Reduction at NLO and beyond II

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Reduction at the integrand level

Over the last few years very important activity to extend unitarity and integrand level reduction ideas beyond one loop

J. Gluza, K. Kajda and D. A. Kosower, "Towards a Basis for Planar Two-Loop Integrals," Phys. Rev. D **83** (2011) 045012 [arXiv:1009.0472 [hep-th]].

D. A. Kosower and K. J. Larsen, "Maximal Unitarity at Two Loops," Phys. Rev. D **85** (2012) 045017 [arXiv:1108.1180 [hep-th]].

P. Mastrolia and G. Ossola, "On the Integrand-Reduction Method for Two-Loop Scattering Amplitudes," JHEP **1111** (2011) 014 [arXiv:1107.6041 [hep-ph]].

S. Badger, H. Frellesvig and Y. Zhang, "Hepta-Cuts of Two-Loop Scattering Amplitudes," JHEP **1204** (2012) 055 [arXiv:1202.2019 [hep-ph]].

Y. Zhang, "Integrand-Level Reduction of Loop Amplitudes by Computational Algebraic Geometry Methods," JHEP **1209** (2012) 042 [arXiv:1205.5707 [hep-ph]].

P. Mastrolia, E. Mirabella, G. Ossola and T. Peraro, "Integrand-Reduction for Two-Loop Scattering Amplitudes through Multivariate Polynomial Division," arXiv:1209.4319 [hep-ph].

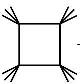
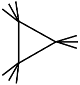
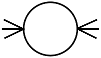
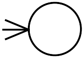
P. Mastrolia, E. Mirabella, G. Ossola and T. Peraro, "Multiloop Integrand Reduction for Dimensionally Regulated Amplitudes," arXiv:1307.5832 [hep-ph].

The one loop paradigm

basis of scalar integrals:

G. Passarino and M. J. G. Veltman, Nucl. Phys. B **160** (1979) 151.

Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **425** (1994) 217 [arXiv:hep-ph/9403226].

$\mathcal{A} = \sum d_{i_1 i_2 i_3 i_4}$  $+ \sum c_{i_1 i_2 i_3}$  $+ \sum b_{i_1 i_2}$  $+ \sum a_{i_1}$  $+ R$

$a, b, c, d \rightarrow$ cut-constructible part $\qquad R \rightarrow$ rational terms

$$\mathcal{A} = \sum_{I \subset \{0,1,\cdots,m-1\}} \int \frac{\mu^{(4-d)} d^d q}{(2\pi)^d} \frac{\bar{N}_I(\bar{q})}{\prod_{i \in I} \bar{D}_i(\bar{q})}$$

OPP integrand level:

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0,i_1,i_2,i_3) + \tilde{d}(q;i_0,i_1,i_2,i_3)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0,i_1,i_2) + \tilde{c}(q;i_0,i_1,i_2)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0,i_1) + \tilde{b}(q;i_0,i_1)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q;i_0)\right] \prod_{i\neq i_0}^{m-1} D_i
\end{aligned}
$$

$\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ are "spurious" terms (vanish upon integration). Their q-dependence is known

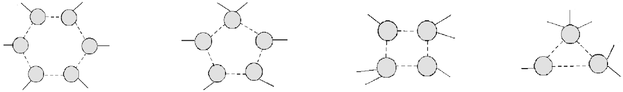
Ossola, Papadopoulos and Pittau, Nucl. Phys. B 763, 147 (2007)

Can be solved either using cuts or simply by polynomial fitting

The one-loop calculation in a nutshell

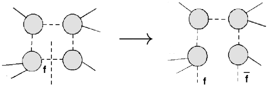
The computation of $pp(p\bar{p}) \to e^{+}\nu_{e}\mu^{-}\bar{\nu}_{\mu}b\bar{b}$ involves up to six-point functions.
The most generic integrand has therefore the form

$$\mathcal{A}(q) = \sum \underbrace{\frac{N_i^{(6)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_5}}} + \underbrace{\frac{N_i^{(5)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_4}}} + \underbrace{\frac{N_i^{(4)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_3}}} + \underbrace{\frac{N_i^{(3)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\bar{D}_{i_2}}} + \cdots$$



In order to apply the OPP reduction, `HELAC` evaluates numerically the numerators $N_i^6(q), N_i^5(q), \ldots$ with the values of the loop momentum q provided by `CutTools`

- generates all inequivalent partitions of 6,5,4,3... blobs attached to the loop, and check all possible flavours (and colours) that can be consistently running inside
- hard-cuts the loop (q is fixed) to get a $n+2$ *tree-like* process



The R_2 contributions (rational terms) are calculated in the same way as the tree-order amplitude, taking into account *extra vertices*

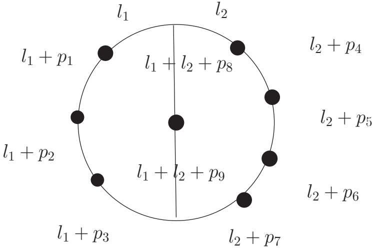
Two-loop amplitudes

- Reduction at the integrand level

Two-loop amplitudes

- Reduction at the integrand level
- Generic two-loop graph: iGraph

R. H. P. Kleiss, I. Malamos, C. G. Papadopoulos and R. Verheyen, arXiv:1206.4180 [hep-ph].



$$D(l_1+p_i)\ ,\ D(l_2+p_j)\ ,\ D(l_1+l_2+p_k)$$

Two-loop amplitudes

The general strategy consists in finding function $x_j \equiv x_j(l_1, l_2)$

$$\sum_{j=1}^{n_1} x_j D(l_1 + p_j) + \sum_{j=n_1+1}^{n_1+n_2} x_j D(l_1 + l_2 + p_j) + \sum_{j=n_1+n_2+1}^{n} x_j D(l_2 + p_j) = 1 \quad .$$

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Let us go a step back at one loop

$$1 = T_1(q) D_1 + T_2(q) D_2 + \cdots + T_n(q) D_n$$

W. L. van Neerven and J. A. M. Vermaseren, Phys. Lett. B **137** (1984) 241.

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Hilbert's Nullstellensatz theorem

Hilbert's Nullstellensatz (German for "theorem of zeros," or more literally, "zero-locus-theorem" see Satz) is a theorem which establishes a fundamental relationship between geometry and algebra. This relationship is the basis of algebraic geometry, an important branch of mathematics. It relates algebraic sets to ideals in polynomial rings over algebraically closed fields. This relationship was discovered by David Hilbert who proved Nullstellensatz and several other important related theorems named after him (like Hilbert's basis theorem).

$$1 = g_1 f_1 + \cdots + g_s f_s \quad g_i, f_i \in k[x_1, \ldots, x_n]$$

Janos Kollar, J. Amer. Math. Soc., Vol. 1, No. 4. (Oct., 1988), pp 963-975

$$\deg g_i f_i \leq \max\{3, d\}^n \quad d = \max \deg f_i \quad 3^8 = 6561$$

M. Sombra, Adv. in Appl. Math. 22 (1999), 271-295

$$\deg g_i f_i \leq 2^{n+1} \quad 2^9 = 512$$

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Constant terms: $T_j(q) = x_j$

$$q^2 \sum_{j=1}^{n} x_j + 2 q_\mu \sum_{j=1}^{n} x_j {p_j}^\mu + \sum_{j=1}^{n} x_j \mu_j = 1 \quad .$$

$$\sum_{j=1}^{n} x_j = 0 \quad , \quad \sum_{j=1}^{n} x_j {p_j}^\mu = 0 \quad , \sum_{j=1}^{n} x_j \mu_j = 1$$

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$$\sum_{j=1}^{n} x_j = 0 \quad , \quad \sum_{j=1}^{n} x_j p_j{}^\mu = 0 \ , \sum_{j=1}^{n} x_j \mu_j = 1$$

- solution exists for $n = 6$ $d = 4$

Two-loop amplitudes

Linear terms $T(q) = P_1(q)$, count tensor structures:

$$1 \ , \ q^\mu \ , \ q^\mu q^\nu \ , \ q^2 q^\mu \ .$$

There are, for $d = 4$, therefore $1+4+10+4 = 19$ independent tensor structures.

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In d dimensions, tensor up to rank k, $N(d, k)$ number of independent tensor structures

$$N(d,k) = \binom{d-1+k}{k} + \sum_{p=0}^{k+1} \binom{d-1+p}{p} \ . \tag{1}$$

In the table below we give the results for various ranks and dimensionalities.

k	0	1	2	3	4
d =1	3	4	5	6	7
2	4	8	13	19	26
3	5	13	26	45	71
4	6	19	45	90	161
5	7	26	71	161	322
6	8	34	105	266	588

Values of $N(d, k)$

The OPP-"miracle" is that the OPP equation works with only 10(6) different coefficients

$$1 = \sum_{i=1}^{5} D_i(q)(c_i^{(0)} + c_i^{(1)}\epsilon_i(q))$$

all $c_i^{(1)}$ being equal! rank deficient problems

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Back to two loops: iGraphs can be denoted by the triplet (n_1, n_2, n_3), $n = n_1 + n_2 + n_3$

$$n_{1,2,3} \leq 4\ (= d)\ , \quad n_1 + n_2 + n_3 \leq 11\ (= 2d + 3)\ .$$

Linear terms

$$x_i = a_i + \sum_j b_{ij}(l_1 \cdot t_j) + \sum_j c_{ij}(l_2 \cdot t_j)$$

$$T(d) = (4d^2 + 18d + 2)/2$$

n	$d=6$	$d=5$	$d=4$	$d=3$	$d=2$	$d=1$
3	39-0	33-0	27-0	21-0	15-0	9-0
4	52-0	44-0	36-0	28-0	20-0	12-2
5	65-1	55-1	45-1	35-1	25-1	15-5
6	78-3	66-3	54-3	42-3	30-3	
7	91-6	77-6	63-6	49-6	35-8	
8	104-10	88-10	72-10	56-10		
9	111-15	99-15	81-15	63-17		
10	130-21	110-21	90-21			
11	143-28	121-28	99-30			
12	156-36	132-36				
13	169-45	143-47				
14	182-55					
15	195-55					
$T(d)$	127	96	69	46	27	10

Quadratic terms

$$x_i = a_i + \sum_j b_{ij}(l_1 \cdot t_j) + \sum_j c_{ij}(l_2 \cdot t_j) + \sum_{j \le k} d_{ijk}(l_1 \cdot t_j)(l_1 \cdot t_k) + \cdots$$

$$T(d) = 4d^3/3 + 10d^2 + 20d/3 - 2 \qquad (2)$$

n	$d=4$	$d=3$	$d=2$
3	135-4	84-3	45-3
4	180-6	128-6	60-6
5	225-18	140-16	75-15
6	270-38	168-32	90-30
7	315-65	196-53	
8	360-98	224-80	
9	405-136	252-108	
10	450-180		
11	495-225		
$T(d)$	270	144	60

Cubic terms

$$x_i = a_i + \sum_j b_{ij}(l_1 \cdot t_j) + \cdots + \sum_{j \le k} g_{ijkl}(l_1 \cdot t_j)(l_1 \cdot t_k)(l_1 \cdot t_l) + \cdots$$

$$T(d) = 2d^4/3 + 22d^3/3 + 71d^2/6 + d/6 + 1$$

n	$d = 6$	$d = 5$	$d = 4$	$d = 3$
5				420/332
6				504/352
7			1155/803	588/360
8			1320/823	672/360
9		2574/1603	1485/831	
10		2860/1623	1650/831	
11	5005/2848	3146/1631		
12	5460/2868	3432/1631		
13	5915/2876			
14	6370/2876			
$T(d)$	2876	1631	831	360

A bit of algebraic geometry

D. Cox, J. Little, D. O'Shea *Ideals, Varieties and Algorithms*

• Set of all polynomials $k[x_1, \ldots, x_n]$ $f = \sum a_\alpha x^\alpha, a_\alpha \in k$,
$x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$

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• Ideal: I subset of $k[x_1, \ldots, x_n]$, (i) $0 \in I$, (ii) $f, g \in I$ then $f + g \in I$, (iii) $f \in I$ and $h \in k[x_1, \ldots, x_n]$ then $hf \in I$

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• Ideal generator: $f_1,f_2,\dots,f_s\in k[x_1,\dots,x_n]$ then
$<f_1,f_2,\dots,f_s>=\{\sum_{i=1}^s h_i f_i : h_i\in k[x_1,\dots,x_n]\}$

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$< f_1, f_2, \ldots, f_s >= \{\sum_{i=1}^{s} h_i f_i : h_i \in k[x_1, \ldots, x_n]\}$
• Division: $f = a_1 f_1 + a_2 f_2 + \ldots a_s f_s + r$

$$x^2y + xy^2 + y^2 = (x + y)(xy - 1) + (y^2 - 1) + x + y + 1$$

$$x^2y + xy^2 + y^2 = x(xy - 1) + (x + 1)(y^2 - 1) + 2x + 1$$

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$$x^2y+xy^2+y^2=(x+y)(xy-1)+(y^2-1)+x+y+1$$

$$x^2y+xy^2+y^2=x(xy-1)+(x+1)(y^2-1)+2x+1$$

- Ordering: Let $\alpha,\beta\in\mathbb{Z}^n_{\geq 0}$ then $\alpha >_{grevlex} \beta$ if $|\alpha|=\sum\alpha_i>|\beta|=\sum\beta_i$ or $|\alpha|=|\beta|$ and $\alpha-\beta\in\mathbb{Z}^n$ the rightmost non-zero entry is negative.
$(1,0,\ldots,0)>_{grevlex}(0,0,\ldots,1)$

• Hilbert Basis Theorem: every ideal I has a finite generating set, $I =< g_1, \ldots, g_s >$

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- Ideal quotient: $I : J$ the set $\{f \in k[x_1, \ldots, x_n] : fg \in I \text{ for all } g \in J\}$. $I : f$ instead of $I :< f >$.

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• Maximal Ideal: $I \subset k[x_1, \dots, x_n]$ is said to be maximal if $I \neq k[x_1, \dots, x_n]$ and any ideal J containing I is either $J = I$ or $J = k[x_1, \dots, x_n]$

Proper ideal $I \neq k[x_1, \dots, x_n]$

the ideal $I =< x_1 - a_1, x_2 - a_2, \dots, x_n - a_n >$, is maximal

A BIT OF ALGEBRAIC GEOMETRY

• Primary Ideal: whenever $fg \in I$, then either $f \in I$ or some power $g^m \in I, m > 0$. Every ideal can be written as a finite intersection of primary ideals.

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• A primary decomposition $I = \cap_{i=1}^r Q_i$ is called minimal or irredundant if $\sqrt{Q_i}$ are all distinct and $\cap_{j \neq i} Q_j \not\subset Q_i$. Lask-Noether: every ideal has a minimal primary decomposition.

$$< x^2, xy >=< x > \cap < x^2, xy, y^2 >=< x > \cap < x^2, y >$$

The radical of the ideals in the above decomposition are uniquely determined: let $P_i = \sqrt{Q_i}$ then P_i the proper prime ideals occurring in the set $\left\{\sqrt{I : f} : f \in k[x_1, \ldots, x_n]\right\}$

Multivariate Division and Groebner Basis

• Given any ideal I we can define a unique Groebner basis up to ordering $< g_1, \ldots, g_s >$

$$f = h_1 g_1 + \ldots + h_n g_n + r$$

Strategy:

• Start with a set of polynomials $I =< d_1, \ldots, d_n >$

• Find the GB, $G =< g_1, \ldots, g_s >$

• Perform the division of an arbitrary polynomial N

$$N = h_1 g_1 + \ldots + h_n g_s + v$$

• Express back g_i in terms of d_i

$$N = \tilde{h}_1 d_1 + \ldots + \tilde{h}_n d_n + v$$

- Repeating the above procedure

$$\frac{N\left(l_1,l_2;\{p_i\}\right)}{D_1 D_2 \ldots D_n} = \sum_{m=1}^{\min(n,8)} \sum_{S_{m;n}} \frac{\Delta_{i_1 i_2 \ldots i_m}\left(l_1,l_2;\{p_i\}\right)}{D_{i_1} D_{i_2} \ldots D_{i_m}}$$

$S_{m;n}$ stands for all subsets of m indices out of the n ones: for instance if $S_{n;n} = S_{4;4} = \{\{1,2,3,4\}\}$ then $S_{3;4} = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$ and so on.

$$\begin{aligned}
\frac{N\left(q;\{p_i\}\right)}{D_1 D_2 D_3 D_4} &= \frac{\Delta_{1234}\left(q;\{p_i\}\right)}{D_1 D_2 D_3 D_4} \\
&+ \frac{\Delta_{123}\left(q;\{p_i\}\right)}{D_1 D_2 D_3} + \frac{\Delta_{124}\left(q;\{p_i\}\right)}{D_1 D_2 D_4} + \ldots \\
&+ \frac{\Delta_{12}\left(q;\{p_i\}\right)}{D_1 D_2} + \frac{\Delta_{13}\left(q;\{p_i\}\right)}{D_1 D_3} + \ldots \\
&+ \frac{\Delta_{1}\left(q;\{p_i\}\right)}{D_1} + \frac{\Delta_{2}\left(q;\{p_i\}\right)}{D_2} + \ldots
\end{aligned}$$

OPP at two loops

- Planar topology (4,1,4)

$$
\begin{aligned}
&D_1 = l_1^2 - M_1^2, D_2 = (l_1 + p_1)^2 - M_2^2,\\
&D_3 = (l_1 + p_2)^2 - M_3^2, D_4 = (l_1 + p_3)^2 - M_4^2,\\
&D_5 = l_2^2 - M_5^2, D_6 = (l_2 + p_4)^2 - M_6^2,\\
&D_7 = (l_2 + p_5)^2 - M_7^2, D_8 = (l_2 + p_6)^2 - M_8^2,\\
&D_9 = (l_1 + l_2)^2 - M_9^2
\end{aligned}
$$

- : l_1

$$
v_1^\mu = \frac{\delta^{\mu p_2 p_3}_{p_1 p_2 p_3}}{\Delta} \quad v_2^\mu = \frac{\delta^{p_1 \mu p_3}_{p_1 p_2 p_3}}{\Delta} \quad v_3^\mu = \frac{\delta^{p_1 p_2 \mu}_{p_1 p_2 p_3}}{\Delta} \quad \eta^\mu = \frac{\varepsilon^{\mu p_1 p_2 p_3}}{\sqrt{\Delta}}
$$

with

$$
\Delta = \delta^{p_1 p_2 p_3}_{p_1 p_2 p_3} = \varepsilon^{p_1 p_2 p_3} \varepsilon_{p_1 p_2 p_3} = \begin{vmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 & p_1 \cdot p_3 \\ p_2 \cdot p_1 & p_2 \cdot p_2 & p_2 \cdot p_3 \\ p_3 \cdot p_1 & p_3 \cdot p_2 & p_3 \cdot p_3 \end{vmatrix}
$$

OPP AT TWO LOOPS

•: l_2, the same as above with p_4, p_5, p_6 replacing p_1, p_2, p_3 accordingly. The momenta $p_i, i = 1, \ldots, 6$ are arbitrary. The basis coefficients may be read as $l_1^\mu = \sum_{i=1}^3 z_i v_i^\mu + z_4 \eta^\mu$, with $z_i = l_1 \cdot p_i, i = 1 \ldots, 3$ (l_2, with w_i replacing z_i).

$$1 = \sum_{i=1}^{9} x_i(l_1, l_2) D_i$$

• polynomial equation with 8 variables $(z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4)$, $x_i(l_1, l_2)$ polynomials of degree 3

• x_i, being a degree 3 polynomial in these variables, consists of 165 terms, total of $9 \times 165 = 1485$

• 831 out of 1485 are "independent"

• on the cut (OPP) l_1^c, l_2^c, $x_i(l_1^c, l_2^c)$ 4 coefficients: $\{1, z_4, w_4, z_4 w_4\}$

• $x_i = V_i + R_i$ with $R_i \sim D_j$

- Demo 1: one-loop

OPP at two loops

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- Demo 2: $9 \to 8$

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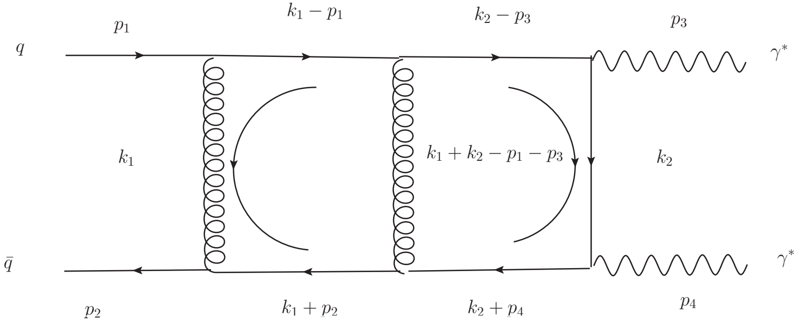
- Demo 2: $9 \to 8$
- Demo 3: $q\bar{q} \to \gamma^*\gamma^*$



- Rational terms

OPP AT TWO LOOPS

- Rational terms

$$l_1 \to l_1 + l_1^{(2\varepsilon)},\ \ l_2 \to l_2 + l_2^{(2\varepsilon)},\ \ l_{1,2} \cdot l_{1,2}^{(2\varepsilon)} = 0$$

$$\left(l_1^{(2\varepsilon)}\right)^2 = \mu_{11}, \quad \left(l_2^{(2\varepsilon)}\right)^2 = \mu_{22}, \quad l_1^{(2\varepsilon)} \cdot l_2^{(2\varepsilon)} = \mu_{12}$$

$$\left\{l_1^{(4)}, l_2^{(4)}\right\} \to \left\{l_1^{(4)}, l_2^{(4)}, \mu_{11}, \mu_{22}, \mu_{12}\right\}$$

- R_2 terms

OPP AT TWO LOOPS

- Rational terms
- IBP

OPP at two loops

- Rational terms
- IBP
  - Laporta: `FIRE`, `AIR`, `Reduze`
  - New algorithms: speed, numerical, integrand level

OPP AT TWO LOOPS

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OPP AT TWO LOOPS

- Rational terms
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- MI
  - Library of MI à la one-loop
  - Recycling through DE
  - Iterated Integrals K. T. Chen, Iterated path integrals, Bull. Amer. Math. Soc. 83 (1977) 831
  - Polylogarithms, Symbol algebra

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NNLO in the future

- The NNLO revolution to come

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  - In a few years a new "wish list"
    $pp \to t\bar{t}$, $pp \to W^+W^-$, $pp \to W/Z + nj$, $pp \to H + nj$

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- A HELAC-NNLO framework ?