Variational approach to neutrino mass matrices

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Aim of variational approach to neutrino mass matrices:

- Investigation of relations between neutrino mass matrix textures, scales and neutrino masses.
- Deeper understanding of a relation between the PMNS mixing matrix and heavy neutrino mass matrix textures.
- Establishing new restrictions on light-heavy neutrino mixings.

Neutrino mass matrices, diagonalization, mixing

$$\mathcal{L}^{D+M} = \frac{1}{2}\tilde{N}^T \mathcal{C}^\dagger M \tilde{N} + H.c.$$

$$M = \begin{pmatrix} M_L & M_D \\ M_D^T & M_R \end{pmatrix}$$

$$\tilde{N} = \begin{pmatrix} \nu_L \\ \nu_R^C \end{pmatrix}$$

$$\tilde{N} = UN$$

$$U = \begin{pmatrix} K^T \\ U_R \end{pmatrix}$$

$$U^T M U = m_{diag}$$

$$m_{heavy} \simeq M_R$$

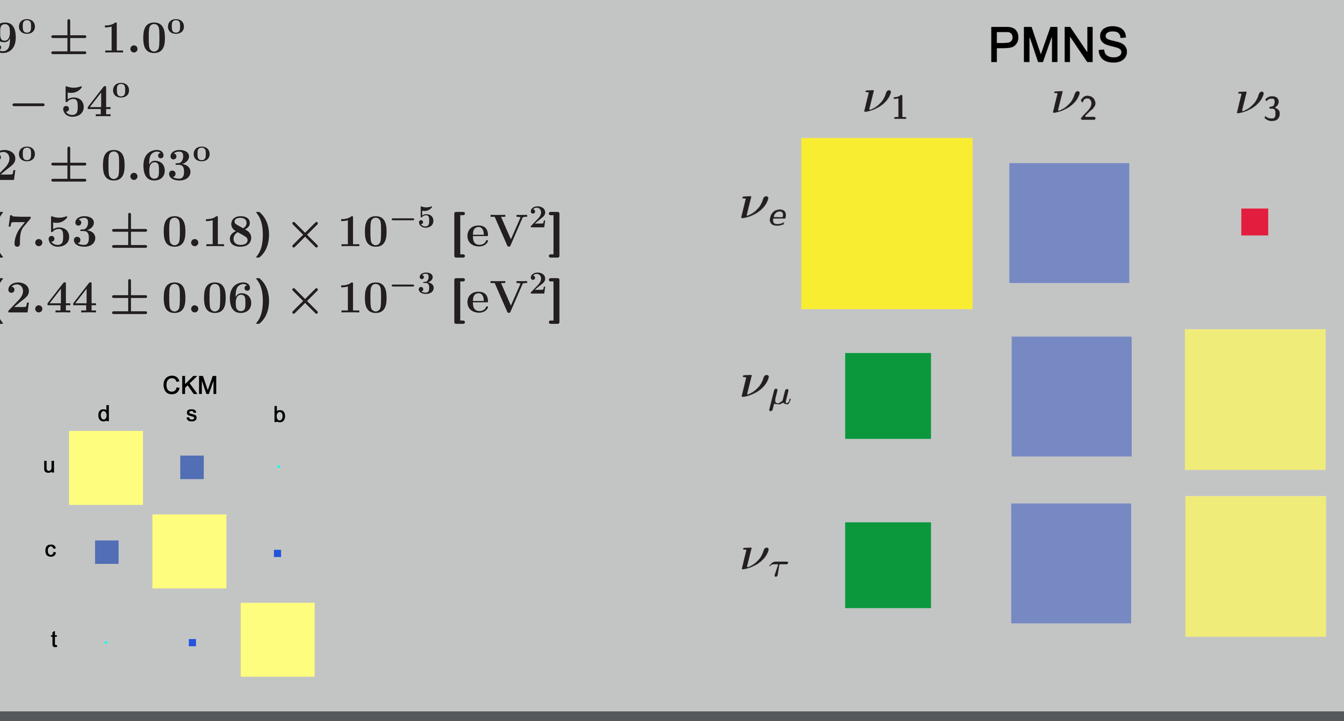
$$m_{light} \simeq -M_D^T M_R^{-1} M_D$$

$$K = \begin{array}{c} e\,\mu\,\tau \\ \left.\begin{matrix} \cdot\,\cdot\,\cdot \\ \cdot\,\cdot\,\cdot \\ \cdot\,\cdot\,\cdot \end{matrix}\right\}\text{light neutrinos} \\ \left.\begin{matrix} \square\,\cdot\,\cdot \\ \square\,\cdot\,\cdot \\ \square\,\cdot\,\cdot \end{matrix}\right\}\text{heavy neutrinos} \end{array}$$

Current status of masses and mixings [4]

$$U_{PMNS} = \begin{bmatrix} (0.810, 0.829) & (0.539, 0.562) & (0.147, 0.169) \\ (-0.485, -0.479) & (0.467, 0.563) & (0.669, 0.743) \\ (0.278, 0.339) & (-0.683, -0.626) & (0.647, 0.728) \end{bmatrix}$$

$$\theta_{12} = 33.9^o \pm 1.0^o$$
$$\theta_{23} = 36^o - 54^o$$
$$\theta_{13} = 9.12^o \pm 0.63^o$$
$$\Delta m_{21}^2 = (7.53 \pm 0.18) \times 10^{-5}\ [\mathrm{eV}^2]$$
$$\Delta m_{32}^2 = (2.44 \pm 0.06) \times 10^{-3}\ [\mathrm{eV}^2]$$



(A) Heavy neutrinos and physical consequences

Theory [1] :

$$\mathcal{L}_{CC} = \frac{g}{\sqrt{2}}\bar{N} K P_L \hat{l} W_\mu^+ + H.c.$$

$$\hat{l} = (e, \mu, \tau)$$

$$P_L = \frac{1-\gamma_5}{2},\ P_R = ?$$

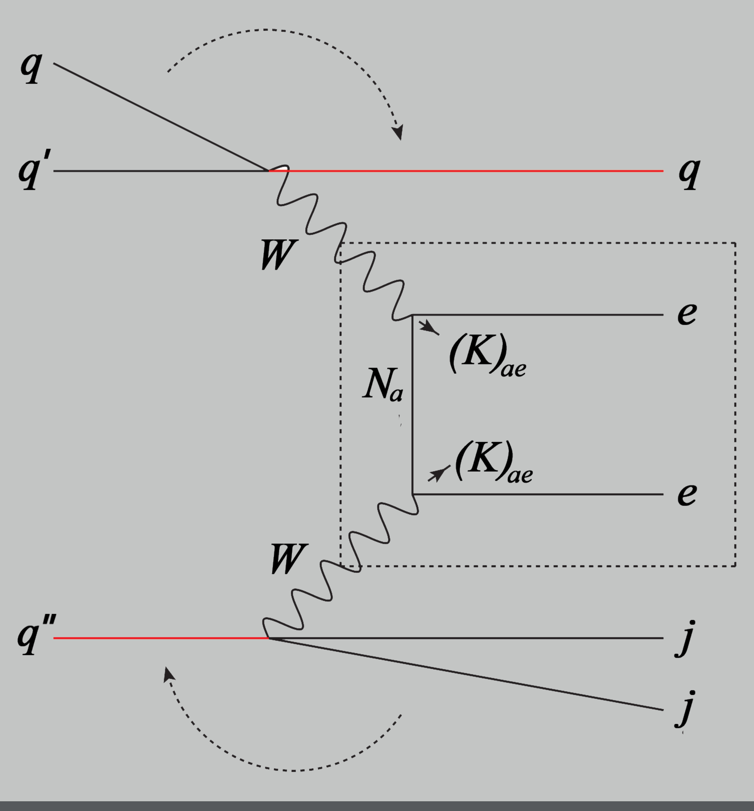
Physical processes, examples [3] :

(i) $p\,p \to e^- e^- j\,j$

(ii) $e^- e^- \to W^- W^-$

(iii) $(\beta\beta)_{0\nu}$:

$$(A, Z) \to (A, Z+2) + 2e^-$$



(B) Variational method - a basic mathematical description

1) Let A be an $n \times n$ Hermitan matrix with eigenvalues

$$\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$$

Then for each $1 \leq k \leq n$

$$\alpha_k = \max_{\substack{S \subset C^n \\ dim S = k}} \min_{\substack{x \in S \\ \|x\|=1}} x^\dagger A x = \min_{\substack{T \subset C^n \\ dim T = n-k+1}} \max_{\substack{x \in T \\ \|x\|=1}} x^\dagger A x$$

2) Let A and B be $n \times n$ Hermitian matrices and $Y = A + B$. Let

$$\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n, \quad \beta_1 \leq \beta_2 \leq \ldots \leq \beta_n, \quad \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n$$

be the eigenvalues of A, B and C, respectively. Then for $1 \leq i + j - 1 \leq n$

$$\alpha_i + \beta_j \leq \gamma_{i+j-1} \quad and \quad \gamma_{n-i-j} \leq \alpha_{n-i+1} + \beta_{n-j+1}$$

3) Let A be an $n \times n$ Hermitian matrix partitioned as

$$A = \begin{pmatrix} B & X \\ X^T & D \end{pmatrix}$$

where matrix B is of order $m \times m$, $m < n$. Let

$$\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n, \quad \beta_1 \leq \beta_2 \leq \ldots \leq \beta_m$$

be the eigenvalues of A and B, respectively. Then for $1 \leq i \leq m$

$$\alpha_i \leq \beta_i \leq \alpha_{n-m+i}$$

(C) Matrix Decomposition

$$M = \begin{pmatrix} 0 & M_D \\ M_D^T & M_R \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & M_R \end{pmatrix} + \begin{pmatrix} 0 & M_D \\ M_D^T & 0 \end{pmatrix} \equiv \hat{M}_R + \hat{M}_D$$

(D) Maximally simplified pattern

$$M_D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.8 & 0.9 & 1 \end{pmatrix} [\mathrm{GeV}], \quad \mathrm{M_R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 150 & 0 \\ 0 & 0 & 200 \end{pmatrix} [\mathrm{GeV}]$$

Direct calculation:

$$\lambda(\mathrm{M}) = \{0, 0, 0.793, 0.807, 150, 200\}[\mathrm{GeV}] \longleftarrow \text{Typical problem! [1]}$$

Variational approach:

We use (2B) and (3B) to estimate neutrino masses:

$$\lambda(\hat{\mathrm{M}}_\mathrm{D}) = \{-1.565, 0, 0, 0, 0, 1.565\}\ [\mathrm{GeV}]$$

$$\lambda(\hat{\mathrm{M}}_\mathrm{R}) = \{0, 0, 0, 0, 150, 200\}\ [\mathrm{GeV}]$$

From (2B) we have a quick estimation

$$\lambda(M) = \{0 \pm 1.565, 0 \pm 1.565, 0 \pm 1.565, 0 \pm 1.565, 150 \pm 1.565, 200 \pm 1.565\}\ [\mathrm{GeV}]$$

Not a quite impressive result, can we get more from that?

(E) Variational method and neutrinos: first conclusive outcomes

- In **[2]** a formal proof is given that for $M_R \gg M_D$ we can not get a fourth light sterile neutrino.
- In the CP invariance seesaw scenario with two mass scales $M_R \gg M_D$, $M_D \in M_{3\times n}$, $M_R \in M_{n \times n}$ and $\lambda(M_R) \gg |m_D|$ exactly 3 light neutrinos are present.

Proof scheme:

From Weyl's inequalities (2B)

$$|\lambda_i(M) - \lambda_i(\hat{M}_R)| \leq \rho(\hat{M}_D) \leq \|\hat{M}_D\|_F = \sum_{i,j} m_{ij}^2 \ , \ \rho(\hat{M}_D) = \max\{|\lambda(\hat{M}_D)|\}$$

But

$$\sum_{i,j} m_{ij}^2 \sim |m_D| \Rightarrow \rho(\hat{M}_D) \leq |m_D| \ , \ m_D \in M_D$$

On the other hand matrix $\hat{M}_R$ has at least 3 eigenvalues equal 0.

Thus $\quad |\lambda_i(M) - 0| \leq \rho(\hat{M}_D) \quad for\ \lambda_i(\hat{M}_R) = 0$

Hence, three eigenvalues of M must be smaller than $|m_D|$.

Conclusion (I): At least 3 light neutrinos exist.

Similar steps with assumption that $\lambda(M_R) \gg |m_D|$ gives
Conclusion (II): Remaining masses must be heavy.

Conclusions (I) and (II) implies Conclusion (III):
Heavy masses $N_{1,\ldots,n}$ are maximally shifted by $|m_D|$ from eigenvalues of M_R.

References

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Acknowledgments

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