Progress in Analytical Calculations for g-2 at Four Loops

Peter Marquard

DESY
in collaboration with

R. Lee, A.V. Smirnov, V.A. Smirnov and M. Steinhauser `JHEP03(2013)162`
A. Kurz, T. Liu and M. Steinhauser
P. Baikov and A. Maier `arXiv:1307.6105`

Matter To The Deepest, Ustron, 2013

Outline

Lepton anomalous magnetic moment

- Best experimentally measured and theoretically predicted quantity

$$
\begin{aligned}
a_e|_{\mathrm{exp}} &= 0.00115965218073(28) \\
a_e|_{\mathrm{theo}} &= 0.00115965218178(6)(4)(3)(77)
\end{aligned}
$$

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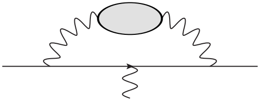
$$

$$
\begin{aligned}
a_\mu|_{\mathrm{exp}} &= 1.16592080(54)(33)[63] \cdot 10^{-3} \\
a_\mu|_{\mathrm{theo}} &= 1.16591790(65) \cdot 10^{-3} \qquad 3.2\sigma \text{ diff.}
\end{aligned}
$$

- QED contributions known numerically up to 5 loops but starting from four loops not checked by an independent calculation
- $a_\mu|_{\mathrm{exp}} - a_\mu|_{\mathrm{theo}} \approx \mathcal{O}(\text{4-loop QED contribution})$

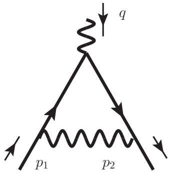
Status QED contributions

- analytical results
  - one loop: $a_\mu^{(1)} = \frac{1}{2}$ [Schwinger 1948]
  - two loop [Petermann; Sommerfeld 1957]
  - three loop [Laporta, Remiddi 1996]
  - four loop: only partial results, mainly contributions due to corrections to the vacuum polarization function of the photon



- numerical results
  - four loop [Kinoshita et al]
  - five loop [Aoyama,Hayakawa,Kinoshita,Nio 2012]

Definition and Notation



$$= \quad (-ie)\overline{u}(p_2)\left\{\gamma^\mu F_E(q^2) + i\frac{\sigma^{\mu\nu}q^\nu}{2m}F_M(q^2)\right\}u(p_1)$$

$$a_\mu = F_M(0)$$

a_μ can be obtained by using suitable projektor, but needs expansion up to first order in q^2

$$a_\mu = \frac{\alpha}{\pi}a_\mu^{(1)} + \left(\frac{\alpha}{\pi}\right)^2 a_\mu^{(2)} + \left(\frac{\alpha}{\pi}\right)^3 a_\mu^{(3)} + \left(\frac{\alpha}{\pi}\right)^4 a_\mu^{(4)}$$

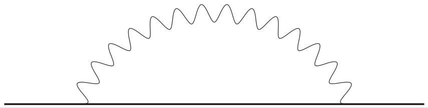
$$a_\mu^{(4)} = a_\mu^{(40)} + n_l a_\mu^{(41)} + n_l^2 a_\mu^{(42)} + n_l^3 a_\mu^{(43)}$$

Integral Classes

- At four-loop level two classes of integrals have been studied extensively: massive tadpoles and massless propagators.
- Both classes have many phenomenological applications
- Here we need a new class of integrals: on-shell integrals!
- New class also important for the calculation of the $\overline{\text{MS}}$ – on-shell relation for quark masses in QCD

Order of Complexity

$Z_m : \Sigma(q^2, M^2)|_{q^2=M^2}$



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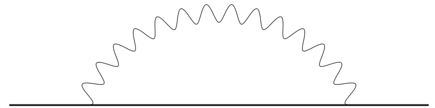
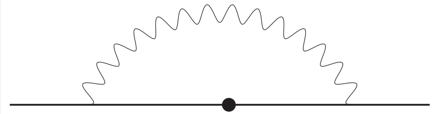


$Z_2 : \frac{\mathrm{d}}{\mathrm{d}q^2}\Sigma(q^2, M^2)|_{q^2=M^2}$



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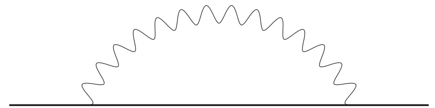

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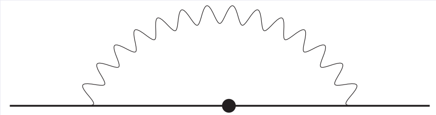
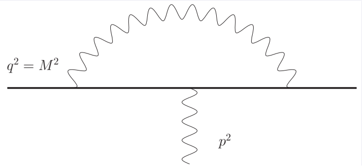


$g - 2 : \frac{\mathrm{d}}{\mathrm{d}p^2}\Gamma(p^2, q^2 = M^2, M^2)|_{p^2=0}$



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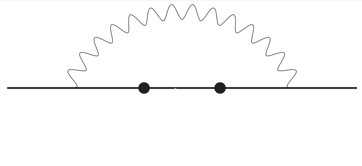
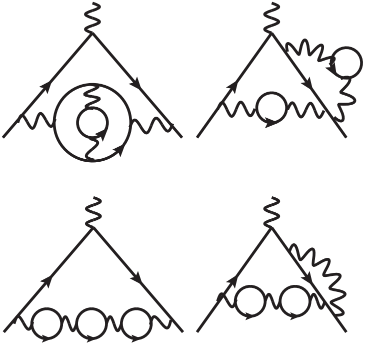


Outline

Setup I

- calculation done for massless electrons, leading logarithmic effects recovered through renormalization
- later extend calculation to obtain universal contributions and power suppressed terms
- sample diagrams for n_l^3, n_l^2 part

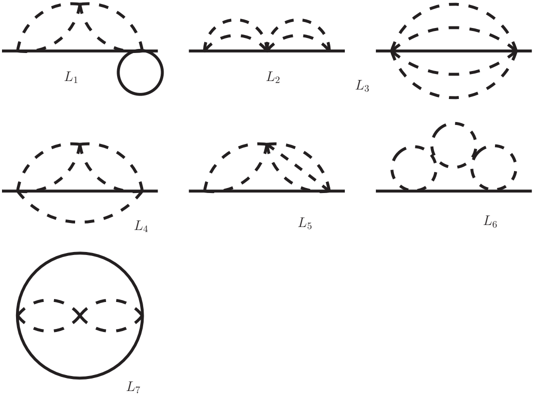


Setup II

tools used include

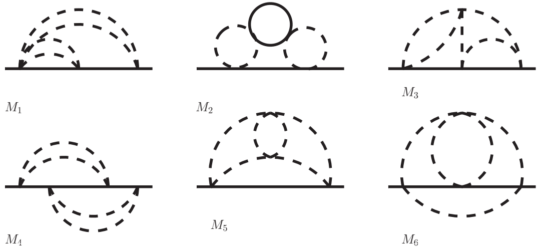
- `qgraf` [Nogueira]
  Generation of Feynman diagrams
- `q2e,exp` [Harlander, Seidensticker, Steinhauser]
  Expansion / Mapping to topologies
- `FORM` [Vermaseren]
  Algebra
- `CRUSHER,FIRE` [Seidel,PM / Smirnov,Smirnov]
  Reduction to master integrals
- `FIESTA` [Smirnov,Smirnov]
  Calculation of master integrals

Master Integrals n_l^3, n_l^2: simple



- Expressible through Gamma functions for arbitrary dimension D!

Master Integrals n_l^3, n_l^2: difficult



- Calculated analytically in expansion in $\epsilon = (4 - D)/2$ using the DRA (dimensional recurrence and analyticity) method and checked using `FIESTA`!
- Calculated up to $\mathcal{O}(\epsilon^3)$

Results

$$a_\mu^{(43)} = \frac{1}{54} L_{\mu e}^3 - \frac{25}{108} L_{\mu e}^2 + \left(\frac{317}{324} + \frac{\pi^2}{27} \right) L_{\mu e} - \frac{2\zeta_3}{9} - \frac{25\pi^2}{162} - \frac{8609}{5832}$$
$$\approx 7.196\,66\,,$$

[Laporta; Aguilar,Greynat,De Rafael]

Results

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[Laporta; Aguilar,Greynat,De Rafael]

$$a_\mu^{(42)} = a_\mu^{(42)a} + n_h a_\mu^{(42)b}$$

$$a_\mu^{(42)a} = L_{\mu e}^2\left[\pi^2\left(\frac{5}{36} - \frac{\log 2}{6}\right) + \frac{\zeta_3}{4} - \frac{13}{24}\right] + \ldots \approx -3.624\,27\,,$$

$$a_\mu^{(42)a}\Big|_{\mathrm{num}} = -3.642\,04(1\,12)\,,$$

[Aoyama,Hayakawa,Kinoshita,Nio 2012]

Results

$$
\begin{aligned}
a_\mu^{(43)} &= \frac{1}{54}L_{\mu e}^3 - \frac{25}{108}L_{\mu e}^2 + \left(\frac{317}{324}+\frac{\pi^2}{27}\right)L_{\mu e} - \frac{2\zeta_3}{9} - \frac{25\pi^2}{162} - \frac{8609}{5832} \\
&\approx 7.196\,66\,,
\end{aligned}
$$

[Laporta; Aguilar,Greynat,De Rafael]

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a_\mu^{(42)} = a_\mu^{(42)a} + n_h a_\mu^{(42)b}
$$

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a_\mu^{(42)a} = L_{\mu e}^2\left[\pi^2\left(\frac{5}{36}-\frac{\log 2}{6}\right)+\frac{\zeta_3}{4}-\frac{13}{24}\right]+\ldots \approx -3.624\,27\,,
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a_\mu^{(42)a}\Big|_{\mathrm{num}} = -3.642\,04(1\,12)\,,
$$

[Aoyama,Hayakawa,Kinoshita,Nio 2012]

$$
\begin{aligned}
a_\mu^{(42)b} &= \left(\frac{119}{108}-\frac{\pi^2}{9}\right)L_{\mu e}^2 + \left(\frac{\pi^2}{27}-\frac{61}{162}\right)L_{\mu e} - \frac{4\pi^4}{45} + \frac{13\pi^2}{27} + \frac{7627}{1944} \\
&\approx 0.494\,05
\end{aligned}
$$

[Laporta; Aguilar,Greynat,De Rafael]

Results: n_l part

- decompose $a_\mu^{(41)}$ further

$$a_\mu^{(41)} = a_\mu^{(41)a} + n_h a_\mu^{(41)b} + n_h^2 a_\mu^{(41)c}$$

- Preliminary result for $a_\mu^{(41)b}$ and $a_\mu^{(41)c}$

$$\begin{aligned} a_\mu^{(41)b} &= -1.06(5) \\ a_\mu^{(41)c} &= 0.0280 \end{aligned}$$

compare with [Aoyama,Hayakawa,Kinoshita,Nio 2012]

$$\begin{aligned} a_\mu^{(41)b} &= -1.046 \\ a_\mu^{(41)c} &= 0.0280 \end{aligned}$$

- $n_l n_h^0$ part calculated but further cross checks necessary !

Outline

Method

- contributions due to heavy leptons can be calculated in an asymptotic expansion in M_μ/M_τ
- asymptotic expansion leads to at most vacuum diagrams at four loops
- expansion up to including terms of order $(M_\mu/M_\tau)^7$
- all required master integrals are known
- same tool set used as in case of electron contributions

Asymptotic expansion

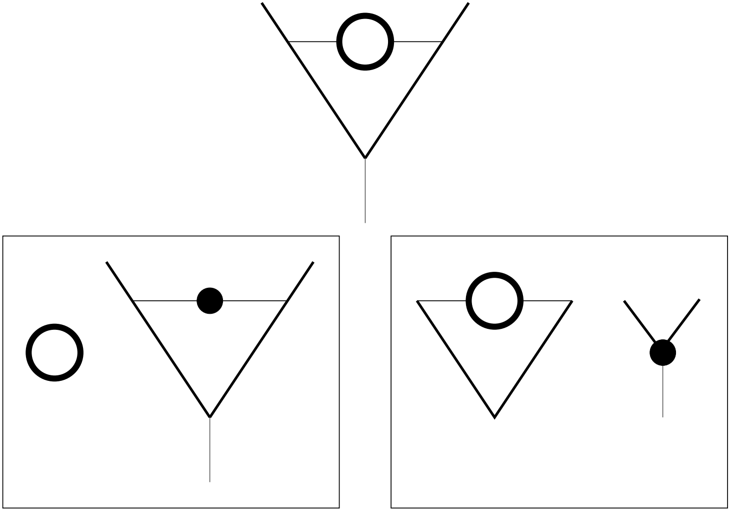
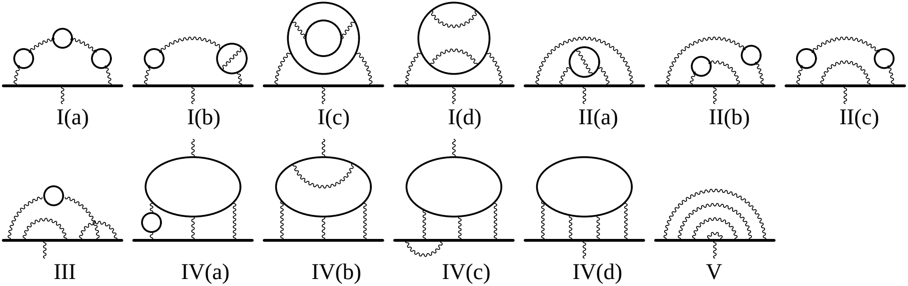


Diagram classes at four loops



[Aoyama et al ’2012]

Results

Preliminary!

group	$10^2 \cdot a^{(4)}_{2,\mu}(M_\mu/M_\tau)$	
	our work	Kinoshita et al
I(a)	0.00324281(XX)	0.0032(0)
I(b) + I(c) + II(b) + II(c)	-0.6292808(XX)	-0.6293(1)
I(d)	0.0367796(XX)	0.0368(0)
III	4.5208986(XX)	4.504(14)
II(a) + IV(d)	-2.316756(XX)	-2.3197(37)
IV(a)	3.851967(XX)	3.8513(11)
IV(b)	0.612661(XX)	0.6106(31)
IV(c)	-1.83010(XX)	-1.823(11)

fast convergence

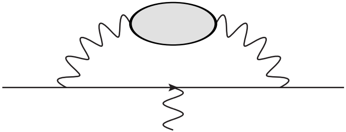
$$a^{(4)}_{2,\mu}(M_\mu/M_\tau) = 0.0421670 + 0.0003257 + 0.0000015 = 0.0424941$$

Outline

Method

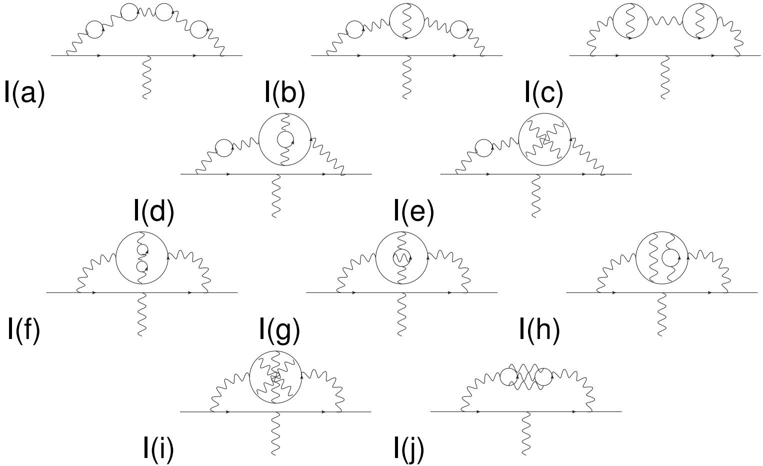
- Certain contributions to g-2 can be obtained by integration over the vacuum polarisation function $\Pi(q^2)$

$$a_\mu = \frac{\alpha}{\pi}\int_0^1 \mathrm{d}x(1-x)\left[-\Pi(s_x)\right]\ ,\ s_x = -\frac{x^2}{1-x}m_\mu^2\ .$$



- use a suitable approximating function for $\Pi(q^2)$ at four loops and obtain g-2 at five loops

Accessible classes



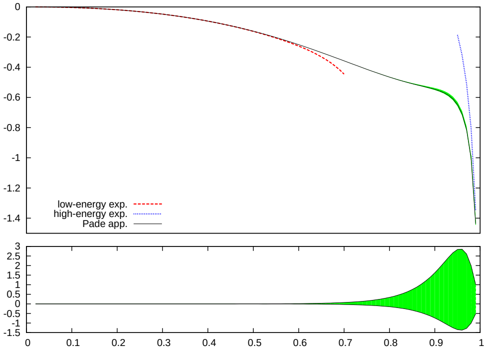
Reconstruction of $\Pi(q^2)$ I

- first step: take only leading term in high-energy expansion

  [Baikov et al 2013]
- for certain classes large deviation from numerical results of Kinohita et al.
- Improve by using all available information in the low- and high-energy and the threshold region to obtain best possible approximation for $\Pi(q^2)$ in form of a Padé approximation
- input used: 3 low-energy, 2 high-energy and 2 threshold constants

Reconstruction of $\Pi(q^2)$ II

- To obtain an error estimate constructed $\approx$ 800 Padé approximants

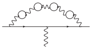
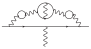
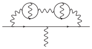
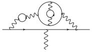
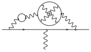


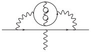
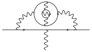
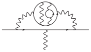
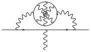
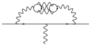
- few percent uncertainty in the high energy region

Results

	our work	Baikov et al	Kinoshita et al
I(a)	20.142 813	20.183 2	20.142 93(23)
I(b)	27.690 061	27.718 8	27.690 38(30)
I(c)	4.742 149	4.817 59	4.742 12(14)
I(d)+I(e)	6.241 470	6.117 77	6.243 32(101)(70)
I(e)	-1.211 249	-1.331 41	-1.208 41(70)
I(f)+I(g)+I(h)	$4.446\,8^{+6}_{-4}$	4.391 31	4.446 68(9)(23)(59)
I(i)	$0.074\,6^{+8}_{-19}$	0.252 37	0.087 1(59)
I(j)	$-1.246\,9^{+4}_{-3}$	-1.214 29	-1.247 26(12)

Improved agreement with previous works by Kinoshita et al.

I(a)  I(b)  I(c)  I(d)  I(e) 

I(f)  I(g)  I(h)  I(i)  I(j) 

Conclusions and Outlook

- calculation of the leading contribution from diagrams containing electron loops finished
- contributions from diagrams containing tau loops calculated in an expansion in M_μ/M_τ
- improved prediction for certain class of five-loop diagrams leading to better agreement with literature
- ToDo
  - extend calculations to include higher orders in M_μ/M_τ
  - calculate the universal contribution at four loops