Roots of unity and lepton mixing patterns from finite flavour symmetries

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Talk based on the paper

Renato M. Fonseca, Walter Grimus
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Introduction

- 3×3 mixing matrix U in lepton sector:
  two large and one small mixing angle
  explanation through underlying flavour symmetry?
- Notation: $|U|^2 \equiv \left(|U_{ij}|^2\right)$, $|T| \equiv (|T_{ij}|)$, flavour group G
- Idea by C.S. Lam (2008):
  "residual symmetries" in mass matrices
  with G non-abelian
  - Diagonalization of mass matrices effectively replaced by diagonalization of symmetry transformation matrices
  - Three possibilities:
    - one row of $|U|^2$ determined
    - one column of $|U|^2$ determined
    - $|U|^2$ completely determined

Introduction

Complete classification of possible $|U|^2$ under the following assumptions:

- Three flavours
- Majorana neutrinos
- G finite

Result:

17 sporadic mixing patterns and one infinite series (modulo permutations)

NOTE:
Finiteness of G is an ad hoc assumption for the mathematical treatment of the problem!

Residual symmetries

Fixing the notation:

Mass terms: Majorana neutrinos $\Rightarrow M_\nu^T = M_\nu$

$$\mathcal{L}_{\mathrm{mass}} = -\bar{\ell}_L M_\ell \ell_R + \frac{1}{2}\nu_L^T C^{-1} M_\nu \nu_L + \mathrm{H.c.}$$

Diagonalization:

$$U_\ell^\dagger M_\ell M_\ell^\dagger U_\ell = \mathrm{diag}\left(m_e^2, m_\mu^2, m_\tau^2\right), \quad U_\nu^T M_\nu U_\nu = \mathrm{diag}\left(m_1, m_2, m_3\right)$$

Mixing matrix: $U = U_\ell^\dagger U_\nu$

Residual symmetries

Idea of residual symmetries: C.S. Lam

- Weak basis $\Rightarrow \ell_L, \nu_L$ in same multiplet of G
- Flavour group G broken to subgroup G_ℓ in the charged-lepton and G_ν in the neutrino sector
- Charged-lepton and neutrino mass spectrum non-degenerate $\Rightarrow G_\ell$ and G_ν abelian
- Invariance of mass matrices under residual groups:

$$T \in G_\ell \Rightarrow T^\dagger M_\ell M_\ell^\dagger T = M_\ell M_\ell^\dagger$$
$$S \in G_\nu \Rightarrow \quad S^T M_\nu S = M_\nu$$

$$G_\ell \subseteq U(1) \times U(1) \times U(1), \quad G_\nu \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

- All $T \in G_\ell$ and $M_\ell M_\ell^\dagger$ simultaneously diagonalizable!
  All $S \in G_\ell$ and M_ν simultaneously diagonalizable!

Residual symmetries

In essence:

Diagonalization of $M_\ell M_\ell^\dagger$ replaced by diagonalization of the $T \in G_\ell$
Diagonalization of M_ν replaced by diagonalization of the $S \in G_\nu$

Remarks:

- If a single $T \in G_\ell$ has non-generate eigenvalues, then U_ℓ uniquely determined and $G_\ell \cong \mathbb{Z}_N$ (finiteness of G!)
- One can show:
  If all $T \in G_\ell$ degenerate, one can confine oneself to two generators T_1, T_2 of G_ℓ and
  $G_\ell \cong K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (Klein's four group)
- Without loss of generality $G_\nu \cong K$

Residual symmetries

Note:

- This method determines entries of $|U|^2$ as pure numbers, independent of parameters of any underlying theory.
- Therefore, residual symmetries cannot fix Majorana phases.
- $|U|^2$ is only determined up to independent permutations from the left and right.
- This method gives no information at all on the lepton masses!

Residual symmetries

From groups to mixing matrices:

1. Choose group G which has subgroup $G_\nu = K$
2. Find all subgroups G_ℓ which completey fix U_ℓ
3. Compute $|U|^2$ for all these subgroups

Many authors: (incomplete list)
C.S. Lam (2008); Ge, Dicus, Repko; He, Yin; Hernandez, Smirnov; B. Hu; de Adelhart Toorop, Feruglio, Hagedorn; Holthausen, Lim, Lindner; Hagedorn, Meroni, Vitale;. . .

General analysis: group-independent!

General analysis

Determination of possible forms of T

Preliminaries:

- Basis where $G_\nu = \{\mathbb{1}, S_1, S_2, S_3\}$ with $S_1 = \mathrm{diag}\,(1, -1, -1)$, $S_2 = \mathrm{diag}\,(-1, 1, -1)$, $S_3 = S_1 S_2$
- Consequently $U_\nu = \mathbb{1}$, $U = U_\ell^\dagger$, $UTU^\dagger = \hat{T}$ diagonal
- $T \Rightarrow |U|^2$

General analysis

Series of steps: (3×3 permutation matrices P_1, P_2, P)

1. 5 basic forms of $|T|$ modulo permutations $P_1|T|P_2$
2. Internal (CKM-type) phase of T
3. Inequivalent forms of $|T|$ through $|T| \to |T|P$
4. Exclusion of forms 1 and 4 which do not lead to finite groups
5. External (Majorana-type) phases of T
6. Possible patterns of $|U|^2$ modulo permutations $P_1|U|^2P_2$

General analysis

Basic forms of $|T|$:

1. $Y^{(ij)} \equiv T^\dagger S_i T S_j \in G$

   $\Rightarrow S_j^{-1} Y^{(ij)} S_j = (Y^{(ij)})^\dagger$, $\det Y^{(ij)} = 1$

   $\Rightarrow$ eigenvalues $1,\ \lambda^{(ij)},\ (\lambda^{(ij)})^*$

2. $\sum_{k=1}^{3} S_k = -\mathbb{1} \Rightarrow \sum_{k=1}^{3} \mathrm{Tr}\, Y^{(kj)} = \sum_{k=1}^{3} \mathrm{Tr}\, Y^{(ik)} = 1$ or

   $$\sum_{k=1}^{3} \left(\lambda^{(kj)} + \lambda^{(kj)*}\right) + 2 = \sum_{k=1}^{3} \left(\lambda^{(ik)} + \lambda^{(ik)*}\right) + 2 = 0$$

   for $i, j = 1, 2, 3$

3. $|T_{ij}|^2 = \frac{1}{2}\left(1 + \mathrm{Re}\,\lambda^{(ij)}\right)$

General analysis

Generic equation:

$$\sum_{k=1}^{3} (\lambda_k + \lambda_k^*) + 2 = 0$$

group G finite $\Rightarrow$ all $\lambda^{(ij)}$ roots of unity
vanishing sum of roots of unity $\Rightarrow$ find solutions by using a Theorem of Conway and Jones

Only three solutions:

$$(\lambda_1, \lambda_2, \lambda_3) = \begin{cases} (i, \omega, \omega) & \text{(A)} \\ (\omega, \beta, \beta^2) & \text{(B)} \\ (-1, \lambda, -\lambda) & \text{(C)} \end{cases}$$

$\omega = e^{2\pi i/3}$, $\beta = e^{2\pi i/5}$, $\lambda = e^{i\vartheta}$ (arbitrary root of unity)

General analysis

Form 1: $$|T| = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Form 2: $$|T| = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Form 3: $$|T| = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} & \frac{1}{2} \end{pmatrix}$$

Form 4: $$|T| = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \\ \frac{1}{2} & \frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} \end{pmatrix}$$

Form 5: $$|T| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \quad [\sin^2\theta = (1-\cos\vartheta)/2]$$

Results

General solution: Fonseca, Grimus (2014), arXiv:1405.3678

- Complete solution: 17 sporadic patterns of $|U|^2$, one series
- All sporadic cases are ruled out!
- Infinite series: depends on $\sigma = e^{2\pi i p/n}$ with $p, n \in \mathbb{Z}$
  $\exists$ range of σ such that $|U|^2$ compatible with data

Results

Infinite series:

$$\text{Case } \mathcal{C}_2 : \quad |U|^2 = \frac{1}{3} \begin{pmatrix} 1 & 1 + \mathrm{Re}\,\sigma & 1 - \mathrm{Re}\,\sigma \\ 1 & 1 + \mathrm{Re}\,(\omega\sigma) & 1 - \mathrm{Re}\,(\omega\sigma) \\ 1 & 1 + \mathrm{Re}\,(\omega^2\sigma) & 1 - \mathrm{Re}\,(\omega^2\sigma) \end{pmatrix}$$

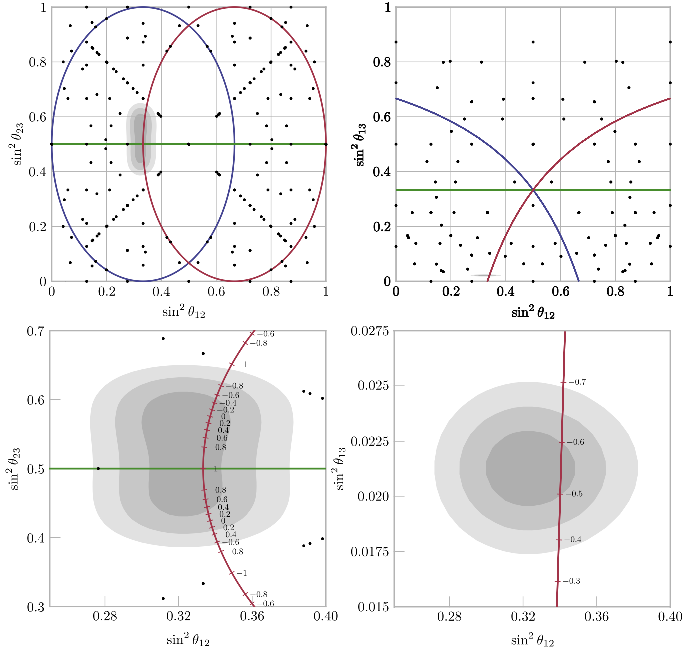
$$\omega = \exp(2\pi i/3), \quad \sigma = \exp(2i\pi p/n) \text{ with } p \text{ coprime to } n$$

Three choices:

- red: $\cos^2\theta_{13}\sin^2\theta_{12} = 1/3$
- blue: $\cos^2\theta_{13}\cos^2\theta_{12} = 1/3$
- green: $\sin^2\theta_{13} = 1/3$

Properties of $|U|^2$ in red infinite series:

- $|U|^2$ depends on $\mathrm{Re}\,\sigma^6$: $-0.69 \lesssim \mathrm{Re}\,\sigma^6 \lesssim -0.37$ (Forero et al., thanks to M. Tórtola)
- CKM-type phase in $|U|^2$ trivial ($\pm\pi$)
- $\sin^2\theta_{12} \geq 1/3$



$\sin^2\theta_{23}$
$\sin^2\theta_{12}$
$\sin^2\theta_{13}$
$\sin^2\theta_{12}$
$\sin^2\theta_{23}$
$\sin^2\theta_{12}$
$\sin^2\theta_{13}$
$\sin^2\theta_{12}$
0 0.2 0.4 0.6 0.8 1
0.3 0.4 0.5 0.6 0.7
0.28 0.32 0.36 0.40
0.015 0.0175 0.020 0.0225 0.025 0.0275

Results

Minimal groups for the infinite series:

- $\Delta(6m^2) \equiv (\mathbb{Z}_m \times \mathbb{Z}_m) \rtimes S_3$ with $m = \mathrm{lcm}(6, n)/3$ when $9 \nmid n$
- $(\mathbb{Z}_m \times \mathbb{Z}_{m/3}) \rtimes S_3$ with $m = \mathrm{lcm}(2, n)$ when $9 \mid n$

For instance:

- $n = 9,\ 18$ with $m = 18$, $G = (\mathbb{Z}_{18} \times \mathbb{Z}_6) \rtimes S_3$ and $\mathrm{order}(G) = 648$
- $n = 11,\ 22,\ 33,\ 66$ with $m = 22$, $G = \Delta(6 \times 22^2)$ and $\mathrm{order}(G) = 2904$

Results

Sporadic mixing pattern with minimal groups:

One non-degenerate $T \hookrightarrow \mathcal{C}_i$, two degenerate T_1, $T_2 \hookrightarrow \mathcal{CD}_j$

- S_4 for $\mathcal{C}_1/\mathcal{CD}_2$
- $PSL\,(2,7)$ for $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{C}_5$, $\mathcal{CD}_1$
- $\Sigma\,(360 \times 3)$ for $\mathcal{C}_6/\mathcal{C}_{15}$, $\mathcal{C}_7$, $\mathcal{C}_8/\mathcal{C}_{17}$, $\mathcal{C}_9$, $\mathcal{C}_{10}$, $\mathcal{C}_{14}$, $\mathcal{C}_{16}$, $\mathcal{CD}_4$
- A_5 for $\mathcal{C}_{11}/\mathcal{C}_{13}$, $\mathcal{C}_{12}$, $\mathcal{CD}_3$
- A_4 for $\mathcal{C}_{30}$

de Adelhart Toorop, Feruglio, Hagedorn, arXiv:1112.1340
Hagedorn, Meroni, Vitale, arXiv:1307.5308

Example of a sporadic case: $(5-\sqrt{21})/12 \simeq 0.035 > s_{13}^2$

$$\mathcal{C}_5: \quad |U|^2 = \begin{pmatrix} \frac{1}{12}\left(5+\sqrt{21}\right) & \frac{1}{6} & \frac{1}{12}\left(5-\sqrt{21}\right) \\ \frac{1}{12}\left(5-\sqrt{21}\right) & \frac{1}{6} & \frac{1}{12}\left(5+\sqrt{21}\right) \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{pmatrix}$$

Summary

- Hypothesis of $|U|^2$ determined by residual symmetries leads to 17 sporadic mixing pattern and one series
- All 17 sporadic mixing patterns are ruled out
- Series depends on parameter $\sigma = \exp(2\pi i p/n)$ with rational number p/n, $|U|^2$ compatible with data for range of σ
- Predictions of series:
  $\sin^2\theta_{12} \geq 1/3$, trivial CKM-type phase

Thank you for your attention!