Master Formulae for $\Delta F = 2$ NLO-QCD Factors in the Standard Model and Beyond

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Abstract

We present analytic formulae for the QCD renormalization group factors relating the Wilson coefficients $C_i(\mu_t)$ and $C_i(\mu)$, with $\mu_t = \mathcal{O}(m_t)$ and $\mu < \mu_t$, of the $\Delta F = 2$ dimension six four-quark operators Q_i in the Standard Model and in all of its extensions. Analogous analytic formulae for the QCD factors relating the matrix elements $\langle Q_i(2 \text{ GeV})\rangle$ and $\langle Q_i(\mu_K)\rangle$ with $\mu_K < 2$ GeV are also presented. The formulae are given in the NDR scheme. The strongest renormalization-group effects are found for the operators with the Dirac structures $(1 - \gamma_5) \otimes (1+\gamma_5)$ and $(1-\gamma_5)\otimes(1-\gamma_5)$. We calculate the matrix elements $\langle \overline{K}^0|Q_i|K^0\rangle$ in the NDR scheme using the lattice results in the LRI scheme. We give expressions for the mass differences ΔM_K and ΔM_B and the CP-violating parameter ϵ_K in terms of the non-perturbative parameters B_i and the Wilson coefficients $C_i(\mu_t)$. The latter summarize the dependence on new physics contributions.

1 Introduction

Renormalization group short-distance QCD effects play an important role in $K^0 - \overline{K}^0$ and $B^0_{d,s} - \overline{B}^0_{d,s}$ mixing within the Standard Model (SM) and its extensions [1, 2]. They can be calculated by solving renormalization group equations that govern the scale dependence of the Wilson coefficients $C_i(\mu)$ of the relevant $\Delta F = 2$ operators Q_i. The resulting effective weak Hamiltonian reads

$$H_{\text{eff}}^{\Delta F=2} = \frac{G_F^2}{16\pi^2} M_W^2 \sum_i V_{\text{CKM}}^i C_i(\mu) Q_i \ . \tag{1.1}$$

Here G_F is the Fermi constant and V_{CKM}^i the Cabibbo-Kobayashi-Maskawa (CKM) factor equal to $(V_{tb}^* V_{td})^2$ in the case of $B^0_d - \overline{B}^0_d$ mixing in the SM. Beyond the SM other factors not proportional to CKM elements are generally present. Using this Hamiltonian one can calculate $\Delta F = 2$ amplitudes, in particular the mass differences ΔM_K and $\Delta M_{d,s}$ in the $K^0 - \overline{K}^0$ and $B^0_{d,s} - \overline{B}^0_{d,s}$ systems and the CP-violating parameter ε_K.

Within the SM there is only one single operator

$$Q_1^{\text{VLL}} = \left(\bar{s}^\alpha \gamma_\mu P_L d^\alpha\right)\left(\bar{s}^\beta \gamma^\mu P_L d^\beta\right) \tag{1.2}$$

relevant for $K^0 - \overline{K}^0$ mixing, with analogous operators for $B^0_{d,s} - \overline{B}^0_{d,s}$ mixing obtained from (1.2) through the appropriate change of flavours. Beyond the SM the full set of dimension six operators contributing to $K^0 - \overline{K}^0$ mixing consists of 8 operators that can be split into 5 separate sectors according to the chirality of the quark fields they contain. These operators are listed in (2.1). Corresponding operators contributing to $B^0_{d,s} - \overline{B}^0_{d,s}$ mixing exist.

The general expression for $C_i(\mu)$ is given by

$$\vec{C}(\mu) = \hat{U}(\mu, \mu_t)\vec{C}(\mu_t) \tag{1.3}$$

where $\vec{C}$ is a column vector built out of the C_i's and $\hat{U}(\mu, \mu_t)$ is the renormalization group matrix. $\vec{C}(\mu_t)$, with $\mu_t = \mathcal{O}(m_t)$, are the initial conditions which depend on the short distance physics at high energy scales. In particular they depend on the top quark mass and the couplings and masses of new particles in extensions of the SM. We will later briefly discuss the

case of scales much higher than m_t. Otherwise μ_t denotes a high energy scale in the range, say, $M_W \leq \mu_t \leq 2m_t$.

While the initial conditions $C_i(\mu_t)$ at the NLO level are known only in the SM [3, 4] and in some of its extensions [4], all the ingredients are available to compute the NLO evolution matrix $\hat{U}(\mu, \mu_t)$ for all possible extensions of the SM. Indeed, the two-loop anomalous dimension matrix for all $\Delta F = 2$ four-quark dimension six operators has been calculated in the regularization independent renormalization scheme (RI) in [5] and in the NDR scheme in [6]. Together with the known one-loop anomalous dimension matrix [7, 8] and the known β function, the evolution matrix can be straightforwardly computed by means of the methods reviewed in [1, 2].

The LO analytic expressions for $\hat{U}(\mu, \mu_t)$ can be found in [8]. For phenomenological applications it is useful to derive analogous expressions including NLO corrections. The first step in this direction has been made in [9] where $\hat{U}(\mu, \mu_s)$ with $\mu_s > m_t$ has been written as

$$\hat{U}(\mu, \mu_s) = \hat{U}(\mu, \mu_t)\hat{U}(\mu_t, \mu_s) \tag{1.4}$$

with $\hat{U}(\mu_t, \mu_s)$ given analytically in the Landau RI scheme (LRI) but $\hat{U}(\mu, \mu_t)$ evaluated numerically for $\mu = 2$ GeV, $\mu_t = m_t$ and particular values of m_c, m_b and α_s. The corresponding results for $B_{d,s}^0 - \overline{B}_{d,s}^0$ mixing have not been presented in [9].

It should be emphasized that NLO corrections are necessary for a satisfactory matching of the Wilson coefficients to the matrix elements obtained from lattice calculations. Moreover as demonstrated in [8, 9] the inclusion of QCD corrections at the LO and the NLO level is mandatory in order to place reliable constraints on the parameters in the extensions of the SM, in particular on the squark mass matrices in supersymmetric theories.

The purpose of our paper is to present NLO analytic formulae for the matrix $\hat{U}(\mu, \mu_t)$ relevant for $B_{d,s}^0 - \overline{B}_{d,s}^0$ mixing ($\mu = \mathcal{O}(m_b)$), and $K^0 - \overline{K}^0$ mixing ($\mu = \mathcal{O}(1-2 \text{ GeV})$). These formulae when combined with the initial conditions $\vec{C}(\mu_t)$ and the hadronic matrix elements $\langle \vec{Q}(\mu) \rangle$ will allow to calculate in the future the $\Delta F = 2$ amplitudes for any extension of the SM.

The formulae given below for $\hat{U}(\mu, \mu_t)$ apply to the situation in which the initial conditions for the Wilson coefficients are known at $\mu_t = \mathcal{O}(m_t)$ and the evolution down to scales $\mu < \mu_t$ is performed in an effective theory with the top quark and the heavy new particles integrated

out. Whether the top quark and the new particles have been integrated out at a single scale μ_t or at different scales, say μ_t, μ_{s_1}, μ_{s_2} with $\mu_t < \mu_{s_1} < \mu_{s_2}$, is immaterial here. What matters are the values of the Wilson coefficients at μ_t and not how they have been evaluated from the contributions at scales higher than μ_t. On the other hand in the process of the evaluation of $C_i(\mu_t)$ large logarithms $\log \mu_{s_1}/\mu_t$, $\log \mu_{s_2}/\mu_{s_1}$ may appear. These logarithms have to be resummed which results in new evolution functions $\hat{U}(\mu_t, \mu_{s_1})$, $\hat{U}(\mu_{s_1}, \mu_{s_2})$, etc. As discussed in [8, 9] the structure of these matrices is model dependent and consequently beyond the scope of the present paper. We will, however, provide an analytic formula for the evolution $\hat{U}(\mu_t, \mu_s)$ with $\mu_t \ll \mu_s$ in an effective $f = 6$ theory in which only SM degrees of freedom are present and all new particles have been integrated out.

Now, the lattice results for the matrix elements $\langle \vec{Q}(\mu) \rangle$ are usually given at $\mu = 2$ GeV. In what follows we will denote this scale by μ_L. On the other hand large-N approaches, the chiral quark model and any non-perturbative method in which the low-energy degrees of freedom are mesons provide these matrix elements at scales $\mu_K \leq 1$GeV. In our opinion it would be useful to have the matrix elements obtained by means of different methods at a common "standard" scale, which we will choose to be μ_L in the following. This is achieved using the formula

$$\langle \vec{Q}(\mu_L) \rangle = \hat{U}^T(\mu_K, \mu_L) \langle \vec{Q}(\mu_K) \rangle \tag{1.5}$$

where $\langle \vec{Q}(\mu_K) \rangle$ are the matrix elements calculated for scales $\mu_K < \mu_L$ and $\hat{U}(\mu_K, \mu_L)$ is the renormalization group evolution matrix. In our paper we provide analytic formulae for $\hat{U}(\mu_K, \mu_L)$.

At this point we would like to stress that our paper is addressed first of all to the practitioners of weak decays who do not want to get involved with the details of NLO calculations but rather would like to use the final QCD factors in phenomenological applications. On the other hand it should also be useful to experts. Indeed, having explicit analytic formulae, rather than numerical values, not only gives the freedom to change input parameters but also makes possible the checking of a given calculation. In particular when multiplying the $\hat{U}$ matrices like in (1.4) one easily generates higher-order terms in α_s which really do not belong to NLO corrections. While these corrections should be removed from NLO expressions, this is not always done in

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the literature. Consequently, already at this stage unnecessary discrepancies of the order of 5% between calculations performed by different groups may arise. These higher-order terms in α_s are consistently removed in the present paper. We are aware of the fact that some of the formulae presented below are rather long. Nevertheless we believe that they should turn out to be useful in future phenomenological applications.

The paper is organized as follows. In Section 2 we give the list of the $\Delta F = 2$ operators in question and establish our notation. In Section 3 we give analytic formulae for the QCD factors $[\eta_{ij}(\mu)]_a$ that represent the evolution matrix $\hat{U}(\mu, \mu_t)$ in (1.3) in five different sectors, $a = (\text{VLL, LR, SLL, VRR, SRR})$, in the leading order (LO) and the next-to-leading (NLO) approximation in the NDR scheme. In Section 4 we give the analogous formulae for the QCD factors $[\rho_{ij}(\mu_K)]_a$ which represent the evolution matrix $\hat{U}(\mu_K, \mu_L)$ in (1.5). In Section 5 we provide numerical results for $[\eta_{ij}(\mu)]_a$ and $[\rho_{ij}(\mu)]_a$ in the NDR scheme. In section 6 we discuss the transformation rules for obtaining the corresponding results in other renormalization schemes and we present the relation between the QCD factors calculated here and the QCD factors η_B and η_2 used in phenomenological applications. In Section 7 we calculate the matrix elements $\langle \overline{K}^0 | Q_i | K^0 \rangle$ in the NDR scheme using the lattice results in the LRI scheme [9, 10]. We give general expressions for the mass differences ΔM_K and ΔM_B and the CP-violating parameter ϵ_K in terms of the non-perturbative parameters B_i^a and the Wilson coefficients $C_i(\mu_t)$. We conclude in Section 8. For completeness we list in appendix A the one-loop and two-loop anomalous dimension matrices that we have used in our paper. Appendix B gives the general formulae for the $\hat{U}$ matrices which have been used to obtain the analytic formulae of sections 3 and 4. Finally in Appendix C we give analytic formulae for the evolution matrix $\hat{U}(\mu_t, \mu_s)$.

2 Basic Formulae

For definiteness, we will give explicit expressions for the operators responsible for the $K^0 - \overline{K}^0$ mixing. The operators belonging to the VLL, LR and SLL sectors read

$$Q_1^{\text{VLL}} = (\bar{s}^\alpha \gamma_\mu P_L d^\alpha)(\bar{s}^\beta \gamma^\mu P_L d^\beta),$$

$$\begin{aligned}
Q_1^{\mathrm{LR}} &= (\bar{s}^\alpha \gamma_\mu P_L d^\alpha)(\bar{s}^\beta \gamma^\mu P_R d^\beta), \\
Q_2^{\mathrm{LR}} &= (\bar{s}^\alpha P_L d^\alpha)(\bar{s}^\beta P_R d^\beta), \\[4pt]
Q_1^{\mathrm{SLL}} &= (\bar{s}^\alpha P_L d^\alpha)(\bar{s}^\beta P_L d^\beta), \\
Q_2^{\mathrm{SLL}} &= (\bar{s}^\alpha \sigma_{\mu\nu} P_L d^\alpha)(\bar{s}^\beta \sigma^{\mu\nu} P_L d^\beta),
\end{aligned} \tag{2.1}$$

where α, β are colour indices, $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$ and $P_{L,R} = \frac{1}{2}(1 \mp \gamma_5)$. The operators belonging to the two remaining sectors (VRR and SRR) are obtained from Q_1^{VLL} and Q_i^{SLL} by interchanging P_L and P_R. Since QCD preserves chirality, there is no mixing between different sectors. Moreover, the anomalous dimension matrices and the evolution matrices in the VRR and SRR sectors are the same as in the VLL and SLL sectors, respectively. Therefore, in the following, we shall consider only the VLL, LR and SLL sectors. However, one should remember that the initial conditions $C_i(\mu_t)$ are generally changed when P_L and P_R are interchanged. The operators in the case of $B_d^0 - \overline{B}_d^0$ mixing are obtained from (2.1) through the replacement $s \to b$. Performing the subsequent replacement $d \to s$ gives the operators contributing to $B_s^0 - \overline{B}_s^0$ mixing. The one-loop and two-loop anomalous dimension matrices of the operators (2.1) are given in appendix A.

Restricting the discussion to the VLL, LR and SLL sectors $\hat{U}(\mu_1, \mu_2)$ takes the following form

$$\hat{U}(\mu_1, \mu_2) = \begin{pmatrix} [\eta(\mu_1, \mu_2)]_{\mathrm{VLL}} & 0 & 0 \\ 0 & [\hat{\eta}(\mu_1, \mu_2)]_{\mathrm{LR}} & 0 \\ 0 & 0 & [\hat{\eta}(\mu_1, \mu_2)]_{\mathrm{SLL}} \end{pmatrix} \tag{2.2}$$

where $[\hat{\eta}(\mu_1, \mu_2)]_{\mathrm{LR}}$ and $[\hat{\eta}(\mu_1, \mu_2)]_{\mathrm{SLL}}$ are 2×2 matrices and $\mu_1 < \mu_2$. In what follows we will use a short-hand notation, denoting the QCD factors representing $\hat{U}(\mu, \mu_t)$ and $\hat{U}(\mu_K, \mu_L)$ by

$$[\hat{\eta}(\mu, \mu_t)]_a \equiv [\hat{\eta}(\mu)]_a = \left[\hat{\eta}^{(0)}(\mu)\right]_a + \frac{\alpha_s^{(f)}(\mu)}{4\pi}\left[\hat{\eta}^{(1)}(\mu)\right]_a, \tag{2.3}$$

$$[\hat{\rho}(\mu_K, \mu_L)]_a \equiv [\hat{\rho}(\mu_K)]_a = \left[\hat{\rho}^{(0)}(\mu_K)\right]_a + \frac{\alpha_s^{(3)}(\mu_K)}{4\pi}\left[\hat{\rho}^{(1)}(\mu_K)\right]_a, \tag{2.4}$$

respectively. That is, we will suppress the high-energy scale μ_t in the argument of the η-factors. Similarly, we will suppress the "lattice scale" μ_L in the argument of the ρ-factors. Using this

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notation we have for instance

$$C_1^{\text{VLL}}(\mu_b) = [\eta(\mu_b)]_{\text{VLL}}\, C_1^{\text{VLL}}(\mu_t), \tag{2.5}$$

$$\begin{pmatrix} C_1^{\text{LR}}(\mu_b) \\ C_2^{\text{LR}}(\mu_b) \end{pmatrix} = \begin{pmatrix} [\eta_{11}(\mu_b)]_{\text{LR}} & [\eta_{12}(\mu_b)]_{\text{LR}} \\ [\eta_{21}(\mu_b)]_{\text{LR}} & [\eta_{22}(\mu_b)]_{\text{LR}} \end{pmatrix} \begin{pmatrix} C_1^{\text{LR}}(\mu_t) \\ C_2^{\text{LR}}(\mu_t) \end{pmatrix}, \tag{2.6}$$

$$\langle Q_1^{\text{VLL}}(\mu_L)\rangle = [\rho(\mu_K)]_{\text{VLL}}\, \langle Q_1^{\text{VLL}}(\mu_K)\rangle, \tag{2.7}$$

$$\begin{pmatrix} \langle Q_1^{\text{LR}}(\mu_L)\rangle \\ \langle Q_2^{\text{LR}}(\mu_L)\rangle \end{pmatrix} = \begin{pmatrix} [\rho_{11}(\mu_K)]_{\text{LR}} & [\rho_{21}(\mu_K)]_{\text{LR}} \\ [\rho_{12}(\mu_K)]_{\text{LR}} & [\rho_{22}(\mu_K)]_{\text{LR}} \end{pmatrix} \begin{pmatrix} \langle Q_1^{\text{LR}}(\mu_K)\rangle \\ \langle Q_2^{\text{LR}}(\mu_K)\rangle \end{pmatrix} \tag{2.8}$$

and analogous formulae for the SLL sector. Note that in accordance with (1.5), the transpose of $[\hat{\rho}(\mu_K)]_{\text{LR}}$ enters the transformation (2.8).

In Section 3 we will give analytic formulae for the LO factors $\left[\eta^{(0)}(\mu)\right]_{\text{VLL}}$, $\left[\eta_{ij}^{(0)}(\mu)\right]_{\text{LR}}$, $\left[\eta_{ij}^{(0)}(\mu)\right]_{\text{SLL}}$ and the NLO factors $\left[\eta^{(1)}(\mu)\right]_{\text{VLL}}$, $\left[\eta_{ij}^{(1)}(\mu)\right]_{\text{LR}}$, $\left[\eta_{ij}^{(1)}(\mu)\right]_{\text{SLL}}$. The corresponding expressions for the ρ-factors are given in Section 4. $\alpha_s^{(f)}(\mu)$ is the QCD coupling constant in an effective theory with f flavours: $f = 5$ for $\mu_b < \mu < \mu_t$, $f = 4$ for $\mu_c < \mu < \mu_b$, and $f = 3$ for $\mu < \mu_c$, where $\mu_t = \mathcal{O}(m_t)$, $\mu_b = \mathcal{O}(m_b)$ and $\mu_c = \mathcal{O}(m_c)$. We impose the continuity relations

$$\alpha_s^{(3)}(\mu_c) = \alpha_s^{(4)}(\mu_c), \qquad \alpha_s^{(4)}(\mu_b) = \alpha_s^{(5)}(\mu_b) \ . \tag{2.9}$$

The general expression for $\alpha_s^{(f)}$ reads

$$\frac{\alpha_s^{(f)}(\mu)}{4\pi} = \frac{1}{\beta_0 \ln\left(\mu^2/\Lambda_{\overline{MS}}^{(f)\,2}\right)} - \frac{\beta_1}{\beta_0^3} \frac{\ln\ln\left(\mu^2/\Lambda_{\overline{MS}}^{(f)\,2}\right)}{\ln^2\left(\mu^2/\Lambda_{\overline{MS}}^{(f)\,2}\right)} \tag{2.10}$$

with

$$\beta_0 = 11 - \frac{2}{3}f, \qquad \beta_1 = 102 - \frac{38}{3}f. \tag{2.11}$$

$\Lambda_{\overline{MS}}^{(f)}$ is the QCD scale parameter in a theory with f quark flavours [11]. The existing analyses of high energy processes give $\alpha_{\overline{MS}}^{(5)}(M_Z) = 0.118 \pm 0.003$ [12] or equivalently $\Lambda_{\overline{MS}}^{(5)} = (226\ {}^{+41}_{-36}\ )\text{MeV}$.

The evolution matrix, $\hat{U}(\mu, \mu_t)$, is given as follows:

$$\hat{U}(\mu, \mu_t) = T_g \exp\left[\int_{g(\mu_t)}^{g(\mu)} dg' \frac{\hat{\gamma}^T(g')}{\beta(g')}\right] \tag{2.12}$$

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with g denoting the QCD effective coupling constant and T_g an ordering operation defined in [1]. $\beta(g)$ governs the evolution of g and $\hat{\gamma}$ is the anomalous dimension matrix of the operators involved.

We also have

$$\hat{U}(\mu_K, \mu_t) = \hat{U}(\mu_K, \mu_c)\hat{U}(\mu_c, \mu_b)\hat{U}(\mu_b, \mu_t) \tag{2.13}$$

with the three factors on the r.h.s. evaluated in $f = 3$, $f = 4$ and $f = 5$ effective theories, respectively. Now,

$$\hat{U}(\mu_L, \mu_t) = \hat{U}(\mu_L, \mu_b)\hat{U}(\mu_b, \mu_t), \qquad \hat{U}(\mu_K, \mu_t) = \hat{U}(\mu_K, \mu_L)\hat{U}(\mu_L, \mu_t). \tag{2.14}$$

This means that knowing $\eta_{ij}(\mu_L)$ and $\rho_{ij}(\mu_K)$ allows to calculate $\eta_{ij}(\mu_K)$.

Keeping the first two terms in the expansions of $\hat{\gamma}(g)$ and $\beta(g)$ in powers of g

$$\hat{\gamma}(g) = \hat{\gamma}^{(0)}\frac{\alpha_s}{4\pi} + \hat{\gamma}^{(1)}\left(\frac{\alpha_s}{4\pi}\right)^2, \qquad \beta(g) = -\beta_0\frac{g^3}{16\pi^2} - \beta_1\frac{g^5}{(16\pi^2)^2}, \tag{2.15}$$

inserting these expansions into (2.12) and (2.13) and expanding in α_s one can calculate the η- and ρ-factors defined in (2.3) and (2.4), respectively. To this end, one has to remove terms $\mathcal{O}(\alpha_s^2)$ and higher-order terms. We discuss this point in appendix B where the expansions of $\hat{U}(\mu, \mu_t)$ for $\mu = \mu_b$, $\mu = \mu_L$ and $\mu = \mu_K$ are given.

3 η-Factors in the NDR Scheme

3.1 η-Factors for $B^0_{d,s} - \overline{B}^0_{d,s}$ mixing

VLL-Sector

$$\left[\eta^{(0)}(\mu_b)\right]_{\text{VLL}} = \eta_5^{6/23}, \tag{3.1}$$

$$\left[\eta^{(1)}(\mu_b)\right]_{\text{VLL}} = 1.6273(1 - \eta_5)\eta_5^{6/23}. \tag{3.2}$$

LR-Sector

$$\left[\eta_{11}^{(0)}(\mu_b)\right]_{\text{LR}} = \eta_5^{3/23}, \tag{3.3}$$

$$\left[\eta_{12}^{(0)}(\mu_b)\right]_{\text{LR}} = 0, \tag{3.4}$$

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$$\left[\eta_{21}^{(0)}(\mu_b)\right]_{\mathrm{LR}} = \frac{2}{3}(\eta_5^{3/23} - \eta_5^{-24/23}), \tag{3.5}$$

$$\left[\eta_{22}^{(0)}(\mu_b)\right]_{\mathrm{LR}} = \eta_5^{-24/23}, \tag{3.6}$$

$$\left[\eta_{11}^{(1)}(\mu_b)\right]_{\mathrm{LR}} = 0.9250\,\eta_5^{-24/23} + \eta_5^{3/23}\left(-2.0994 + 1.1744\,\eta_5\right), \tag{3.7}$$

$$\left[\eta_{12}^{(1)}(\mu_b)\right]_{\mathrm{LR}} = 1.3875\left(\eta_5^{26/23} - \eta_5^{-24/23}\right), \tag{3.8}$$

$$\left[\eta_{21}^{(1)}(\mu_b)\right]_{\mathrm{LR}} = \left(-11.7329 + 0.7829\,\eta_5\right)\eta_5^{3/23} + \eta_5^{-24/23}\left(-5.3048 + 16.2548\,\eta_5\right), \tag{3.9}$$

$$\left[\eta_{22}^{(1)}(\mu_b)\right]_{\mathrm{LR}} = \left(7.9572 - 8.8822\,\eta_5\right)\eta_5^{-24/23} + 0.9250\,\eta_5^{26/23}. \tag{3.10}$$

SLL-Sector

$$\left[\eta_{11}^{(0)}(\mu_b)\right]_{\mathrm{SLL}} = 1.0153\eta_5^{-0.6315} - 0.0153\eta_5^{0.7184}, \tag{3.11}$$

$$\left[\eta_{12}^{(0)}(\mu_b)\right]_{\mathrm{SLL}} = 1.9325(\eta_5^{-0.6315} - \eta_5^{0.7184}), \tag{3.12}$$

$$\left[\eta_{21}^{(0)}(\mu_b)\right]_{\mathrm{SLL}} = 0.0081(\eta_5^{0.7184} - \eta_5^{-0.6315}), \tag{3.13}$$

$$\left[\eta_{22}^{(0)}(\mu_b)\right]_{\mathrm{SLL}} = 1.0153\eta_5^{0.7184} - 0.0153\eta_5^{-0.6315}, \tag{3.14}$$

$$\left[\eta_{11}^{(1)}(\mu_b)\right]_{\mathrm{SLL}} = \left(4.8177 - 5.2272\,\eta_5\right)\eta_5^{-0.6315} + \left(0.3371 + 0.0724\,\eta_5\right)\eta_5^{0.7184}, \tag{3.15}$$

$$\left[\eta_{12}^{(1)}(\mu_b)\right]_{\mathrm{SLL}} = \left(9.1696 - 38.8778\,\eta_5\right)\eta_5^{-0.6315} + \left(42.5021 - 12.7939\,\eta_5\right)\eta_5^{0.7184}, \tag{3.16}$$

$$\left[\eta_{21}^{(1)}(\mu_b)\right]_{\mathrm{SLL}} = \left(0.0531 + 0.0415\,\eta_5\right)\eta_5^{-0.6315} - \left(0.0566 + 0.0380\,\eta_5\right)\eta_5^{0.7184}, \tag{3.17}$$

$$\left[\eta_{22}^{(1)}(\mu_b)\right]_{\mathrm{SLL}} = \left(0.1011 + 0.3083\,\eta_5\right)\eta_5^{-0.6315} + \eta_5^{0.7184}\left(-7.1314 + 6.7219\,\eta_5\right), \tag{3.18}$$

where

$$\eta_5 \equiv \frac{\alpha_s^{(5)}(\mu_t)}{\alpha_s^{(5)}(\mu_b)}. \tag{3.19}$$

3.2 η-Factors for $K^0 - \overline{K}^0$ mixing with $\mu = \mu_L$

These factors are relevant for $K^0 - \overline{K}^0$ mixing but can also be used in $D^0 - \overline{D}^0$ mixing. They can be expressed in terms of η_5 defined in (3.19) and

$$\eta_4 \equiv \frac{\alpha_s^{(4)}(\mu_b)}{\alpha_s^{(4)}(\mu_L)}. \tag{3.20}$$

VLL-Sector

$$\left[\eta^{(0)}(\mu_L)\right]_{\mathrm{VLL}} = \eta_4^{6/25}\eta_5^{6/23}, \tag{3.21}$$

$$\left[\eta^{(1)}(\mu_L)\right]_{\mathrm{VLL}} = \eta_4^{6/25}\eta_5^{6/23}(1.7917 - 0.1644\eta_4 - 1.6273\eta_4\eta_5). \tag{3.22}$$

LR-Sector

$$\left[\eta_{11}^{(0)}(\mu_L)\right]_{\mathrm{LR}} = \eta_4^{3/25}\eta_5^{3/23}, \tag{3.23}$$

$$\left[\eta_{12}^{(0)}(\mu_L)\right]_{\mathrm{LR}} = 0, \tag{3.24}$$

$$\left[\eta_{21}^{(0)}(\mu_L)\right]_{\mathrm{LR}} = \frac{2}{3}(\eta_4^{3/25}\eta_5^{3/23} - \eta_4^{-24/25}\eta_5^{-24/23}), \tag{3.25}$$

$$\left[\eta_{22}^{(0)}(\mu_L)\right]_{\mathrm{LR}} = \eta_4^{-24/25}\eta_5^{-24/23}, \tag{3.26}$$

$$\begin{aligned}\left[\eta_{11}^{(1)}(\mu_L)\right]_{\mathrm{LR}} = &\; 0.9279\,\eta_4^{-24/25}\,\eta_5^{-24/23} - 0.0029\,\eta_4^{28/25}\,\eta_5^{-24/23} \\ &+\eta_4^{3/25}\,\eta_5^{3/23}\,(-2.0241 - 0.0753\,\eta_4 + 1.1744\,\eta_4\,\eta_5)\,,\end{aligned} \tag{3.27}$$

$$\begin{aligned}\left[\eta_{12}^{(1)}(\mu_L)\right]_{\mathrm{LR}} = &\; -1.3918\,\eta_4^{-24/25}\,\eta_5^{-24/23} + 0.0043\,\eta_4^{28/25}\,\eta_5^{-24/23} \\ &+1.3875\,\eta_4^{28/25}\,\eta_5^{26/23}\,,\end{aligned} \tag{3.28}$$

$$\begin{aligned}\left[\eta_{21}^{(1)}(\mu_L)\right]_{\mathrm{LR}} = &\; -0.0019\,\eta_4^{28/25}\,\eta_5^{-24/23} + 5.0000\,\eta_4^{1/25}\,\eta_5^{3/23} \\ &+\eta_4^{3/25}\,\eta_5^{3/23}\,(-16.6828 - 0.0502\,\eta_4 + 0.7829\,\eta_4\,\eta_5) \\ &+\eta_4^{-24/25}\,\eta_5^{-24/23}\,(-4.4701 - 0.8327\,\eta_4 + 16.2548\,\eta_4\,\eta_5)\,,\end{aligned} \tag{3.29}$$

$$\begin{aligned}\left[\eta_{22}^{(1)}(\mu_L)\right]_{\mathrm{LR}} = &\; 0.0029\,\eta_4^{28/25}\,\eta_5^{-24/23} + 0.9250\,\eta_4^{28/25}\,\eta_5^{26/23} \\ &+\eta_4^{-24/25}\,\eta_5^{-24/23}\,(6.7052 + 1.2491\,\eta_4 - 8.8822\,\eta_4\,\eta_5)\,.\end{aligned} \tag{3.30}$$

SLL-Sector

$$\left[\eta_{11}^{(0)}(\mu_L)\right]_{\mathrm{SLL}} = 1.0153\eta_4^{-0.5810}\eta_5^{-0.6315} - 0.0153\eta_4^{0.6610}\eta_5^{0.7184}, \tag{3.31}$$

$$\left[\eta_{12}^{(0)}(\mu_L)\right]_{\mathrm{SLL}} = 1.9325(\eta_4^{-0.5810}\eta_5^{-0.6315} - \eta_4^{0.6610}\eta_5^{0.7184}), \tag{3.32}$$

$$\left[\eta_{21}^{(0)}(\mu_L)\right]_{\mathrm{SLL}} = 0.0081(\eta_4^{0.6610}\eta_5^{0.7184} - \eta_4^{-0.5810}\eta_5^{-0.6315}), \tag{3.33}$$

$$\left[\eta_{22}^{(0)}(\mu_L)\right]_{\mathrm{SLL}} = 1.0153\eta_4^{0.6610}\eta_5^{0.7184} - 0.0153\eta_4^{-0.5810}\eta_5^{-0.6315}, \tag{3.34}$$

$$\begin{aligned}\left[\eta_{11}^{(1)}(\mu_L)\right]_{\mathrm{SLL}} = &\; 0.0020\,\eta_4^{1.6610}\,\eta_5^{-0.6315} - 0.0334\,\eta_4^{0.4190}\,\eta_5^{0.7184} \\ &+\eta_4^{-0.5810}\,\eta_5^{-0.6315}\,(4.2458 + 0.5700\,\eta_4 - 5.2272\,\eta_4\,\eta_5) \\ &+\eta_4^{0.6610}\,\eta_5^{0.7184}\,(0.3640 + 0.0064\,\eta_4 + 0.0724\,\eta_4\,\eta_5)\,,\end{aligned} \tag{3.35}$$

$$\begin{aligned}\left[\eta_{12}^{(1)}(\mu_L)\right]_{\mathrm{SLL}} = &\; 0.0038\,\eta_4^{1.6610}\,\eta_5^{-0.6315} - 4.2075\,\eta_4^{0.4190}\,\eta_5^{0.7184} \\ &+\eta_4^{-0.5810}\,\eta_5^{-0.6315}\,(8.0810 + 1.0848\,\eta_4 - 38.8778\,\eta_4\,\eta_5) \\ &+\eta_4^{0.6610}\,\eta_5^{0.7184}\,(45.9008 + 0.8087\,\eta_4 - 12.7939\,\eta_4\,\eta_5)\,,\end{aligned} \tag{3.36}$$

$$\left[\eta_{21}^{(1)}(\mu_L)\right]_{\text{SLL}} = -0.0011\,\eta_4{}^{1.6610}\,\eta_5{}^{-0.6315} + 0.0003\,\eta_4{}^{0.4190}\,\eta_5{}^{0.7184} \tag{3.37}$$

$$+\eta_4{}^{0.6610}\,\eta_5{}^{0.7184}\,(-0.0534 - 0.0034\,\eta_4 - 0.0380\,\eta_4\,\eta_5)$$

$$+\eta_4{}^{-0.5810}\,\eta_5{}^{-0.6315}\,(0.0587 - 0.0045\,\eta_4 + 0.0415\,\eta_4\,\eta_5)\,,$$

$$\left[\eta_{22}^{(1)}(\mu_L)\right]_{\text{SLL}} = -0.0020\,\eta_4{}^{1.6610}\,\eta_5{}^{-0.6315} + 0.0334\,\eta_4{}^{0.4190}\,\eta_5{}^{0.7184} \tag{3.38}$$

$$+\eta_4{}^{-0.5810}\,\eta_5{}^{-0.6315}\,(0.1117 - 0.0086\,\eta_4 + 0.3083\,\eta_4\,\eta_5)$$

$$+\eta_4{}^{0.6610}\,\eta_5{}^{0.7184}\,(-6.7398 - 0.4249\,\eta_4 + 6.7219\,\eta_4\,\eta_5)\,.$$

3.3 η-Factors for $K^0 - \overline{K}^0$ mixing with $\mu_K = \mathcal{O}(1\text{GeV})$

The formulae for the QCD factors $\hat{\eta}(\mu_K, \mu_t) \equiv \hat{\eta}(\mu_K)$ relating the coefficients $C_i(\mu_K)$ and $C_i(\mu_t)$ are rather long and will not be presented. These factors can be obtained using the relation

$$\hat{\eta}(\mu_K) = \hat{\rho}(\mu_K)\hat{\eta}(\mu_L) \tag{3.39}$$

with $\hat{\eta}(\mu_L)$ given in section 3.2 and $\hat{\rho}(\mu_K)$ given below. When calculating (3.39), terms of $\mathcal{O}(\alpha_s^2)$ should be removed.

4 ρ-Factors in the NDR Scheme

These factors allow to calculate $\langle Q_i(\mu_L)\rangle$ from $\langle Q_i(\mu_K)\rangle$ with $\mu_K < \mu_c$. They can be expressed in terms of

$$\eta_3 \equiv \frac{\alpha_s^{(3)}(\mu_c)}{\alpha_s^{(3)}(\mu_K)} \qquad \text{and} \qquad \tilde{\eta}_4 \equiv \frac{\alpha_s^{(4)}(\mu_L)}{\alpha_s^{(4)}(\mu_c)}. \tag{4.1}$$

VLL-Sector

$$\left[\rho^{(0)}(\mu_K)\right]_{\text{VLL}} = \eta_3^{2/9}\tilde{\eta}_4^{6/25}, \tag{4.2}$$

$$\left[\rho^{(1)}(\mu_K)\right]_{\text{VLL}} = 1.8951\eta_3^{2/9}\tilde{\eta}_4^{6/25} - 0.1033\eta_3^{11/9}\tilde{\eta}_4^{6/25} - 1.7917\eta_3^{11/9}\tilde{\eta}_4^{31/25}. \tag{4.3}$$

LR-Sector

$$\left[\rho_{11}^{(0)}(\mu_K)\right]_{\text{LR}} = \eta_3^{1/9}\tilde{\eta}_4^{3/25}, \tag{4.4}$$

$$\left[\rho_{12}^{(0)}(\mu_K)\right]_{\text{LR}} = 0, \tag{4.5}$$

$$\left[\rho_{21}^{(0)}(\mu_K)\right]_{\text{LR}} = \frac{2}{3}(\eta_3^{1/9}\tilde{\eta}_4^{3/25} - \eta_3^{-8/9}\tilde{\eta}_4^{-24/25}), \tag{4.6}$$

$$\left[\rho_{22}^{(0)}(\mu_K)\right]_{\text{LR}} = \eta_3^{-8/9}\tilde{\eta}_4^{-24/25}, \tag{4.7}$$

$$\left[\rho_{11}^{(1)}(\mu_K)\right]_{\text{LR}} = 0.9306\,\eta_3^{-8/9}\,\tilde{\eta}_4^{-24/25} - 0.0027\,\eta_3^{10/9}\,\tilde{\eta}_4^{-24/25} \tag{4.8}$$
$$+\eta_3^{1/9}\,\tilde{\eta}_4^{3/25}\,(-1.9784 - 0.0457\,\eta_3 + 1.0962\,\eta_3\,\tilde{\eta}_4)\,,$$

$$\left[\rho_{12}^{(1)}(\mu_K)\right]_{\text{LR}} = -1.3958\,\eta_3^{-8/9}\,\tilde{\eta}_4^{-24/25} + 0.0040\,\eta_3^{10/9}\,\tilde{\eta}_4^{-24/25} \tag{4.9}$$
$$+1.3918\,\eta_3^{10/9}\,\tilde{\eta}_4^{28/25}\,,$$

$$\left[\rho_{21}^{(1)}(\mu_K)\right]_{\text{LR}} = -3.8570\,\eta_3^{-8/9}\,\tilde{\eta}_4^{-24/25} \tag{4.10}$$
$$+\eta_3^{1/9}\,\tilde{\eta}_4^{-24/25}\,(-0.6113 - 0.0018\,\eta_3 + 20.4220\,\tilde{\eta}_4)$$
$$+\eta_3^{1/9}\,\tilde{\eta}_4^{3/25}\,(-16.6523 - 0.7407\,\log(\eta_3) - 0.0305\,\eta_3 + 0.7308\,\eta_3\,\tilde{\eta}_4)\,,$$

$$\left[\rho_{22}^{(1)}(\mu_K)\right]_{\text{LR}} = 5.7855\,\eta_3^{-8/9}\,\tilde{\eta}_4^{-24/25} + 0.9279\,\eta_3^{10/9}\,\tilde{\eta}_4^{28/25} \tag{4.11}$$
$$+\eta_3^{1/9}\,\tilde{\eta}_4^{-24/25}\,(0.9170 + 0.0027\,\eta_3 - 7.6331\,\tilde{\eta}_4)\,.$$

SLL-Sector

$$\left[\rho_{11}^{(0)}(\mu_K)\right]_{\text{SLL}} = 1.0153\eta_3^{-0.5379}\tilde{\eta}_4^{-0.5810} - 0.0153\eta_3^{0.6120}\tilde{\eta}_4^{0.6610}, \tag{4.12}$$

$$\left[\rho_{12}^{(0)}(\mu_K)\right]_{\text{SLL}} = 1.9325(\eta_3^{-0.5379}\tilde{\eta}_4^{-0.5810} - \eta_3^{0.6120}\tilde{\eta}_4^{0.6610}), \tag{4.13}$$

$$\left[\rho_{21}^{(0)}(\mu_K)\right]_{\text{SLL}} = 0.0081(\eta_3^{0.6120}\tilde{\eta}_4^{0.6610} - \eta_3^{-0.5379}\tilde{\eta}_4^{-0.5810}), \tag{4.14}$$

$$\left[\rho_{22}^{(0)}(\mu_K)\right]_{\text{SLL}} = 1.0153\eta_3^{0.6120}\tilde{\eta}_4^{0.6610} - 0.0153\eta_3^{-0.5379}\tilde{\eta}_4^{-0.5810}, \tag{4.15}$$

$$\left[\rho_{11}^{(1)}(\mu_K)\right]_{\text{SLL}} = 0.0019\,\eta_3^{1.6120}\,\tilde{\eta}_4^{-0.5810} - 0.0663\,\eta_3^{0.4621}\,\tilde{\eta}_4^{0.6610} \tag{4.16}$$
$$+\eta_3^{-0.5379}\,\tilde{\eta}_4^{-0.5810}\,(3.8487 + 0.3952\,\eta_3 - 4.6906\,\eta_3\,\tilde{\eta}_4)$$
$$+\eta_3^{0.6120}\,\tilde{\eta}_4^{0.6610}\,(0.4264 + 0.0039\,\eta_3 + 0.0808\,\eta_3\,\tilde{\eta}_4)\,,$$

$$\left[\rho_{12}^{(1)}(\mu_K)\right]_{\text{SLL}} = 0.0036\,\eta_3^{1.6120}\,\tilde{\eta}_4^{-0.5810} - 8.3647\,\eta_3^{0.4621}\,\tilde{\eta}_4^{0.6610} \tag{4.17}$$
$$+\eta_3^{-0.5379}\,\tilde{\eta}_4^{-0.5810}\,(7.3253 + 0.7521\,\eta_3 - 42.0005\,\eta_3\,\tilde{\eta}_4)$$
$$+\eta_3^{0.6120}\,\tilde{\eta}_4^{0.6610}\,(53.7722 + 0.4933\,\eta_3 - 11.9813\,\eta_3\,\tilde{\eta}_4)\,,$$

$$\left[\rho_{21}^{(1)}(\mu_K)\right]_{\text{SLL}} = -0.0010\,\eta_3^{1.6120}\,\tilde{\eta}_4^{-0.5810} + 0.0005\,\eta_3^{0.4621}\,\tilde{\eta}_4^{0.6610} \tag{4.18}$$
$$+\eta_3^{0.6120}\,\tilde{\eta}_4^{0.6610}\,(-0.0519 - 0.0021\,\eta_3 - 0.0425\,\eta_3\,\tilde{\eta}_4)$$
$$+\eta_3^{-0.5379}\,\tilde{\eta}_4^{-0.5810}\,(0.0628 - 0.0031\,\eta_3 + 0.0372\,\eta_3\,\tilde{\eta}_4)\,,$$

$$\left[\rho_{22}^{(1)}(\mu_K)\right]_{\text{SLL}} = -0.0019\,\eta_3^{1.6120}\,\tilde{\eta}_4^{-0.5810} + 0.0663\,\eta_3^{0.4621}\,\tilde{\eta}_4^{0.6610} \tag{4.19}$$

$$+\eta_3^{-0.5379}\,\tilde{\eta}_4^{-0.5810}\,(0.1196 - 0.0060\,\eta_3 + 0.3331\,\eta_3\,\tilde{\eta}_4)$$

$$+\eta_3^{0.6120}\,\tilde{\eta}_4^{0.6610}\,(-6.5470 - 0.2592\,\eta_3 + 6.2950\,\eta_3\,\tilde{\eta}_4).$$

Regarding the appearance of $\log(\eta_3)$ in eq. (4.10) in the LR sector, we direct the reader's attention to eq. (B.4) and the fact that in the LR sector for $f = 3$ the form of the evolution operator given there breaks down, as the matrix $\hat{J}$ has a singularity at $f = 3$. However, taking the limit $f \to 3$ of the complete expression (B.4), a finite result exhibiting the aforementioned term $\mathcal{O}(\alpha_s)\log(\eta_3)$ is obtained [13]. In accordance with the convention of (B.6), (B.7) we treat it as an NLO contribution to the evolution operator.

5 Numerical Results

In tables 1–4 we give the numerical values for the η_{ij} and ρ_{ij} factors in the NDR scheme. To this end we have used $\alpha_s^{(5)}(M_Z) = 0.118 \pm 0.003$ with the corresponding values of $\alpha_s^{(f)}$ and $\Lambda_{\overline{MS}}^{(f)}$ for $f = 4$ and $f = 3$ theories, see table 2 in the first paper in [2]. Moreover we have set $\mu_t = m_t(m_t) = 166$ GeV, $\mu_b = 4.4$ GeV, $\mu_L = 2.0$ GeV, $\mu_c = 1.3$ GeV and $\mu_K = 1.0$ GeV. In order to illustrate the effect of the NLO corrections we show also the results in the LO. In doing this we have, however, used the two-loop expression for α_s in both the LO and the NLO parts. In figs. 1 and 2 we show the factors $[\eta_{ij}(\mu)]_{\text{LR}}$ and $[\eta_{ij}(\mu)]_{\text{SLL}}$ versus μ setting $\alpha_s^{(5)}(M_Z) = 0.118$.

Let us recall that in the absence of renormalization group effects $[\eta(\mu)]_{\text{VLL}} = [\rho(\mu)]_{\text{VLL}} = 1$ and $[\hat{\eta}(\mu)]_a$ and $[\hat{\rho}(\mu)]_a$ are unit matrices. Renormalization group effects generate non-diagonal elements in these matrices and renormalize $[\eta(\mu)]_{\text{VLL}}$ and the diagonal elements in $[\hat{\eta}(\mu)]_a$ and $[\hat{\rho}(\mu)]_a$ away from unity.

Inspecting tables 1–4 and figs. 1–2 we observe the following pattern:

- Large renormalization group effects are found for the diagonal entries $[\eta_{22}(\mu)]_{\text{LR}}$ and $[\eta_{11}(\mu)]_{\text{SLL}}$ which for $\alpha_s^{(5)}(M_Z) = 0.118$ and $\mu = \mu_K = 1$ GeV are enhanced by factors of 5.6 and 2.9, respectively. On the other hand $[\eta_{22}(\mu)]_{\text{SLL}}$ is strongly suppressed down to 0.26 at $\mu_K = 1$ GeV.

- Similarly the renormalization group effects in the non-diagonal entries $[\eta_{21}(\mu)]_{\mathrm{LR}}$ and $[\eta_{12}(\mu)]_{\mathrm{SLL}}$ are large. For $\mu_K = 1.0$ GeV and $\alpha_s^{(5)}(M_Z) = 0.118$ they become as large as -3.2 and 4.8, respectively. This implies that $C_2^{\mathrm{LR}}(\mu)$ is strongly affected by the presence of the operator Q_1^{LR}. Similarly $C_1^{\mathrm{SLL}}(\mu)$ is strongly affected by the presence of Q_2^{SLL}.

- These enhancements and suppressions are more pronounced after the inclusion of NLO corrections. The largest NLO corrections, in the ball park of 25%, are found for the elements $[\eta_{21}(\mu_K)]_{\mathrm{LR}}$, $[\eta_{22}(\mu_K)]_{\mathrm{LR}}$ and $[\eta_{22}(\mu_K)]_{\mathrm{SLL}}$.

- $[\eta_{11}(\mu)]_{\mathrm{LR}}$ and $\eta_{\mathrm{VLL}}(\mu)$ are both suppressed but this suppression is at most by 10% and 30%, respectively.

- $[\eta_{12}(\mu)]_{\mathrm{LR}}$ is small and $[\eta_{21}(\mu)]_{\mathrm{SLL}}$ negligible in the full range of μ considered. This implies that $C_1^{\mathrm{LR}}(\mu)$ and $C_2^{\mathrm{SLL}}(\mu)$ are essentially unaffected by the presence of the operators Q_2^{LR} and Q_1^{SLL}, respectively.

- Similar patterns are observed for the $[\rho_{ij}(\mu)]_a$ factors.

On the basis of this pattern we conclude that the renormalization group effects strongly enhance the Wilson coefficients C_2^{LR} and C_1^{SLL} and strongly suppress C_2^{SLL} with respect to their values at μ_t. The corresponding effects in C_1^{VLL} and C_1^{LR} are substantially smaller.

	$\alpha_s^{(5)}(M_Z) = 0.115$		$\alpha_s^{(5)}(M_Z) = 0.118$		$\alpha_s^{(5)}(M_Z) = 0.121$	
	LO	NLO	LO	NLO	LO	NLO
$[\eta(\mu_b)]_{\mathrm{VLL}}$	0.835	0.847	0.829	0.842	0.823	0.836
$[\eta_{11}(\mu_b)]_{\mathrm{LR}}$	0.914	0.923	0.911	0.921	0.907	0.919
$[\eta_{12}(\mu_b)]_{\mathrm{LR}}$	0	-0.037	0	-0.041	0	-0.045
$[\eta_{21}(\mu_b)]_{\mathrm{LR}}$	-0.760	-0.835	-0.801	-0.885	-0.845	-0.939
$[\eta_{22}(\mu_b)]_{\mathrm{LR}}$	2.054	2.181	2.112	2.254	2.176	2.334
$[\eta_{11}(\mu_b)]_{\mathrm{SLL}}$	1.560	1.621	1.587	1.654	1.616	1.690
$[\eta_{12}(\mu_b)]_{\mathrm{SLL}}$	1.809	1.910	1.884	1.993	1.962	2.082
$[\eta_{21}(\mu_b)]_{\mathrm{SLL}}$	-0.008	-0.006	-0.008	-0.007	-0.008	-0.007
$[\eta_{22}(\mu_b)]_{\mathrm{SLL}}$	0.595	0.563	0.583	0.549	0.570	0.535

Table 1: Numerical values for the η-factors for $B_{d,s}^0 - \overline{B}_{d,s}^0$ mixing.

	$\alpha_s^{(5)}(M_Z) = 0.115$		$\alpha_s^{(5)}(M_Z) = 0.118$		$\alpha_s^{(5)}(M_Z) = 0.121$	
	LO	NLO	LO	NLO	LO	NLO
$[\eta(\mu_L)]_{\mathrm{VLL}}$	0.778	0.796	0.768	0.788	0.757	0.780
$[\eta_{11}(\mu_L)]_{\mathrm{LR}}$	0.882	0.906	0.876	0.906	0.870	0.906
$[\eta_{12}(\mu_L)]_{\mathrm{LR}}$	0	-0.076	0	-0.087	0	-0.101
$[\eta_{21}(\mu_L)]_{\mathrm{LR}}$	-1.236	-1.401	-1.336	-1.530	-1.449	-1.677
$[\eta_{22}(\mu_L)]_{\mathrm{LR}}$	2.735	3.009	2.879	3.200	3.043	3.420
$[\eta_{11}(\mu_L)]_{\mathrm{SLL}}$	1.859	1.976	1.918	2.052	1.984	2.138
$[\eta_{12}(\mu_L)]_{\mathrm{SLL}}$	2.586	2.775	2.732	2.946	2.892	3.137
$[\eta_{21}(\mu_L)]_{\mathrm{SLL}}$	-0.011	-0.009	-0.011	-0.009	-0.012	-0.009
$[\eta_{22}(\mu_L)]_{\mathrm{SLL}}$	0.480	0.438	0.461	0.417	0.442	0.394

Table 2: Numerical values for the η-factors for $K^0 - \overline{K}^0$ mixing with $\mu = \mu_L = 2$ GeV.

	$\alpha_s^{(5)}(M_Z) = 0.115$		$\alpha_s^{(5)}(M_Z) = 0.118$		$\alpha_s^{(5)}(M_Z) = 0.121$	
	LO	NLO	LO	NLO	LO	NLO
$[\eta(\mu_K)]_{\mathrm{VLL}}$	0.701	0.735	0.681	0.720	0.658	0.705
$[\eta_{11}(\mu_K)]_{\mathrm{LR}}$	0.837	0.921	0.825	0.941	0.811	0.978
$[\eta_{12}(\mu_K)]_{\mathrm{LR}}$	0	-0.194	0	-0.254	0	-0.347
$[\eta_{21}(\mu_K)]_{\mathrm{LR}}$	-2.199	-2.657	-2.545	-3.159	-3.006	-3.861
$[\eta_{22}(\mu_K)]_{\mathrm{LR}}$	4.136	4.875	4.643	5.630	5.320	6.688
$[\eta_{11}(\mu_K)]_{\mathrm{SLL}}$	2.392	2.663	2.566	2.912	2.787	3.243
$[\eta_{12}(\mu_K)]_{\mathrm{SLL}}$	3.836	4.273	4.223	4.782	4.702	5.442
$[\eta_{21}(\mu_K)]_{\mathrm{SLL}}$	-0.016	-0.011	-0.018	-0.012	-0.020	-0.012
$[\eta_{22}(\mu_K)]_{\mathrm{SLL}}$	0.346	0.291	0.314	0.255	0.279	0.217

Table 3: Numerical values for the η-factors for $K^0 - \overline{K}^0$ mixing with $\mu_K = 1$ GeV.

	$\alpha_s^{(5)}(M_Z) = 0.115$		$\alpha_s^{(5)}(M_Z) = 0.118$		$\alpha_s^{(5)}(M_Z) = 0.121$	
	LO	NLO	LO	NLO	LO	NLO
$[\rho(\mu_K)]_{\mathrm{VLL}}$	0.902	0.923	0.887	0.914	0.870	0.905
$[\rho_{11}(\mu_K)]_{\mathrm{LR}}$	0.950	0.955	0.942	0.951	0.933	0.947
$[\rho_{12}(\mu_K)]_{\mathrm{LR}}$	0	-0.045	0	-0.060	0	-0.083
$[\rho_{21}(\mu_K)]_{\mathrm{LR}}$	-0.375	-0.448	-0.447	-0.548	-0.544	-0.688
$[\rho_{22}(\mu_K)]_{\mathrm{LR}}$	1.512	1.620	1.613	1.762	1.748	1.963
$[\rho_{11}(\mu_K)]_{\mathrm{SLL}}$	1.293	1.357	1.345	1.431	1.413	1.533
$[\rho_{12}(\mu_K)]_{\mathrm{SLL}}$	1.029	1.176	1.190	1.381	1.394	1.650
$[\rho_{21}(\mu_K)]_{\mathrm{SLL}}$	-0.004	-0.003	-0.005	-0.003	-0.006	-0.003
$[\rho_{22}(\mu_K)]_{\mathrm{SLL}}$	0.744	0.688	0.710	0.641	0.670	0.584

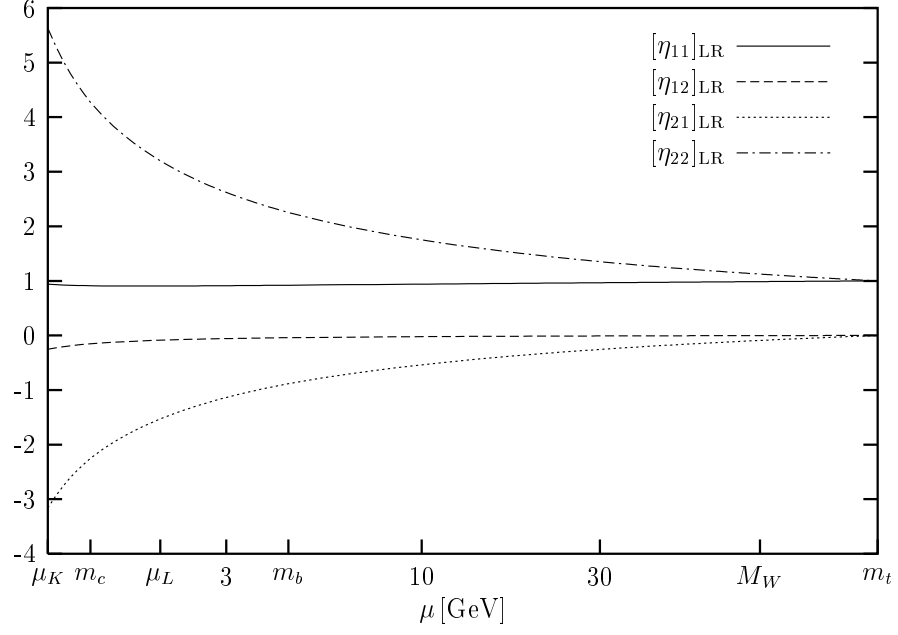Table 4: Numerical values for the ρ-factors with $\mu_K = 1$ GeV and $\mu_L = 2$ GeV.

Figure 1: The $[\eta_{ij}]_{\mathrm{LR}}$ factors as functions of μ for $\alpha_s^{(5)}(M_Z) = 0.118$.
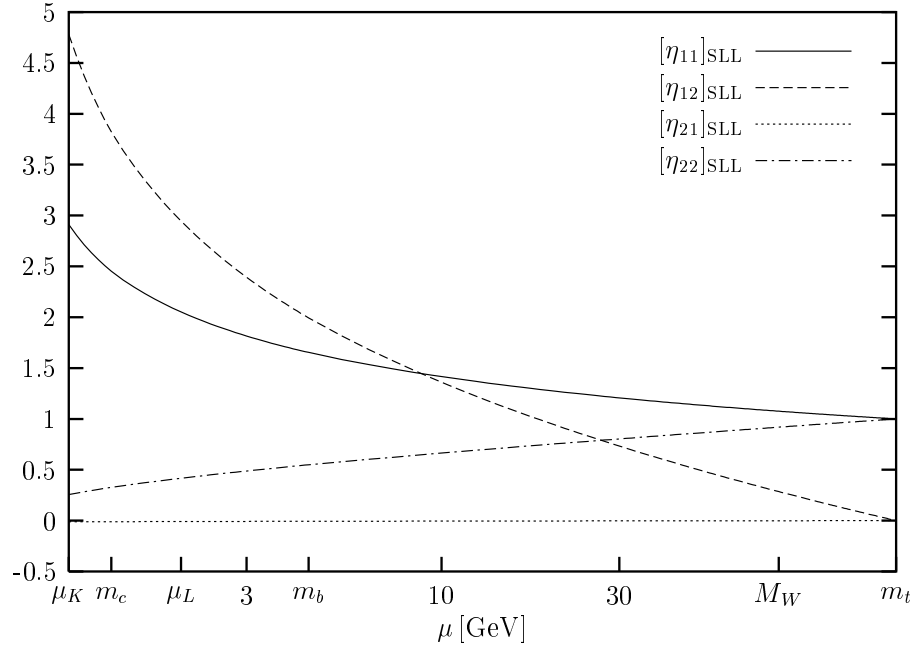

Figure 2: The $[\eta_{ij}]_{\mathrm{SLL}}$ factors as functions of μ for $\alpha_s^{(5)}(M_Z) = 0.118$

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6 General Remarks

6.1 Renormalization Scheme Dependence

The evolution matrix $\hat{U}$ is renormalization scheme dependent. It is instructive to recall how this scheme dependence is canceled in physical amplitudes by considering a single operator Q. Then the $\Delta F = 2$ amplitude reads

$$A(\Delta F = 2) = \langle H_{\text{eff}} \rangle = \frac{G_F^2}{16\pi^2} M_W^2 V_{\text{CKM}} C(\mu) \langle Q(\mu) \rangle . \tag{6.1}$$

The Wilson coefficient is given by

$$C(\mu) = U(\mu, \mu_t) C(\mu_t) \tag{6.2}$$

where

$$U(\mu, \mu_t) = \left[1 + \frac{\alpha_s(\mu)}{4\pi} J\right] \left[\frac{\alpha_s(\mu_t)}{\alpha_s(\mu)}\right]^P \left[1 - \frac{\alpha_s(\mu_t)}{4\pi} J\right] \tag{6.3}$$

with

$$P = \frac{\gamma^{(0)}}{2\beta_0}, \qquad J = \frac{P}{\beta_0} \beta_1 - \frac{\gamma^{(1)}}{2\beta_0} \tag{6.4}$$

and

$$C(\mu_t) = C_0 + \frac{\alpha_s(\mu_t)}{4\pi} C_1. \tag{6.5}$$

C_0 and C_1 depend generally on m_t, M_W and the masses of new particles in extensions of the SM.

Now, the renormalization scheme dependence of C_1 is canceled by the one of J in the last square bracket in (6.3). The scheme dependence of J in the first square bracket in (6.3) is canceled by the scheme dependence of $\langle Q(\mu) \rangle$. The power P and the coefficient C_0 are scheme independent.

6.2 Transformation to Different Renormalization Schemes

Once the Wilson coefficients $C_i(\mu)$ have been calculated in the NDR scheme, they can be transformed to a different renormalization scheme A by means of

$$\vec{C}^{\mathrm{A}}(\mu) = \left(1 - \frac{\alpha_s(\mu)}{4\pi}\Delta\hat{r}^T_{\mathrm{NDR}\to\mathrm{A}}\right)\vec{C}^{\mathrm{NDR}}(\mu) \; . \tag{6.6}$$

Likewise the matrix elements $\langle Q_i(\mu)\rangle$ can be transformed from scheme A to the NDR scheme:

$$\langle\vec{Q}(\mu)\rangle_{\mathrm{NDR}} = \left(1 + \frac{\alpha_s(\mu)}{4\pi}\Delta\hat{r}_{\mathrm{A}\to\mathrm{NDR}}\right)\langle\vec{Q}(\mu)\rangle_{\mathrm{A}} \; , \qquad \Delta\hat{r}_{\mathrm{A}\to\mathrm{NDR}} = -\Delta\hat{r}_{\mathrm{NDR}\to\mathrm{A}} \; . \tag{6.7}$$

The transformation matrices $\Delta\hat{r}_{\mathrm{NDR}\to\mathrm{RI}}$ from the NDR scheme to the RI schemes of [5] can be found in section 5 of [6].

6.3 η_B and η_2 Factors in the SM

At this point we would like to warn the reader that the QCD factors $\eta_B = 0.55$ [3, 4] and η_2 [3] used in the analysis of $B^0_{d,s} - \overline{B}^0_{d,s}$ mixing and of the top quark contribution to ε_K in the SM should not be identified with the factors $[\eta(\mu_b)]_{\mathrm{VLL}}$ and $[\eta(\mu_K)]_{\mathrm{VLL}}$ presented in this paper.

The factors η_B and η_2 are discussed in detail in [1, 2, 3]. See in particular the expressions (12.10) and (13.3) in [1] for η_2 and η_B, respectively. Using these expressions it is straightforward to find the relation between η_B and $[\eta(\mu_b)]_{\mathrm{VLL}}$. It reads

$$[\eta(\mu_b)]_{\mathrm{VLL}}\, C^{\mathrm{VLL}}_{\mathrm{SM}}(\mu_t) = \left[\alpha^{(5)}_s(\mu_b)\right]^{-6/23}\left[1 + \frac{\alpha^{(5)}_s(\mu_b)}{4\pi}J_5\right]\eta_B 4S_0(x_t) \tag{6.8}$$

where $C^{\mathrm{VLL}}_{\mathrm{SM}}(\mu_t)$ includes NLO corrections calculated in [3, 4], $J_5 = 1.627$ in the NDR scheme and $4S_0(x_t)$ with $x_t = m^2_t(\mu_t)/M^2_W$ is the LO expression for $C^{\mathrm{VLL}}_{\mathrm{SM}}(\mu_t)$ that is obtained from box diagrams with top quark exchanges without QCD corrections. η_B in contrast to $[\eta(\mu_b)]_{\mathrm{VLL}}$ is renormalization scheme independent and does not depend on μ_b. The latter dependence has been factored out as seen on the r.h.s of (6.8). Notice that the QCD corrections to $C^{\mathrm{VLL}}_{\mathrm{SM}}(\mu_t)$ have been absorbed into η_B. An analogous relation between η_2 and $[\eta(\mu_K)]_{\mathrm{VLL}}$ can be found.

6.4   Going Beyond the SM

In the SM

$$\langle \bar{B}^0 | H_{\text{eff}}^{\Delta B=2} | B^0 \rangle_{\text{SM}} = \frac{G_F^2}{48\pi^2} M_W^2 m_B F_B^2 \hat{B}_B (V_{tb}^* V_{td})^2 \eta_B 4 S_0(x_t) \tag{6.9}$$

where $\hat{B}_B$ is the renormalization group invariant parameter defined by

$$\hat{B}_B = B_1^{\text{VLL}}(\mu_b) \left[\alpha_s^{(5)}(\mu_b) \right]^{-6/23} \left[1 + \frac{\alpha_s^{(5)}(\mu_b)}{4\pi} J_5 \right], \tag{6.10}$$

with $B_1^{\text{VLL}}(\mu)$ defined in (7.1). F_B is the B-meson decay constant and η_B is the QCD factor defined in (6.8).

In the extensions of the SM with minimal flavour violation (MFV) and without contributions from new operators it is useful to generalize (6.9) to

$$\langle \bar{B}^0 | H_{\text{eff}}^{\Delta B=2} | B^0 \rangle = \frac{G_F^2}{48\pi^2} M_W^2 m_B F_B^2 \hat{B}_B (V_{tb}^* V_{td})^2 \eta_B 4 F_{tt} \tag{6.11}$$

where

$$F_{tt} = S_0(x_t) + \frac{\eta_B^{\text{new}}}{\eta_B} S_0^{\text{new}}. \tag{6.12}$$

Here η_B^{new} and S_0^{new} describe new physics contributions in analogy to η_B and $S_0(x_t)$, respectively. That is

$$[\eta(\mu_b)]_{\text{VLL}} C_{\text{new}}^{\text{VLL}}(\mu_t) = \left[\alpha_s^{(5)}(\mu_b) \right]^{-6/23} \left[1 + \frac{\alpha_s^{(5)}(\mu_b)}{4\pi} J_5 \right] \eta_B^{\text{new}} 4 S_0^{\text{new}}(x_t), \tag{6.13}$$

where $C_{\text{new}}^{\text{VLL}}(\mu_t)$ is the new physics contribution to $C_1^{\text{VLL}}(\mu_t)$ and $4S_0^{\text{new}}$ results from new physics contributions without QCD corrections.

In more complicated models in which new flavour-violating couplings are present and the full set of operators (2.1) is relevant, it appears to be most useful to evaluate new physics contributions using simply

$$\langle \bar{B}^0 | H_{\text{eff}}^{\Delta B=2} | B^0 \rangle_{\text{new}} = \frac{G_F^2}{16\pi^2} M_W^2 \sum C_i(\mu) \langle \bar{B}^0 | Q_i(\mu) | B^0 \rangle \tag{6.14}$$

with $C_i(\mu)$ evaluated by means of the formulae in sections 2 and 3. Similar comments apply to the $K^0 - \bar{K}^0$ system with obvious replacements.

7 Phenomenological Applications

7.1 Hadronic Matrix Elements for $K^0 - \bar{K}^0$ Mixing

The matrix elements $\langle \bar{K}^0 | Q_i(\mu) | K^0 \rangle \equiv \langle Q_i(\mu) \rangle$ contributing to $K^0 - \bar{K}^0$ mixing can be written as

$$\langle Q_1^{\mathrm{VLL}}(\mu) \rangle = \frac{1}{3} m_K F_K^2 B_1^{\mathrm{VLL}}(\mu), \tag{7.1}$$

$$\langle Q_1^{\mathrm{LR}}(\mu) \rangle = -\frac{1}{6} R(\mu)\, m_K F_K^2 B_1^{\mathrm{LR}}(\mu), \tag{7.2}$$

$$\langle Q_2^{\mathrm{LR}}(\mu) \rangle = \frac{1}{4} R(\mu)\, m_K F_K^2 B_2^{\mathrm{LR}}(\mu), \tag{7.3}$$

$$\langle Q_1^{\mathrm{SLL}}(\mu) \rangle = -\frac{5}{24} R(\mu)\, m_K F_K^2 B_1^{\mathrm{SLL}}(\mu), \tag{7.4}$$

$$\langle Q_2^{\mathrm{SLL}}(\mu) \rangle = -\frac{1}{2} R(\mu)\, m_K F_K^2 B_2^{\mathrm{SLL}}(\mu), \tag{7.5}$$

where

$$R(\mu) = \left(\frac{m_K}{m_s(\mu) + m_d(\mu)} \right)^2 \tag{7.6}$$

and F_K is the K-meson decay constant. Let us calculate the non-perturbative parameters $B_i^a(\mu)$ using the lattice results of [10] discussed in [9]. In the Landau RI scheme (LRI) the $B_i^a(\mu)$ factors are given by

$$
\begin{aligned}
\left[B_1^{\mathrm{VLL}}(\mu) \right]_{\mathrm{LRI}} &= \left[B_1(\mu) \right]_{\mathrm{LRI}}, \\
\left[B_1^{\mathrm{LR}}(\mu) \right]_{\mathrm{LRI}} &= \left[B_5(\mu) \right]_{\mathrm{LRI}}, \qquad \left[B_2^{\mathrm{LR}}(\mu) \right]_{\mathrm{LRI}} = \left[B_4(\mu) \right]_{\mathrm{LRI}}, \\
\left[B_1^{\mathrm{SLL}}(\mu) \right]_{\mathrm{LRI}} &= \left[B_2(\mu) \right]_{\mathrm{LRI}}, \qquad \left[B_2^{\mathrm{SLL}}(\mu) \right]_{\mathrm{LRI}} = \left[\frac{5}{3} B_2(\mu) - \frac{2}{3} B_3(\mu) \right]_{\mathrm{LRI}},
\end{aligned}
\tag{7.7}
$$

where $B_i(\mu)$, $i = 1, \ldots, 5$ are the non-perturbative factors entering the matrix elements in the operator basis of [9].

In order to find the matrix elements (7.1)–(7.5) in the NDR scheme we use the results of [6], which allow us to relate the B_i factors in the LRI and NDR schemes. We find

$$\left[B_1^{\mathrm{VLL}}(\mu) \right]_{\mathrm{NDR}} = \left[B_1^{\mathrm{VLL}}(\mu) \right]_{\mathrm{LRI}} + \frac{\alpha_s^{(4)}(\mu)}{4\pi} r^{\mathrm{VLL}} \left[B_1^{\mathrm{VLL}}(\mu) \right]_{\mathrm{LRI}}, \tag{7.8}$$

$$\left[B_1^{\mathrm{LR}}(\mu) \right]_{\mathrm{NDR}} = \left[B_1^{\mathrm{LR}}(\mu) \right]_{\mathrm{LRI}} + \frac{\alpha_s^{(4)}(\mu)}{4\pi} \left[r_{11}^{\mathrm{LR}} B_1^{\mathrm{LR}}(\mu) - \frac{3}{2} r_{12}^{\mathrm{LR}} B_2^{\mathrm{LR}}(\mu) \right]_{\mathrm{LRI}}, \tag{7.9}$$

$$\left[B_2^{\text{LR}}(\mu)\right]_{\text{NDR}} = \left[B_2^{\text{LR}}(\mu)\right]_{\text{LRI}} + \frac{\alpha_s^{(4)}(\mu)}{4\pi}\left[-\frac{2}{3}r_{21}^{\text{LR}}B_1^{\text{LR}}(\mu) + r_{22}^{\text{LR}}B_2^{\text{LR}}(\mu)\right]_{\text{LRI}}, \tag{7.10}$$

$$\left[B_1^{\text{SLL}}(\mu)\right]_{\text{NDR}} = \left[B_1^{\text{SLL}}(\mu)\right]_{\text{LRI}} + \frac{\alpha_s^{(4)}(\mu)}{4\pi}\left[r_{11}^{\text{SLL}}B_1^{\text{SLL}}(\mu) + \frac{12}{5}r_{12}^{\text{SLL}}B_2^{\text{SLL}}(\mu)\right]_{\text{LRI}}, \tag{7.11}$$

$$\left[B_2^{\text{SLL}}(\mu)\right]_{\text{NDR}} = \left[B_2^{\text{SLL}}(\mu)\right]_{\text{LRI}} + \frac{\alpha_s^{(4)}(\mu)}{4\pi}\left[\frac{5}{12}r_{21}^{\text{SLL}}B_1^{\text{SLL}}(\mu) + r_{22}^{\text{SLL}}B_2^{\text{SLL}}(\mu)\right]_{\text{LRI}}, \tag{7.12}$$

where

$$r^{\text{VLL}} \equiv \Delta r_{\text{LRI}\rightarrow\text{NDR}}^{\text{VLL}} = 0.8785, \tag{7.13}$$

$$\hat{r}^{\text{LR}} \equiv \Delta r_{\text{LRI}\rightarrow\text{NDR}}^{\text{LR}} = \begin{pmatrix} -1.1288 & -6.7726 \\ 0.3069 & 10.8712 \end{pmatrix}, \tag{7.14}$$

$$\hat{r}^{\text{SLL}} \equiv \Delta r_{\text{LRI}\rightarrow\text{NDR}}^{\text{SLL}} = \begin{pmatrix} 5.6438 & 0.2140 \\ 12.9387 & 2.6892 \end{pmatrix}. \tag{7.15}$$

Now, the B_i factors presented in [9, 10] read for $\mu = \mu_L = 2$ GeV as follows:

$$[B_1]_{\text{LRI}} = 0.60 \pm 0.06, \qquad [B_2]_{\text{LRI}} = 0.66 \pm 0.04,$$

$$[B_3]_{\text{LRI}} = 1.05 \pm 0.12, \qquad [B_4]_{\text{LRI}} = 1.03 \pm 0.06, \tag{7.16}$$

$$[B_5]_{\text{LRI}} = 0.73 \pm 0.10.$$

Using the central values for these parameters, we find by means of (7.7)

$$\left[B_1^{\text{VLL}}\right]_{\text{LRI}} = 0.60,$$

$$\left[B_1^{\text{LR}}\right]_{\text{LRI}} = 0.73, \qquad\qquad \left[B_2^{\text{LR}}\right]_{\text{LRI}} = 1.03, \tag{7.17}$$

$$\left[B_1^{\text{SLL}}\right]_{\text{LRI}} = 0.66, \qquad\qquad \left[B_2^{\text{SLL}}\right]_{\text{LRI}} = 0.40.$$

Finally, setting $\alpha_s^{(4)}(2\text{ GeV}) = 0.306$ and using (7.8)–(7.12) we obtain in the NDR scheme for $\mu = \mu_L = 2$ GeV

$$\left[B_1^{\text{VLL}}\right]_{\text{NDR}} = 0.61,$$

$$\left[B_1^{\text{LR}}\right]_{\text{NDR}} = 0.96, \qquad\qquad \left[B_2^{\text{LR}}\right]_{\text{NDR}} = 1.30, \tag{7.18}$$

$$\left[B_1^{\text{SLL}}\right]_{\text{NDR}} = 0.76, \qquad\qquad \left[B_2^{\text{SLL}}\right]_{\text{NDR}} = 0.51.$$

We observe that the scheme dependence in the LR and SLL sectors is large, amounting to a shift of the B_i factors by roughly 30% and 20%, respectively. The corresponding scheme dependence in the VLL sector amounts to 2%.

Setting $F_K = 160$ MeV and $m_K = 498$ MeV we obtain at $\mu = 2$ GeV

$$\langle Q_1^{\text{VLL}}\rangle_{\text{NDR}} = 0.26 \cdot 10^{-2}\text{GeV}^3, \tag{7.19}$$

$$\langle Q_1^{\text{LR}}\rangle_{\text{NDR}} = -3.83\left[\frac{115\text{MeV}}{m_s(2\text{GeV}) + m_d(2\text{GeV})}\right]^2 \cdot 10^{-2}\text{GeV}^3, \tag{7.20}$$

$$\langle Q_2^{\text{LR}}\rangle_{\text{NDR}} = 7.77\left[\frac{115\text{MeV}}{m_s(2\text{GeV}) + m_d(2\text{GeV})}\right]^2 \cdot 10^{-2}\text{GeV}^3, \tag{7.21}$$

$$\langle Q_1^{\text{SLL}}\rangle_{\text{NDR}} = -3.79\left[\frac{115\text{MeV}}{m_s(2\text{GeV}) + m_d(2\text{GeV})}\right]^2 \cdot 10^{-2}\text{GeV}^3, \tag{7.22}$$

$$\langle Q_2^{\text{SLL}}\rangle_{\text{NDR}} = -6.10\left[\frac{115\text{MeV}}{m_s(2\text{GeV}) + m_d(2\text{GeV})}\right]^2 \cdot 10^{-2}\text{GeV}^3. \tag{7.23}$$

7.2 ΔM_K, ΔM_B and ε_K

Next we would like to present general formulae for the mass differences ΔM_K and ΔM_B in the $K^0 - \bar{K}^0$ and $B^0 - \bar{B}^0$ systems and for the CP-violating parameter ε_K. In the case of ΔM_K and ε_K our formulae are valid for the contributions of heavy internal particles with masses higher than M_W. The known SM contributions from internal charm quark exchanges and mixed charm-top exchanges [14] have to be added separately.

We have

$$\Delta M_K = 2\text{Re}\langle \bar{K}^0|H_{\text{eff}}^{\Delta S=2}|K^0\rangle , \tag{7.24}$$

$$\Delta M_B = 2|\langle \bar{B}^0|H_{\text{eff}}^{\Delta B=2}|B^0\rangle| , \tag{7.25}$$

$$\varepsilon_K = \frac{\exp(i\pi/4)}{\sqrt{2}\Delta M_K}\text{Im}\langle \bar{K}^0|H_{\text{eff}}^{\Delta S=2}|K^0\rangle . \tag{7.26}$$

The matrix element $\langle \bar{K}^0|H_{\text{eff}}^{\Delta S=2}|K^0\rangle$ can be written as follows

$$
\begin{aligned}
\langle \bar{K}^0|H_{\text{eff}}^{\Delta S=2}|K^0\rangle =\ & \frac{G_F^2}{48\pi^2}M_W^2 m_K F_K^2 \Big[P_1^{\text{VLL}}(C_1^{\text{VLL}}(\mu_t) + C_1^{\text{VRR}}(\mu_t)) \\
& + P_1^{\text{LR}}C_1^{\text{LR}}(\mu_t) + P_2^{\text{LR}}C_2^{\text{LR}}(\mu_t) \\
& + P_1^{\text{SLL}}(C_1^{\text{SLL}}(\mu_t) + C_1^{\text{SRR}}(\mu_t)) + P_2^{\text{SLL}}(C_2^{\text{SLL}}(\mu_t) + C_2^{\text{SRR}}(\mu_t))\Big]
\end{aligned} \tag{7.27}
$$

with

$$P_1^{\text{VLL}} = [\eta(\mu_L)]_{\text{VLL}} \, B_1^{\text{VLL}}(\mu_L), \tag{7.28}$$

$$P_1^{\text{LR}} = -\frac{1}{2} [\eta_{11}(\mu_L)]_{\text{LR}} \left[B_1^{\text{LR}}(\mu_L) \right]_{\text{eff}} + \frac{3}{4} [\eta_{21}(\mu_L)]_{\text{LR}} \left[B_2^{\text{LR}}(\mu_L) \right]_{\text{eff}}, \tag{7.29}$$

$$P_2^{\text{LR}} = -\frac{1}{2} [\eta_{12}(\mu_L)]_{\text{LR}} \left[B_1^{\text{LR}}(\mu_L) \right]_{\text{eff}} + \frac{3}{4} [\eta_{22}(\mu_L)]_{\text{LR}} \left[B_2^{\text{LR}}(\mu_L) \right]_{\text{eff}}, \tag{7.30}$$

$$P_1^{\text{SLL}} = -\frac{5}{8} [\eta_{11}(\mu_L)]_{\text{SLL}} \left[B_1^{\text{SLL}}(\mu_L) \right]_{\text{eff}} - \frac{3}{2} [\eta_{21}(\mu_L)]_{\text{SLL}} \left[B_2^{\text{SLL}}(\mu_L) \right]_{\text{eff}}, \tag{7.31}$$

$$P_2^{\text{SLL}} = -\frac{5}{8} [\eta_{12}(\mu_L)]_{\text{SLL}} \left[B_1^{\text{SLL}}(\mu_L) \right]_{\text{eff}} - \frac{3}{2} [\eta_{22}(\mu_L)]_{\text{SLL}} \left[B_2^{\text{SLL}}(\mu_L) \right]_{\text{eff}}. \tag{7.32}$$

In the case of the SM and MFV models one can use (6.9) and (6.11) instead of (7.28). In writing these formulae we have absorbed the CKM factors into $C_i(\mu_t)$. The effective parameters $[B_i^a(\mu_L)]_{\text{eff}}$ are defined by

$$[B_i^a(\mu_L)]_{\text{eff}} \equiv \left(\frac{m_K}{m_s(\mu_L) + m_d(\mu_L)} \right)^2 B_i^a(\mu_L) = 18.75 \left[\frac{115 \text{ MeV}}{m_s(\mu_L) + m_d(\mu_L)} \right]^2 B_i^a(\mu_L). \tag{7.33}$$

In the case of $B^0 - \bar{B}^0$ mixing one has to make the replacements $\mu_L \to \mu_b$ and $m_K F_K^2 \to m_B F_B^2$. Then in the case of $B_d^0 - \bar{B}_d^0$ system

$$[B_i^a(\mu_b)]_{\text{eff}} \equiv \left(\frac{m_B}{m_b(\mu_b) + m_d(\mu_b)} \right)^2 B_i^a(\mu_b) = 1.44 \left[\frac{4.4 \text{ GeV}}{m_b(\mu_b) + m_d(\mu_b)} \right]^2 B_i^a(\mu_b), \tag{7.34}$$

with an analogous formula for the $B_s^0 - \bar{B}_s^0$ system.

We would like to emphasize that these formulae together with the QCD factors η_{ij} presented in Section 3 are valid for any extension of the SM. In particular the coefficients P_i^a are universal. New physics contributions enter only the coefficients $C_i^a(\mu_t)$. The latter have to be evaluated in the NDR scheme in order to cancel the scheme dependence of the universal coefficients P_i^a. In the process of multiplying $C_i^a(\mu_t)$ and P_i^a terms $\mathcal{O}(\alpha_s^2)$ have to be removed.

It is instructive to evaluate the coefficients P_i^a for the $K^0 - \bar{K}^0$ system. Setting $\mu_L = 2$ GeV, $\Lambda_{\overline{MS}}^{(4)} = 325$ MeV, $m_s(\mu_L) + m_d(\mu_L) = 115$ MeV and using the values for B_i^a in (7.18) we find

$$P_1^{\text{VLL}} = 0.48, \tag{7.35}$$

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$$P_1^{\text{LR}} = -36.1, \qquad P_2^{\text{LR}} = 59.3, \tag{7.36}$$

$$P_1^{\text{SLL}} = -18.1, \qquad P_2^{\text{SLL}} = -32.2. \tag{7.37}$$

We observe that the coefficients P_i^{LR} and P_i^{SLL} are by two orders of magnitude larger than P_1^{VLL}. This originates in the strong enhancement of the QCD factors η_{ij} for the LR and SLL (SRR) operators and in the chiral enhancement of their matrix elements as seen in (7.33). Consequently even small new physics contributions to $C_i^{\text{LR}}(\mu_t)$ and $C_i^{\text{SLL}}(\mu_t)$ can play an important role in the phenomenology [8, 9].

In the case of $B^0 - \bar{B}^0$ mixing the chiral enhancement of the hadronic matrix elements in the LR and SLL sectors is absent. Moreover, the QCD factors η_{ij} are smaller than in the $K^0 - \bar{K}^0$ mixing. Consequently the coefficients P_i^{LR} and P_i^{SLL} are smaller in this case. As lattice results are not yet available for all the relevant hadronic matrix elements in the B system [15] we will set $B_i^a(m_b) = 1$. Taking $m_B = 5.28$ GeV, $\mu_b = 4.4$ GeV, $m_b(\mu_b) + m_d(\mu_b) = 4.4$ GeV and $\Lambda_{\overline{MS}}^{(5)} = 226$ MeV we find

$$P_1^{\text{VLL}} = 0.84, \tag{7.38}$$

$$P_1^{\text{LR}} = -1.62, \qquad P_2^{\text{LR}} = 2.46, \tag{7.39}$$

$$P_1^{\text{SLL}} = -1.47, \qquad P_2^{\text{SLL}} = -2.98. \tag{7.40}$$

8 Summary

We have presented analytic formulae for the QCD renormalization group factors relating the Wilson coefficients $C_i(\mu_t)$ and $C_i(\mu)$, with $\mu_t = \mathcal{O}(m_t)$ and $\mu < \mu_t$, of the $\Delta F = 2$ dimension six four-quark operators Q_i. The formulae presented in section 3 are given in the NDR scheme but are otherwise universal and apply to the Standard Model and all its possible extensions. In order to complete the evaluation of $\Delta F = 2$ amplitudes, the QCD factors presented here have to be combined with the Wilson coefficients $C_i(\mu_t)$ evaluated in a given model at the short distance scale μ_t and with the hadronic matrix elements $\langle Q_i(\mu) \rangle$ evaluated at $\mu = \mu_b$, $\mu = \mu_L$ or $\mu = \mu_K$ dependently on the process considered. $C_i(\mu_t)$ and $\langle Q_i(\mu) \rangle$ have to be computed

in the NDR scheme in order to obtain a renormalization scheme independent answer for the physical amplitudes.

We have also presented analytic formulae for the QCD factors relating the matrix elements $\langle Q_i(2\,\mathrm{GeV})\rangle$ and $\langle Q_i(\mu_K)\rangle$ with $\mu_K < 2$ GeV. These formulae allow the comparison of the matrix elements obtained in lattice simulations with those obtained in approaches which use lower renormalization scales.

Our numerical analysis shows that the renormalization-group effects are very large in the LR and SLL sectors. This in particular is the case for the elements $[\eta_{21}(\mu)]_{\mathrm{LR}}$, $[\eta_{22}(\mu)]_{\mathrm{LR}}$, $[\eta_{11}(\mu)]_{\mathrm{SLL}}$, $[\eta_{12}(\mu)]_{\mathrm{SLL}}$ and $[\eta_{22}(\mu)]_{\mathrm{SLL}}$ and is in accordance with the previous analyses [8, 9]. The NLO corrections amount typically to 5-15% except for the elements $[\eta_{21}(\mu)]_{\mathrm{LR}}$, $[\eta_{22}(\mu)]_{\mathrm{LR}}$ and $[\eta_{22}(\mu)]_{\mathrm{SLL}}$, where in the case of $\mu = 1\mathrm{GeV}$ they can reach 25%. As a result of this pattern the renormalization group effects strongly enhance the Wilson coefficients C_2^{LR} and C_1^{SLL} and strongly suppress C_2^{SLL} with respect to their values at μ_t. The corresponding effects in C_1^{VLL} and C_1^{LR} are substantially smaller.

Finally we have presented expressions for the mass differences ΔM_K and ΔM_B and the CP-violating parameter ϵ_K in terms of the non-perturbative parameters B_i^a and the Wilson coefficients $C_i(\mu_t)$. These formulae include renormalization group effects at the NLO level and allow to calculate straightforwardly ΔM_K, ΔM_B and ϵ_K in any extension of the SM once the Wilson coefficients $C_i(\mu_t)$ and the non-perturbative parameters B_i^a are known in the NDR scheme. In the case of $K^0 - \bar{K}^0$ mixing we have presented the results for $[B_i^a(2\,\mathrm{GeV})]_{\mathrm{NDR}}$ using the lattice results obtained in the LRI scheme. The corresponding results for the B system should be available this year.

Acknowledgements

Appendix

A One-Loop and Two-Loop Anomalous Dimension Matrices

We give below the one-loop and two-loop anomalous dimension matrices. The two-loop expressions are given in the NDR scheme [6].

$$\gamma^{(0)\text{VLL}} = 4, \qquad\qquad \gamma^{(1)\text{VLL}} = -7 + \tfrac{4}{9}f,$$

$$\hat{\gamma}^{(0)\text{LR}} = \begin{pmatrix} 2 & 12 \\ 0 & -16 \end{pmatrix}, \qquad \hat{\gamma}^{(1)\text{LR}} = \begin{pmatrix} \tfrac{71}{3} - \tfrac{22}{9}f & 198 - \tfrac{44}{3}f \\ \tfrac{225}{4} - 2f & -\tfrac{1343}{6} + \tfrac{68}{9}f \end{pmatrix}, \qquad (A.1)$$

$$\hat{\gamma}^{(0)\text{SLL}} = \begin{pmatrix} -10 & \tfrac{1}{6} \\ -40 & \tfrac{34}{3} \end{pmatrix}, \qquad \hat{\gamma}^{(1)\text{SLL}} = \begin{pmatrix} -\tfrac{1459}{9} + \tfrac{74}{9}f & -\tfrac{35}{36} - \tfrac{1}{54}f \\ -\tfrac{6332}{9} + \tfrac{584}{9}f & \tfrac{2065}{9} - \tfrac{394}{27}f \end{pmatrix}.$$

B Expansion of the evolution matrices $\hat{U}$ in α_s

Recall the renormalization group equation to which (2.12) is the formal solution. At NLO it reads, written in terms of α_s,

$$\frac{d\hat{U}(\mu_1, \mu_2)}{d\alpha_s} = \left[-\frac{\hat{\gamma}^{(0)T}}{2\beta_0}\frac{1}{\alpha_s} + \left(\frac{\beta_1}{2\beta_0^2}\hat{\gamma}^{(0)T} - \frac{1}{2\beta_0}\hat{\gamma}^{(1)T} \right)\frac{1}{4\pi} \right] \hat{U}(\mu_1, \mu_2). \qquad (B.2)$$

At the leading order, where only the term $\propto 1/\alpha_s$ is kept, (B.2) has the (exact) solution

$$\hat{U}^{(0)}(\mu_1, \mu_2) = \left(\frac{\alpha_s(\mu_2)}{\alpha_s(\mu_1)} \right)^{\frac{\hat{\gamma}^{(0)T}}{2\beta_0}} = \exp\left(\frac{\hat{\gamma}^{(0)T}}{2\beta_0} \log\frac{\alpha_s(\mu_2)}{\alpha_s(\mu_1)} \right). \qquad (B.3)$$

At the next-to-leading order one has [1]

$$\hat{U}(\mu_1, \mu_2) = \left(\hat{1} + \frac{\alpha_s^{(f)}(\mu_1)}{4\pi}\hat{J}_f \right) \hat{U}_f^{(0)} \left(\hat{1} - \frac{\alpha_s^{(f)}(\mu_2)}{4\pi}\hat{J}_f \right), \qquad (B.4)$$

where higher orders in the parentheses have been omitted. The algorithm for constructing the matrix $\hat{J}_f$ can be found in [1, 2, 13]. Eq. (B.4) holds in a theory with a given number f of

active quark flavours. When evolving across a quark threshold, as for instance in (2.14), one finds

$$\hat{U}(\mu_L, \mu_t) = \left(\hat{1} + \frac{\alpha_s^{(4)}(\mu_L)}{4\pi}\hat{J}_4\right)\hat{U}_{f=4}^{(0)}\left(\hat{1} - \frac{\alpha_s^{(4)}(\mu_b)}{4\pi}\hat{J}_4\right)\left(\hat{1} + \frac{\alpha_s^{(5)}(\mu_b)}{4\pi}\hat{J}_5\right)\hat{U}_{f=5}^{(0)}\left(\hat{1} - \frac{\alpha_s^{(5)}(\mu_t)}{4\pi}\hat{J}_5\right).$$
(B.5)

In light of the fact that $\mathcal{O}(\alpha_s^2)$ and higher terms have been dropped in (B.2) and (B.4), we adopt the convention

$$\eta^{\mathrm{P}} = \left(\frac{\alpha_s(\mu_2)}{\alpha_s(\mu_1)}\right)^{\mathrm{P}} = \mathcal{O}(\alpha_s^0),$$
(B.6)

$$\log(\eta) = \mathcal{O}(\alpha_s^0)$$
(B.7)

and drop all $\mathcal{O}(\alpha_s^2)$ and higher terms in (B.5) and similar expressions. For the desired two- and three-step evolution matrices, one obtains

$$\hat{U}(\mu_L, \mu_t) = \hat{U}_{f=4}^{(0)}(\mu_L, \mu_b)\hat{U}_{f=5}^{(0)}(\mu_b, \mu_t)$$
$$+\frac{\alpha_s^{(4)}(\mu_L)}{4\pi}\left[\hat{J}_4\hat{U}_{f=4}^{(0)}\hat{U}_{f=5}^{(0)} + \eta_4\hat{U}_{f=4}^{(0)}\left(\hat{J}_5 - \hat{J}_4\right)\hat{U}_{f=5}^{(0)} - \eta_4\eta_5\hat{U}_{f=4}^{(0)}\hat{U}_{f=5}^{(0)}\hat{J}_5\right],$$
(B.8)

$$\hat{U}(\mu_K, \mu_L) = \hat{U}_{f=3}^{(0)}(\mu_K, \mu_c)\hat{U}_{f=4}^{(0)}(\mu_c, \mu_L)$$
$$+\frac{\alpha_s^{(3)}(\mu_K)}{4\pi}\left[\hat{J}_3\hat{U}_{f=3}^{(0)}\hat{U}_{f=4}^{(0)} + \eta_3\hat{U}_{f=3}^{(0)}\left(\hat{J}_4 - \hat{J}_3\right)\hat{U}_{f=4}^{(0)} - \eta_3\eta_4\hat{U}_{f=3}^{(0)}\hat{U}_{f=4}^{(0)}\hat{J}_4\right],$$
(B.9)

$$\hat{U}(\mu_K, \mu_t) = \hat{U}_{f=3}^{(0)}(\mu_K, \mu_c)\hat{U}_{f=4}^{(0)}(\mu_c, \mu_b)\hat{U}_{f=5}^{(0)}(\mu_b, \mu_t)$$
$$+\frac{\alpha_s^{(3)}(\mu_K)}{4\pi}\left[\hat{J}_3\hat{U}_{f=3}^{(0)}\hat{U}_{f=4}^{(0)}\hat{U}_{f=5}^{(0)} + \eta_3\hat{U}_{f=3}^{(0)}\left(\hat{J}_4 - \hat{J}_3\right)\hat{U}_{f=4}^{(0)}\hat{U}_{f=5}^{(0)}\right.$$
$$\left.+\eta_3\eta_4\hat{U}_{f=3}^{(0)}\hat{U}_{f=4}^{(0)}\left(\hat{J}_5 - \hat{J}_4\right)\hat{U}_{f=5}^{(0)} - \eta_3\eta_4\eta_5\hat{U}_{f=3}^{(0)}\hat{U}_{f=4}^{(0)}\hat{U}_{f=5}^{(0)}J_5\right],$$
(B.10)

where we have suppressed some obvious arguments in the LO evolution matrices $U^{(0)}$ in order not to unnecessarily clutter the expressions.

C   The Evolution Matrix $\hat{U}(\mu_t, \mu_s)$

For completeness we give here the elements of the evolution matrix $\hat{U}(\mu_t, \mu_s)$ in a $f = 6$ flavour theory with $\mu_s > \mu_t$. The renormalization group evolution from μ_s down to μ_t can even be

included as in (1.4) when μ_s is only by a factor of two higher than m_t. However, it is only necessary when $\mu_s > 4m_t$ in order to avoid large logarithms.

The formulae given below are not as general as the ones given in section 3. They apply only to the evolution of new physics contributions which do not involve SM particles except for the number of quark flavours entering α_s and the anomalous dimensions of the operators (2.1). This is for instance the case considered in [8, 9] in which squarks and gluinos have been integrated out at a scale $\mu_s \gg \mu_t$. On the other hand the renormalization group analysis of charged Higgs contributions with $M_{H^\pm} \gg m_t$ would be more complicated as both $H^\pm$ and top can be simultaneously exchanged in box diagrams. Integrating out first $H^\pm$ and subsequently the top would introduce bilocal structures for $\mu_t < \mu < \mu_s$ quite analogous to the study of charm contributions to $K^0 - \overline{K}^0$ mixing [7, 14]. We find then

VLL-Sector

$$\left[\eta^{(0)}(\mu_t)\right]_{\text{VLL}} = \eta_6^{6/21}, \tag{C.11}$$

$$\left[\eta^{(1)}(\mu_t)\right]_{\text{VLL}} = 1.3707(1 - \eta_6)\eta_6^{6/21}. \tag{C.12}$$

LR-Sector

$$\left[\eta_{11}^{(0)}(\mu_t)\right]_{\text{LR}} = \eta_6^{3/21}, \tag{C.13}$$

$$\left[\eta_{12}^{(0)}(\mu_t)\right]_{\text{LR}} = 0, \tag{C.14}$$

$$\left[\eta_{21}^{(0)}(\mu_t)\right]_{\text{LR}} = \frac{2}{3}(\eta_6^{3/21} - \eta_6^{-24/21}), \tag{C.15}$$

$$\left[\eta_{22}^{(0)}(\mu_t)\right]_{\text{LR}} = \eta_6^{-24/21}, \tag{C.16}$$

$$\left[\eta_{11}^{(1)}(\mu_t)\right]_{\text{LR}} = 0.9219\,\eta_6^{-24/21} + \eta_6^{3/21}\left(-2.2194 + 1.2975\,\eta_6\right), \tag{C.17}$$

$$\left[\eta_{12}^{(1)}(\mu_t)\right]_{\text{LR}} = 1.3828\,(\eta_6^{24/21} - \eta_6^{-24/21}), \tag{C.18}$$

$$\left[\eta_{21}^{(1)}(\mu_t)\right]_{\text{LR}} = \eta_6^{3/21}\left(-10.1463 + 0.8650\,\eta_6\right) + \eta_6^{-24/21}\left(-6.4603 + 15.7415\,\eta_6\right), \tag{C.19}$$

$$\left[\eta_{22}^{(1)}(\mu_t)\right]_{\text{LR}} = 0.9219\,\eta_6^{24/21} + \eta_6^{-24/21}\left(9.6904 - 10.6122\,\eta_6\right). \tag{C.20}$$

SLL-Sector

$$\left[\eta_{11}^{(0)}(\mu_t)\right]_{\text{SLL}} = 1.0153\eta_6^{-0.6916} - 0.0153\eta_6^{0.7869}, \tag{C.21}$$

$$\left[\eta_{12}^{(0)}(\mu_t)\right]_{\text{SLL}} = 1.9325(\eta_6^{-0.6916} - \eta_6^{0.7869}), \tag{C.22}$$

$$\left[\eta_{21}^{(0)}(\mu_t)\right]_{\text{SLL}} = 0.0081(\eta_6^{0.7869} - \eta_6^{-0.6916}), \tag{C.23}$$

$$\left[\eta_{22}^{(0)}(\mu_t)\right]_{\text{SLL}} = 1.0153\eta_6^{0.7869} - 0.0153\eta_6^{-0.6916}, \tag{C.24}$$

$$\left[\eta_{11}^{(1)}(\mu_t)\right]_{\text{SLL}} = \eta_6^{-0.6916}\,(5.6478 - 6.0350\,\eta_6) + \eta_6^{0.7869}\,(0.3272 + 0.0600\,\eta_6), \tag{C.25}$$

$$\left[\eta_{12}^{(1)}(\mu_t)\right]_{\text{SLL}} = \eta_6^{-0.6916}\,(10.7494 - 37.9209\,\eta_6) + \eta_6^{0.7869}\,(41.2556 - 14.0841\,\eta_6), \tag{C.26}$$

$$\left[\eta_{21}^{(1)}(\mu_t)\right]_{\text{SLL}} = \eta_6^{0.7869}\,(-0.0618 - 0.0315\,\eta_6) + \eta_6^{-0.6916}\,(0.0454 + 0.0479\,\eta_6), \tag{C.27}$$

$$\left[\eta_{22}^{(1)}(\mu_t)\right]_{\text{SLL}} = \eta_6^{-0.6916}\,(0.0865 + 0.3007\,\eta_6) + \eta_6^{0.7869}\,(-7.7870 + 7.3999\,\eta_6). \tag{C.28}$$

Here $\mu_t = \mathcal{O}(m_t)$ and $\eta_6 = \alpha_s^{(6)}(\mu_s)/\alpha_s^{(6)}(\mu_t)$. These results together with those presented in section 3 and 4 allow to find $\hat{U}(\mu, \mu_s)$ with $\mu < \mu_t$, see (1.4).

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