Analysis of the breakdown of exponential decay of resonances

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1. Introduction

- Exponential law in alpha decay was first explained by Gamow, in 1928 [1]. In his paper, eigenfunctions with complex eigenenergies are present, which caused some concerns about validity of this approach. Nevertheless, this method leads to correct predictions both about the exponential time evolution in alpha decay and the Geiger-Nutall law.
- The exponential decay law is a very good approximation for times of order of several lifetimes. However, it was shown by Khalfin in 1958 [2], that the exponential decay cannot hold for all times. It breaks down at some time, then it follows the power law dependence [2, 3, 4, 5]:

$$\begin{aligned} t \lesssim t_{\text{breakdown}}: &\quad P(t) \sim \mathrm{e}^{-t/\tau} \\ t \gtrsim t_{\text{breakdown}}: &\quad P(t) \sim t^{-n},\ 1 \le n \le 4 \text{ in the cited papers} \end{aligned} \tag{1}$$

- The subject of this research is analysis of a simple, one-dimensional model of alpha decay, which exhibits breakdown of the exponential behavior [3]:

$$V(x) = \begin{cases} \frac{\lambda\hbar^2}{2ma}\delta(x-a) & \text{for } x > 0, \\ +\infty & \text{for } x \le 0. \end{cases} \tag{2}$$

2. First approach

- One searches for separable solutions $\psi(x,t) = \mathrm{e}^{-iEt/\hbar}\varphi(x)$ of the Schrödinger equation and an outgoing wave ansatz is assumed:

$$\varphi(x) = \begin{cases} A \sin kx & ; 0 < x \le a \\ Be^{ikx} & ; x > a \end{cases} \tag{3}$$

- From this condition and the boundary conditions:

$$ka \cot ka = ika - \lambda \tag{4}$$

- This is an equation with discrete set of complex solutions. Thus:
  - ⇒ For $\lambda \gg 1$, the solution can be found analytically:

$$k_n a \approx \frac{n\pi\lambda}{1+\lambda} - i\left(\frac{n\pi}{\lambda}\right)^2 \qquad (n = 1, 2, \dots, n\pi \ll \lambda) \tag{5}$$

  - ⇒ The spectrum is **discrete**
  - ⇒ Energies $E_n = \hbar^2 k_n^2/2m$ are **complex**
  - ⇒ The wavefunction diverges as $x \to \infty$

3. Second approach

- The Gamow's method approach, i.e. assuming that the wavefunction outside the well is an outgoing wave, leads to unphysical complex eigenenergies and unnormalizable eigenfunctions. Thus, the incoming wave term must be included:

$$\varphi(x) = \begin{cases} A\sin(kx) & ; 0 < x \le a \\ Be^{ikx} + e^{-ikx} & ; x > a \end{cases} \tag{6}$$

- Now, the spectrum is continuous and real. Functions A(k) and B(k), $k \in \mathbb{R}$, can be found:

$$A(k) = -\frac{2ika}{ka + \lambda e^{ika} \sin ka} \tag{7}$$

- The wavefunction can be written as an integral with respect to k, which is dominated by vicinity of the poles of -A(-k)A(k). After rotation of the integration contour, the solution is:

$$\psi(x,t) = e^{-i\pi/4}\int_0^\infty e^{-\frac{\hbar}{2m}k^2 t} f(e^{-i\pi/4}k, x)\, dk + \sum_{n=1}^{\infty} C(k_n, x) e^{-\frac{i\hbar}{2m}k_n^2 t} \tag{8}$$

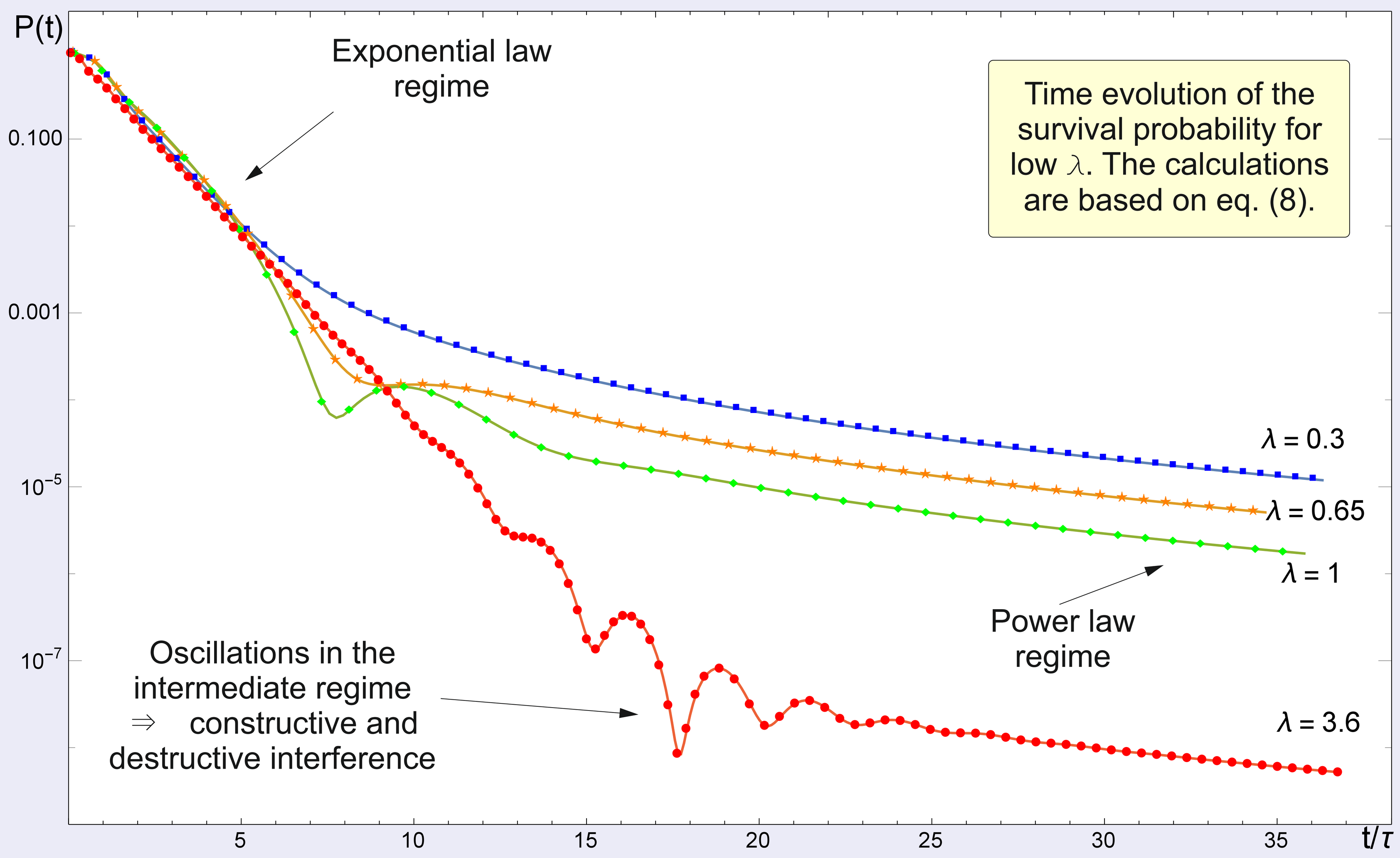
where $\phi(k) \equiv \int_0^a \psi(x',0)\sin kx'\, dx'$ is a function dependent on the initial condition, $f(k,x) \equiv -\frac{1}{2\pi}\phi(k)A(-k)A(k)\sin kx$ and $C(k_n, x)$ are contributions from the poles.

- The three contributions to the survival probability are:

$$P_{poles}(t) = \sum_{n=1}^{\infty} c_n e^{-\Gamma_n t/\hbar}, \quad P_{background}(t) \sim \frac{m^3 a^6}{\lambda^4 t^3}, \quad P_{interf}(t) \tag{9}$$

The second one becomes dominant when $\Gamma_1 t_{breakdown}/\hbar \sim 10 \ln \lambda$, and breakdown of the exponential behavior then occurs.

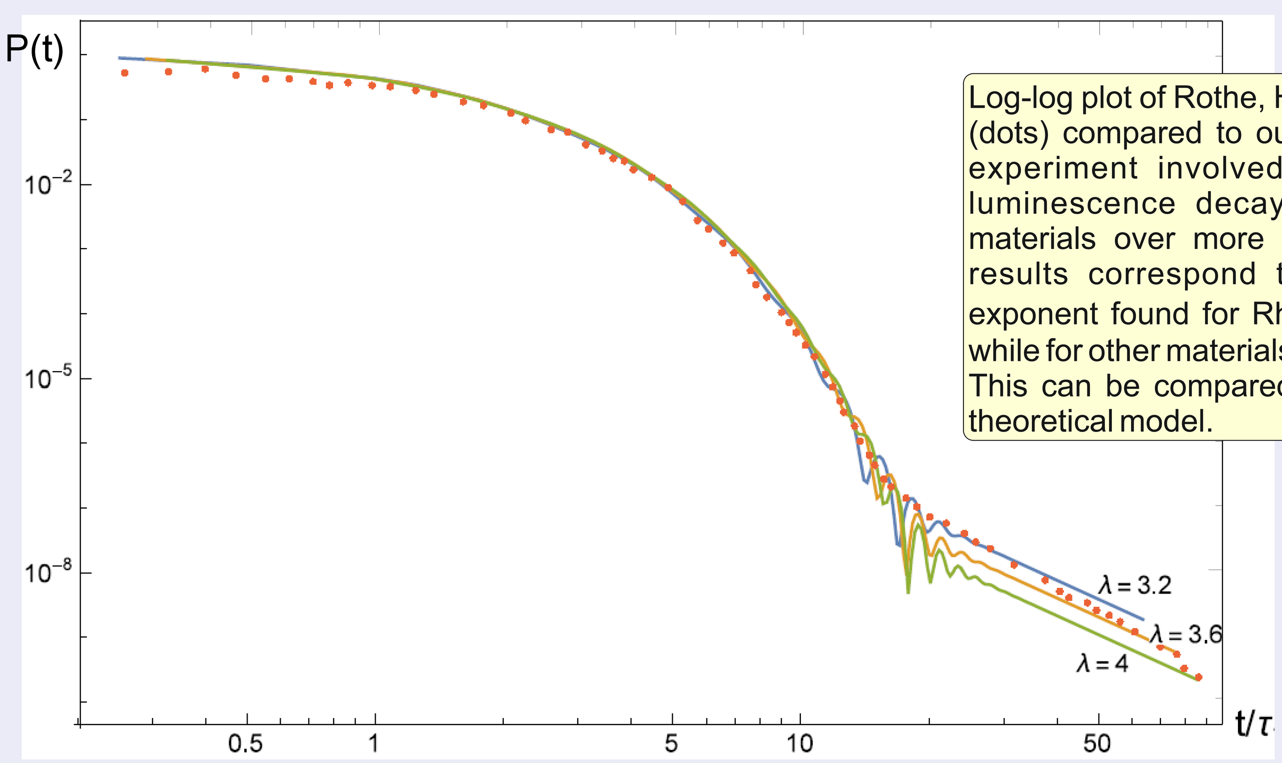
4. Numerical results



Time evolution of the survival probability for low λ. The calculations are based on eq. (8).

5. Experiment

- Rothe, Hintschich, Monkman (2006) - the first experimental proof of the turnover into the nonexponential decay regime
- The simple model studied by us successfully describes this experiment in both the exponential and power law regimes



Log-log plot of Rothe, Hintschich, Monkman data (dots) compared to our theoretical curves. The experiment involved measurements of the luminescence decays of dissolved organic materials over more than 20 lifetimes. These results correspond to Rhodamine 6G. The exponent found for Rhodamine 6G was n≈3.5, while for other materials it varies between 2 and 4. This can be compared to the value n=3 of the theoretical model.

6. Summary

- Power law time-dependence for asymptotic times
- Significance of the constructive and desctructive interference in the intermediate stage of the process
- Problems with experimental observation, recently observed in atomic data
- **This simple model successfully describes experiment in both the exponential and power law regions**

7. Literature

[1] G. Gamow, Z. Phys. 51, 204 (1928)
[2] L. A. Khalfin, Soviet Physics JETP, vol. 6 (33), 6 (1958)
[3] Cavalcanti, Carvalho, Rev. Bras. de Ensino de Fisica 21, 4 (1999)
[4] Rothe, Hintschich, Monkman, Phys. Rev. Lett. 96, 163601 (2006)
[5] Ishkhanyan, Krainov (2014)