Integrand-level reduction at one and higher loops

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Based on:

- P. Mastrolia, E. Mirabella and **T.P.**, *Integrand reduction of one-loop scattering amplitudes through Laurent series expansion,* JHEP **1206**, 095 (2012) [arXiv:1203.0291 [hep-ph]].
- P. Mastrolia, E. Mirabella, G. Ossola and **T.P.**, *Scattering Amplitudes from Multivariate Polynomial Division,* Phys. Lett. B **718**, 173 (2012) [arXiv:1205.7087 [hep-ph]].
- P. Mastrolia, E. Mirabella, G. Ossola and **T.P.**, *Integrand-Reduction for Two-Loop Scattering Amplitudes through Multivariate Polynomial Division,* Phys. Rev. D **87**, 085026 (2013) [arXiv:1209.4319 [hep-ph]].
- H. van Deurzen, N. Greiner, G. Luisoni, P. Mastrolia, E. Mirabella, G. Ossola, **T.P.**, J. F. von Soden-Fraunhofen and F. Tramontano Phys. Lett. B **721**, 74 (2013) [arXiv:1301.0493 [hep-ph]].
- S. Heinemeyer,. . . ,**T.P.** *et al.* [The LHC Higgs Cross Section Working Group Collaboration], *Handbook of LHC Higgs Cross Sections: 3. Higgs Properties,* arXiv:1307.1347 [hep-ph].
- G. Cullen, H. van Deurzen, N. Greiner, G. Luisoni, P. Mastrolia, E. Mirabella, G. Ossola, **T.P.** and F. Tramontano *NLO QCD corrections to Higgs boson production plus three jets in gluon fusion,* arXiv:1307.4737 [hep-ph].
- P. Mastrolia, E. Mirabella, G. Ossola and **T.P.**, *Multiloop Integrand Reduction for Dimensionally Regulated Amplitudes,* arXiv:1307.5832 [hep-ph].
- H. van Deurzen, G. Luisoni, P. Mastrolia, E. Mirabella, G. Ossola and **T.P.**, *NLO QCD corrections to Higgs boson production in association with a top quark pair and a jet,* arXiv:1307.8437 [hep-ph].

Outline

Outline

Introduction and motivation

Motivation

- Understanding the basic analytic and algebraic structure of integrands and integrals of scattering amplitudes
- Exploration of methods for obtaining theoretical predictions in perturbative Quantum Field Theory at higher orders, required for experiments in high-energy physics

We developed a coherent framework for the integrand decomposition of Feynman integrals

- based on simple concepts of algebraic geometry
- applicable at all loops

Outline

Integrand reduction

- Generic ℓ-loop integral:

$$\mathcal{M}_n = \int d^d q_1 \ldots d^d q_\ell \;\; \mathcal{I}_{i_1 \ldots i_n}, \qquad \mathcal{I}_{i_1 \ldots i_n} \equiv \frac{\mathcal{N}_{i_1 \ldots i_n}}{D_{i_1} \ldots D_{i_n}}$$

  - the numerator $\mathcal{N}_{i_1 \ldots i_n}$ is polynomial in q_i
  - the denominators D_i are quadratic polynomials in q_i
- The integrand-reduction method leads to the decomposition:

$$\mathcal{I}_{i_1 \ldots i_n} = \frac{\Delta_{i_1 \cdots i_n}}{D_{i_1} \ldots D_{i_n}} + \ldots + \sum_{k=1}^{n} \frac{\Delta_{i_k}}{D_{i_k}} + \Delta_{\emptyset}$$

- The residues $\Delta_{i_1 \ldots i_k}$ are irreducible polynomials in q_i
  - universal topology-dependent parametric form
  - the coefficients of the parametrization are process-dependent

From integrands to integrals

- By integrating the integrand decomposition

$$\mathcal{M}_n = \int d^d q_1 \dots d^d q_\ell \left(\frac{\Delta_{i_1\cdots i_n}}{D_{i_1}\dots D_{i_n}} + \dots + \sum_{k=1}^{n} \frac{\Delta_{i_k}}{D_{i_k}} + \Delta_\emptyset \right)$$

  - some terms vanish and do not contribute to the amplitude $\Rightarrow$ spurious terms
  - non-vanishing terms give Master Integrals (MIs)
- The amplitude is a linear combination of MIs
- The coefficients of this linear combination can be identified with some of the coefficients which parametrize the polynomial residues

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$\Rightarrow$ reduction to MIs $\equiv$ polynomial fit of the residues

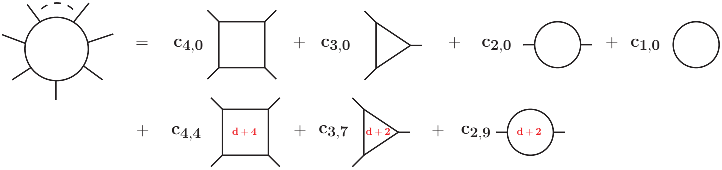
The one-loop decomposition

At one loop the result is well known:

- the integrand decomposition
  [Ossola, Papadopoulos, Pittau (2007); Ellis, Giele, Kunszt, Melnikov (2008)]

$$\mathcal{I}_{i_1\cdots i_n} = \frac{\mathcal{N}_{i_1\cdots i_n}}{D_{i_1}\cdots D_{i_n}} = \sum_{j_1\cdots j_5} \frac{\Delta_{j_1 j_2 j_3 j_4 j_5}}{D_{j_1} D_{j_2} D_{j_3} D_{j_4} D_{j_5}} + \sum_{j_1 j_2 j_3 j_4} \frac{\Delta_{j_1 j_2 j_3 j_4}}{D_{j_1} D_{j_2} D_{j_3} D_{j_4}}$$
$$+ \sum_{j_1 j_2 j_3} \frac{\Delta_{j_1 j_2 j_3}}{D_{j_1} D_{j_2} D_{j_3}} + \sum_{j_1 j_2} \frac{\Delta_{j_1 j_2}}{D_{j_1} D_{j_2}} + \sum_{j_1} \frac{\Delta_{j_1}}{D_{j_1}}$$

- the integral decomposition



Outline

Integrand reduction and polynomials

- At ℓ-loops we want to achieve the integrand decomposition:

$$\mathcal{I}_{i_1\ldots i_n}(q_1,\ldots,q_\ell) \equiv \frac{\mathcal{N}_{i_1\ldots i_n}}{D_{i_1}\ldots D_{i_n}} = \frac{\Delta_{i_1\cdots i_n}}{D_{i_1}\ldots D_{i_n}} + \ldots + \sum_{k=1}^{n} \frac{\Delta_{i_k}}{D_{i_k}} + \Delta_\emptyset$$

- The residues $\Delta_{i_1\ldots i_k}$ must be irreducible
  - can't be written as a combination of denominators $D_{i_1}\ldots D_{i_k}$
- We trade $(q_1,\ldots,q_\ell)$ with their coordinates $\mathbf{z} \equiv (z_1,\ldots,z_m)$

  $\Rightarrow$ numerator and denominators $\equiv$ polynomials in $\mathbf{z}$

$$\mathcal{I}_{i_1\ldots i_n}(\mathbf{z}) \equiv \frac{\mathcal{N}_{i_1\ldots i_n}(\mathbf{z})}{D_{i_1}(\mathbf{z})\ldots D_{i_n}(\mathbf{z})}$$

- We can reformulate the integrand reduction as a problem of multivariate polynomial division

Residues via polynomial division

Y. Zhang (2012), P. Mastrolia, E. Mirabella, G. Ossola, T.P. (2012)

- Define the Ideal of polynomials

$$\mathcal{J}_{i_1\cdots i_n} \equiv \langle D_{i_1}, \ldots, D_{i_n} \rangle = \left\{ p(\mathbf{z}) \,:\, p(\mathbf{z}) = \sum_j h_j(\mathbf{z}) D_j(\mathbf{z}),\; h_j \in P[z] \right\}$$

- Take a Gröbner basis $G_{\mathcal{J}_{i_1\cdots i_n}}$ of $\mathcal{J}_{i_1\cdots i_n}$

$$G_{\mathcal{J}_{i_1\cdots i_n}} = \{g_1, \ldots, g_s\} \quad \text{such that} \quad \mathcal{J}_{i_1\cdots i_n} = \langle g_1, \ldots, g_s \rangle$$

- Perform the multivariate polynomial division $\mathcal{N}_{i_1\ldots i_n}/G_{\mathcal{J}_{i_1\cdots i_n}}$

$$\mathcal{N}_{i_1\cdots i_n}(z) = \sum_{k=1}^{n} \mathcal{N}_{i_1\cdots i_{k-1} i_{k+1}\cdots i_n}(z)\, D_{i_k}(z) + \Delta_{i_1\cdots i_n}(z)$$

- The remainder $\Delta_{i_1\cdots i_n}$ is irreducible $\Rightarrow$ can be identified with the residue

Recursive Relation for the integrand decomposition

P. Mastrolia, E. Mirabella, G. Ossola, T.P. (2012)

The recursive formula

$$\mathcal{N}_{i_1\cdots i_n} = \sum_{k=1}^{n} \mathcal{N}_{i_1\cdots i_{k-1} i_{k+1}\cdots i_n} D_{i_k} + \Delta_{i_1\cdots i_n}$$

$$\mathcal{I}_{i_1\cdots i_n} \equiv \frac{\mathcal{N}_{i_1\cdots i_n}}{D_{i_1}\cdots D_{i_n}} = \sum_{k} \mathcal{I}_{i_1\cdots i_{k-1} i_{k+1}\cdots i_n} + \frac{\Delta_{i_1\cdots i_n}}{D_{i_1}\cdots D_{i_n}}$$

- Fit-on-the-cut approach
  - from a generic $\mathcal{N}$, get the parametric form of the residues Δ
  - determine the coefficients sampling on the cuts (impose $D_i = 0$)
- Divide-and-Conquer approach
  - generate the $\mathcal{N}$ of the process
  - compute the residues by iterating the polynomial division algorithm

Two results from algebraic-geometry techniques

The reducibility criterion

- If a cut $D_{i_1} = \ldots = D_{i_k} = 0$ has no solutions, the associated residue vanishes. In other words, any numerator is completely reducible.
- This generally happens with overdetermined systems i.e. when the number of cut denominators is higher than the one of loop variables.

The maximum-cut theorem

- We define maximum-cut, a cut where the number of cut denominators is equal to the one of the loop variables.
- In non-special kinematic configurations, the residue at the maximum-cut is a polynomial parametrized by n_s coefficients, which admits a univariate representation of degree $(n_s - 1)$.
- The fit-on-the-cut approach therefore gives a number of equations which is equal to the number of unknown coefficients.

Outline

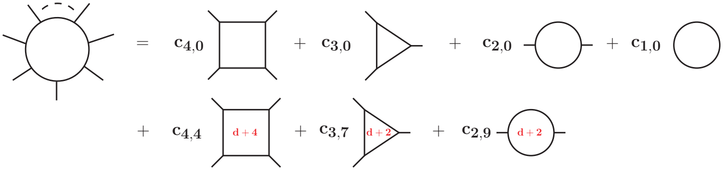
One-loop decomposition from polynomial division

P. Mastrolia, E. Mirabella, G. Ossola, T.P. (2012)

- Start from the most general one-loop amplitude in $d = 4 - 2\epsilon$
- Apply the recursive formula for the integrand decomposition

  $\Rightarrow$ it reproduces the OPP result
  [Ossola, Papadopoulos, Pittau (2007); Ellis, Giele, Kunszt, Melnikov (2008)]
- Drop the spurious terms

$\Rightarrow$ Get the most general integral decomposition (well knwon result)



Integrand Reduction via Laurent series expansion

P. Mastrolia, E. Mirabella, T.P. (2012)

The integrand reduction via Laurent expansion:

- fits residues by taking their asymptotic expansions on the cuts
- yields diagonal systems of equations for the coefficients
- requires the computation of fewer coefficients
  - pentagons are spurious and do not need to be computed
  - spurious terms of boxes and tadpoles do not need to be computed
- subtractions of higher point residues is simplified
  - 4-point (and 5-point) residues are not subtracted
  - subtractions of 3- and 2-point residues at the coefficient level

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★ Implemented in the semi-numerical C++ library NINJA

  - Laurent expansions via a simplified polynomial-division algorithm
  - interfaced with the package GOSAM
  - is a faster and more stable integrand-reduction algorithm

Example: the coefficients of the bubbles

- The residue of a bubble (4-dim for brevity)

$$\Delta_{ij}(q) = b_0 + b_1\,(q\cdot e_2) + b_2\,(q\cdot e_2)^2 + b_3\,(q\cdot e_3) + b_4\,(q\cdot e_3)^2 + b_5\,(q\cdot e_4) + b_6\,(q\cdot e_4)^2 + b_7\,(q\cdot e_2)(q\cdot e_3) + b_8\,(q\cdot e_2)(q\cdot e_4)$$

- solutions of a double cut $D_i = D_j = 0$, parametrized by the free variables t and x

$$q_+ = x\,e_1 + (\alpha_0 + x\,\alpha_1)e_2 + t\,e_3 + \frac{\beta_0 + \beta_1 x + \beta_2 x^2}{t}\,e_4$$

$$q_- = x\,e_1 + (\alpha_0 + x\,\alpha_1)e_2 + \frac{\beta_0 + \beta_1 x + \beta_2 x^2}{t}\,e_3 + t\,e_4$$

- in the limit $t \to \infty$

$$\left.\frac{\mathcal{N}(q_\pm)}{\prod_{m\neq i,j} D_m}\right|_{\text{cut}} = \Delta_{ij} + \sum_k \frac{\Delta_{ijk}}{D_k} + \sum_{kl} \frac{\Delta_{ijkl}}{D_k D_l} + \sum_{klm} \frac{\Delta_{ijklm}}{D_k D_l D_m}$$

$$= \Delta_{ij} + \sum_k \frac{\Delta_{ijk}}{D_k} + \mathcal{O}(1/t)$$

NOTE: Higher point residues are computed in previous steps of the reduction

The coefficients of the bubbles

- In the asymptotic limit $t \to \infty$
  - the integrand

$$\left.\frac{\mathcal{N}(q_\pm)}{\prod_{m\neq i,j,k} D_m}\right|_{\text{cut}} = n_0^\pm + n_1^\pm\, x + n_2^\pm\, x^2 + \left(n_3^\pm + n_4^\pm x\right) t + n_5^\pm\, t^2 + \mathcal{O}(1/t)$$

  - the subtraction term

$$\frac{\Delta_{ijk}(q_\pm)}{D_k} = \tilde{b}_0^{k,\pm} + \tilde{b}_1^{k,\pm}\, x + \tilde{b}_2^{k,\pm}\, x^2 + \left(\tilde{b}_3^{k,\pm} + \tilde{b}_4^{k,\pm} x\right) t + \tilde{b}_5^{k,\pm}\, t^2 + \mathcal{O}(1/t)$$

    - $\tilde{b}_i^{k,\pm}$ are known functions of the triangle coefficients
  - the residue

$$\Delta_{ij}(q_+) = b_0 + b_1\, x + b_2 x^2 - \left(b_5 + b_8 x\right) t + b_6\, t^2 + \mathcal{O}(1/t)$$

$$\Delta_{ij}(q_-) = b_0 + b_1\, x + b_2 x^2 - \left(b_3 + b_7 x\right) t + b_4\, t^2 + \mathcal{O}(1/t)$$

  - by comparison, applying subtractions at the coefficient level

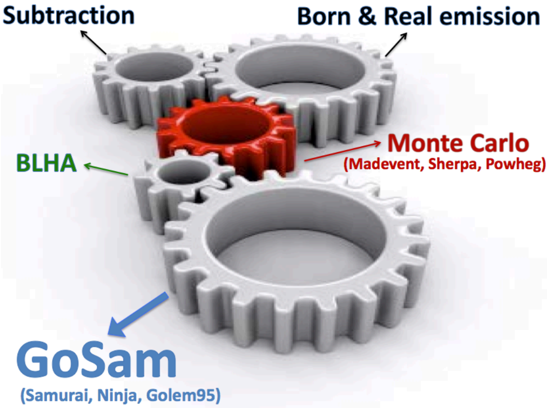
$$b_0 = n_0^\pm - \sum_k \tilde{b}_0^{k,\pm}, \quad b_1 = n_1^\pm - \sum_k \tilde{b}_1^{k,\pm}, \quad b_3 = -n_3^- + \sum_k \tilde{b}_3^{k,-}, \quad \ldots$$

Integrand Reduction with NINJA

Benchmarks: GOSAM + NINJA				
sub-process		# diagrams	# hel. (gen./tot.)	ms/event
$W+3j$	$d\bar{u} \to \bar{\nu}_e e^- ggg$	1 411	1/8	226
$t\bar{t}H$	$q\bar{q} \to t\bar{t}H$	31	8/8	2
	$gg \to t\bar{t}H$	136	4/16	40
$t\bar{t}H+1j$	$q\bar{q} \to t\bar{t}Hg$	320	8/16	93
	$gg \to t\bar{t}Hg$	1 575	4/32	2 070
$H+2j$ in GF (higher rank)	$d\bar{d} \to Hu\bar{u}$	32	4/4	1
	$dd \to Hdd$	60	3/6	4
	$d\bar{d} \to Hgg$	179	2/8	17
	$gg \to Hgg$	651	1/16	166
$H+3j$ in GF (higher rank)	$d\bar{d} \to Hgu\bar{u}$	467	4/7	68
	$dd \to Hgdd$	868	3/12	157
	$d\bar{d} \to Hggg$	2 519	2/16	999
	$gg \to Hggg$	9 325	1/32	11 266

NOTE: Timings refer to full color- and helicity-summed amplitudes

From amplitudes to observables with GoSam



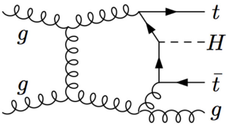
The GoSam collaboration:

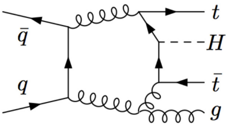
G. Cullen, H. van Deurzen, N. Greiner, G. Heinrich, G. Luisoni, P. Mastrolia, E. Mirabella, G. Ossola, J. Reichel , J. Schlenk, J. F. von Soden-Fraunhofen, T. Reiter, F. Tramontano, T.P.

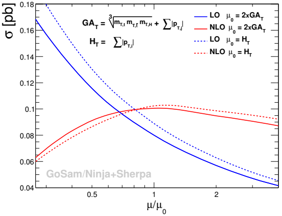
Application: $pp \to t\bar{t}H + jet$ with GoSam + Ninja

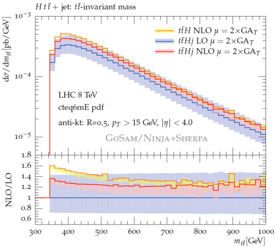
H. van Deurzen, G. Luisoni, P. Mastrolia, E. Mirabella, G. Ossola, T.P. (2013)

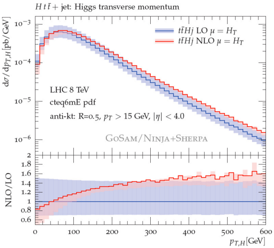
- Interfaced with the Monte Carlo Sherpa

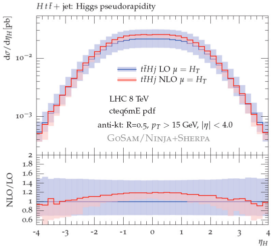












Outline

Extension to higher loops

- The integrand-level approach to scattering amplitudes one-loop
  - can be used to compute any amplitude in any QFT
  - has been implemented in several codes, some of which public [SAMURAI, CUTTOOLS, NGLUONS]
  - has produced (and is still producing) results for LHC [GOSAM (see H. van Deurzen's talk), FORMCALC, BLACKHAT, MADLOOP, NJETS, OPENLOOP ...]
- At two or higher loops
  - no general recipe is available
  - the standard and most successful approach is the Integration By Parts (IBP) method, but it becomes difficult for high multiplicities

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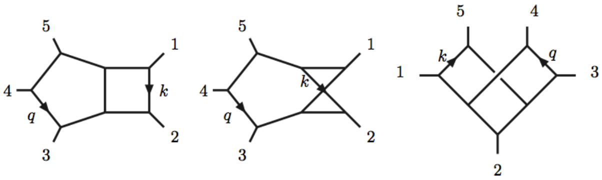
> The integrand-level approach might be a tool for understanding the structure of multi-loop scattering amplitudes and a method for their evaluation.

- ...we are moving the first steps in this direction

$\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA amplitudes

P. Mastrolia, G. Ossola (2011); P. Mastrolia, E. Mirabella, G. Ossola, T.P. (2012)



- Examples in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA amplitudes ($d = 4$)
  - generation of the integrand
    - graph based [Carrasco, Johansson (2011)]
    - unitarity based [U. Schubert (Diplomarbeit)]
  - fit-on-the-cut approach for the reduction
- Results:

$\mathcal{N} = 4$ linear combination of 8 and 7-denominators MIs

$\mathcal{N} = 8$ linear combination of 8, 7 and 6-denominators MIs

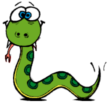
Divide-and-Conquer approach

P. Mastrolia, E. Mirabella, G. Ossola, T.P. (2013)

The divide-and-conquer approach to the integrand reduction

- does not require the knowledge of the solutions of the cut
- can always be used to perform the reduction in a finite number of purely algebraic operations
- has been automated in a PYTHON package which uses MACAULAY2 and FORM for algebraic operations

MACAULAY2 $\Leftarrow$ PYTHON $\Rightarrow$ FORM



- also works in special cases where the fit-on-the-cut approach is not applicable (e.g. in presence of double denominators)

Divide-and-Conquer approach: an explicit example

- A 5-denominator integrand

$$\mathcal{I}_{12345} \equiv \frac{\mathcal{N}_{12345}}{D_1 D_2 D_3 D_4 D_5}, \qquad D_1, \ldots, D_5 \ \underline{\text{not necessarily distinct!}}$$

- After division of $\mathcal{N}_{12345}$ modulo $\mathcal{G}_{\mathcal{J}_{12345}}$ (quotient and remainder)

$$\mathcal{N}_{12345} = \mathcal{N}_{2345} D_1 + \mathcal{N}_{1345} D_2 + \mathcal{N}_{1245} D_3 + \mathcal{N}_{1235} D_4 + \mathcal{N}_{1234} D_5 + \Delta_{12345}$$

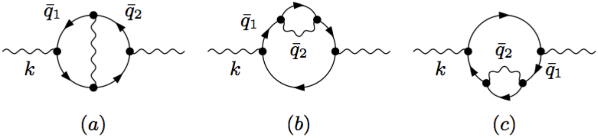
- After division of $\mathcal{N}_{i_1 i_2 i_3 i_4}$ modulo $\mathcal{G}_{\mathcal{J}_{i_1 i_2 i_3 i_4}}$ (quotients and remainders)

$$\begin{aligned} \mathcal{N}_{12345} = {} & \mathcal{N}_{345} D_1 D_2 + \mathcal{N}_{245} D_1 D_3 + \mathcal{N}_{235} D_1 D_4 + \mathcal{N}_{234} D_1 D_5 \\ & + \mathcal{N}_{145} D_2 D_3 + \mathcal{N}_{135} D_2 D_4 + \mathcal{N}_{134} D_2 D_5 \\ & + \mathcal{N}_{125} D_3 D_4 + \mathcal{N}_{124} D_3 D_5 + \mathcal{N}_{123} D_4 D_5 \\ & + \Delta_{2345} D_1 + \Delta_{1345} D_2 + \Delta_{1245} D_3 + \Delta_{1235} D_4 + \Delta_{1234} D_5 \\ & + \Delta_{12345} \end{aligned}$$

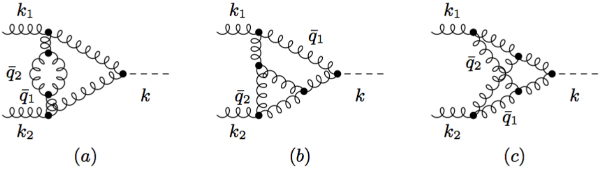
- ... and so forth

Examples of divide-and-conquer approach

- Photon self-energy in massive QED, $(4-2\epsilon)$-dimensions



- Diagrams entering $gg \to H$, in $(4-2\epsilon)$-dimensions



Outline

Conclusions

- We developed a general framework for the reduction at the integrand level
  - can be applied to any amplitude in any QFT
  - is valid at every loop order
- At one-loop
  - naturally reproduces the OPP result
  - allows to express any amplitude in terms of known MIs
  - leads to well established and successful techniques
  - can be improved with the Laurent-expansion approach (NINJA)
- At higher loops
  - it gives a recursive formula for the integrand decomposition
  - generates the form of the residue for every cut
- The divide-and-conquer approach
  - can be used to implement the whole reduction of any integrand with purely algebraic operations
  - has been automated in a PYTHON package

THANK YOU
FOR YOUR ATTENTION

BACKUP SLIDES

One-loop decomposition from polynomial division

At one loop in $4-2\epsilon$ dimensions:

- 5 coordinates $\mathbf{z}=(z_1,z_2,z_3,z_4,z_5)$
  - 4 components (z_1,z_2,z_3,z_4) of q w.r.t. a 4-dimensional basis
  - $z_5=\mu^2$ encodes the $(4-2\epsilon)$-dependence on the loop momentum
- we start with

$$\mathcal{I}_n \equiv \mathcal{I}_{1\ldots n} = \frac{\mathcal{N}_{1\ldots n}(\mathbf{z})}{D_1(\mathbf{z})\ldots D_n(\mathbf{z})}$$

- if $m>5$ any integrand $\mathcal{I}_{i_1\ldots i_m}$ is reducible (reducibility criterion)

$$\mathcal{I}_{i_1\ldots i_m} = \sum_k \mathcal{I}_{i_1\ldots i_{k-1} i_{k+1}\ldots i_m}, \qquad \text{for } m\geq 5$$

- for $m\leq 5$ the polynomial-division algorithm applied to a generic integrand, gives a non-trivial remainder $\Delta_{ijk\ldots}$
  - one finds the already-known parametric form of the residues $\Delta_{ijk\ldots}$

One-loop boxes via Laurent expansion

- The residue of a box reads

$$\Delta_{ijkl}(q,\mu^2) = d_0 + d_2\mu^2 + d_4\,\mu^4 + (d_1 + d_3\mu^2)(q\cdot v_\perp)$$

- d_0 via 4-dimensional 4ple cuts [Britto, Cachazo, Feng (2004)]
- d_4 from d-dimensional 4-ple cuts in the limit $\mu^2 \to \infty$ [S. Badger (2008)]
  - d-dimensional solutions of a 4-ple cut

$$q_\pm = a^\mu \pm \sqrt{\alpha + \frac{\mu^2}{\beta^2}}\; v_\perp^\mu = \pm \frac{\sqrt{\mu^2}}{\beta}\; v_\perp^\mu + \mathcal{O}(1)$$

  - the integrand in the asymptotic limit $\mu^2 \to \infty$ of the cut-solutions

$$\left.\frac{\mathcal{N}(q^i,\mu^2)}{\prod_{m\neq i,j,k,l} D_m}\right|_{\text{cut}} = d_4\,\mu^4 + \mathcal{O}(\mu^3)$$

- d_1, d_2, d_3 are spurious and do not need to be computed

One-loop triangles via Laurent expansion

- The residue of a triangle

$$\Delta_{ijk}(q) = c_0 + c_7\,\mu^2 + (c_1 + c_8\mu^2)\,(q\cdot e_3) + c_2\,(q\cdot e_3)^2 + c_3\,(q\cdot e_3)^3$$
$$+ (c_4 + c_9\mu^2)\,(q\cdot e_4) + c_5\,(q\cdot e_4)^2 + c_6\,(q\cdot e_4)^3$$

- solutions of a triple cut parametrized by the variables t and μ^2

$$q_+^\mu = a^\mu + t\,e_3^\mu + \frac{\alpha + \mu^2}{2\,t}\,e_4^\mu, \qquad q_-^\mu = a^\mu + \frac{\alpha + \mu^2}{2\,t}\,e_3^\mu + t\,e_4^\mu$$

- in the limit $t \to \infty$ [Forde (2007)]

$$\left.\frac{\mathcal{N}(q_\pm)}{\prod_{m\neq i,j,k} D_m}\right|_{\text{cut}} = \Delta_{ijk} + \sum_l \frac{\Delta_{ijkl}}{D_l} + \sum_{lm}\frac{\Delta_{ijklm}}{D_l D_m}$$
$$= \Delta_{ijk} + d_1^\pm + d_2^\pm\,\mu^2 + \mathcal{O}(1/t)$$

with $d_i^+ + d_i^- = 0$

One-loop triangles via Laurent expansion

- In the asymptotic limit $t \to \infty$

$$\left.\frac{\mathcal{N}(q_\pm)}{\prod_{m\neq i,j,k} D_m}\right|_{\text{cut}} = d_1^\pm + d_2^\pm\,\mu^2 + \Delta_{ijk} + \mathcal{O}(1/t) \qquad \text{with } d_i^+ + d_i^- = 0$$

  - the integrand

$$\left.\frac{\mathcal{N}(q_\pm)}{\prod_{m\neq i,j,k} D_m}\right|_{\text{cut}} = n_0^\pm + n_4^\pm\,\mu^2 + (n_1^\pm + n_5^\pm\,\mu^2)\,t + n_2^\pm\,t^2 + n_3^\pm\,t^3 + \mathcal{O}(1/t)$$

  - the residue

$$\Delta_{ijk}(q_+) = c_0 + c_7\,\mu^2 - (c_4 + c_9\,\mu^2)\,t + c_5\,t^2 - c_6\,t^3 + \mathcal{O}(1/t)$$
$$\Delta_{ijk}(q_-) = c_0 + c_7\,\mu^2 - (c_1 + c_8\,\mu^2)\,t + c_2\,t^2 - c_3\,t^3 + \mathcal{O}(1/t)$$

  - by comparison we get

$$c_0 = \frac{n_0^+ + n_0^-}{2}, \quad c_1 = -n_1^-, \quad c_2 = n_2^-, \quad c_3 = -n_3^-, \quad \ldots$$