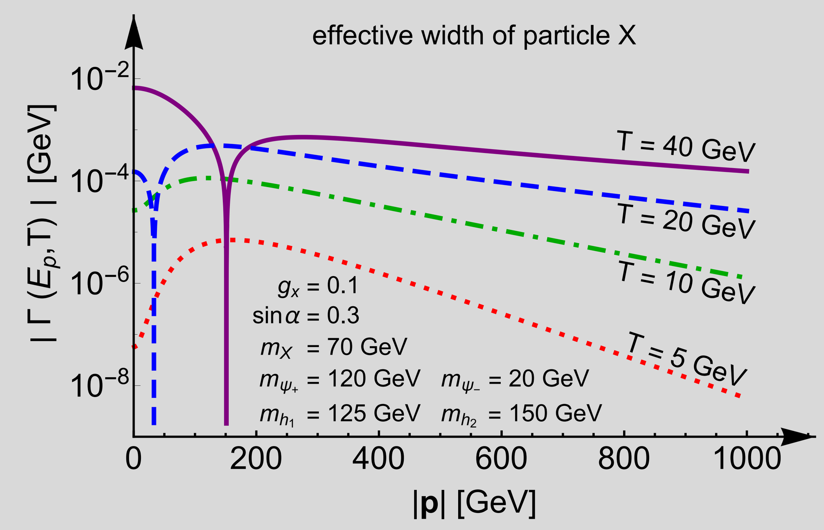
singular processes



singular if $\begin{cases}\mathbf{m_{\psi_+} > m_{\psi_-} + m_X}\\ \mathbf{m_{h_i} > 2\,m_X}\end{cases}$ | singular if $\begin{cases}\mathbf{m_X > m_{\psi_+} + m_{\psi_-}}\\ \mathbf{m_{h_i} > 2\,m_{\psi_\pm}}\end{cases}$ | singular if $\begin{cases}\mathbf{m_{\psi_+} > m_X + m_{\psi_-}}\\ \mathbf{m_{h_i} > 2\,m_{\psi_-}}\end{cases}$

regularizing loops





problem: different contributions can **cancel** each other at some $E = E_0$
$\Rightarrow \quad \Gamma(E_0, T) = 0$

A. Ahmed, M. Duch, B. Grządkowski, M. Iglicki
doi: 10.1140/epjc/s10052-018-6371-2 (model)
stay tuned! → M. Iglicki arXiv:2211.xxxxx (regularization)