Dyson–Schwinger recursion at fixed order

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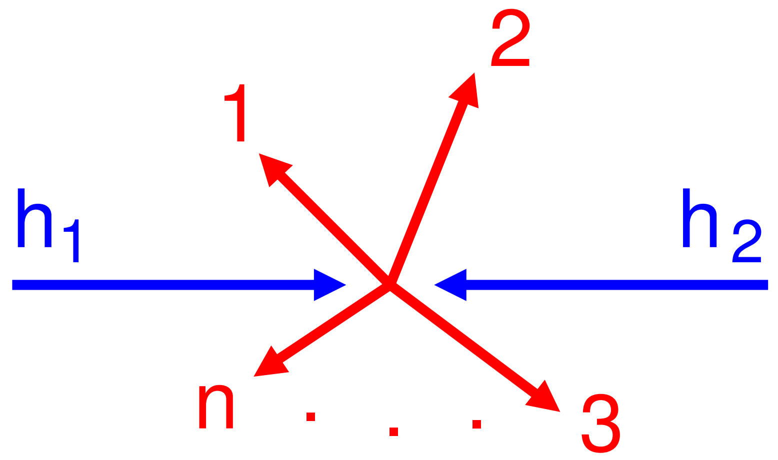
Outline

- introduction
- zero-dimensional field theory
- tree-level recursion
- color treatment
- one-loop recursion

Hard scattering cross sections within collinear factorization



PDFs are related to the structure of the hadrons, universal to the scattering process

$$\sigma_{h_1,h_2\to n}(p_1,p_2) = \sum_{a,b}\int dx_1\,dx_2\; f_a(x_1,\mu)\,f_b(x_2,\mu)\;\hat\sigma_{a,b\to n}(x_1p_1,x_2p_2;\mu)$$

$$\hat\sigma_{a,b\to n}(p_a,p_b;\mu) = \int d\Phi(p_a,p_b\to\{p\}_n)\;|\mathcal{M}_{a,b\to n}(p_a,p_b\to\{p\}_n;\mu)|^2\;\mathcal{O}(p_a,p_b,\{p\}_n)$$

Phase space (includes spin/color summation) governs the kinematics

Matrix element (squared) contains model parameters, governs the dynamics

Observable, imposes phase space cuts

Zero–dimensional QFT

Consider ϕ^3-theory on a single space-time point

$$Z[J] = \int_{-\infty}^{\infty} d\phi \, \exp\left\{ \frac{i}{\hbar}\left[J\phi + S(\phi)\right] \right\} \quad , \quad S(\phi) = -\frac{m^2}{2}\phi^2 - \frac{g}{6}\phi^3 \ , \ \mathrm{Im}(m^2 < 0)$$

We trivially have the linear Dyson-Schwinger equation

$$0 = \int_{-\infty}^{\infty} d\phi \, \frac{\hbar}{i}\frac{d}{d\phi} \, \exp\left\{ \frac{i}{\hbar}\left[J\phi + S(\phi)\right] \right\} = \left(J - \frac{\hbar}{i} m^2 \frac{d}{dJ} + \frac{\hbar^2 g}{2} \frac{d^2}{dJ^2} \right) Z[J]$$

$Z[J]$ generates zero-dimensional “Green functions”, connected “Green functions” generated by

$$W[J] = \ln Z[J]$$

Non-linear Dyson-Schwinger equation

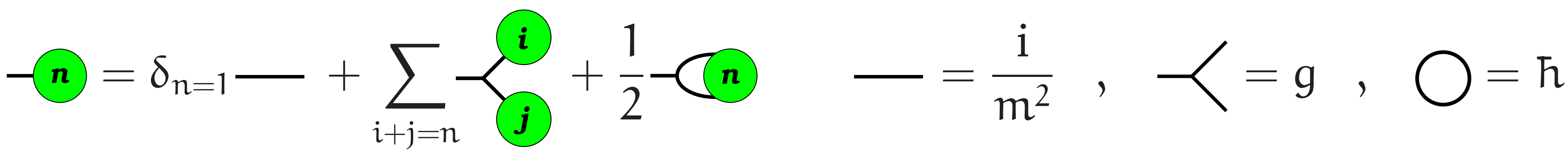
$$0 = J + im^2 \frac{dW[J]}{dJ} + \frac{g}{2}\left[\hbar \frac{d^2 W[J]}{dJ^2} + \left(\frac{dW[J]}{dJ} \right)^2 \right]$$

Zero–dimensional QFT

Dyson-Schwinger equation for Green functions from $\dfrac{dW[J]}{dJ} = \sum_{n=0}^{\infty} \dfrac{C_{n+1} J^n}{n!}$

$$\frac{C_{n+1}}{n!} = \frac{i}{m^2}\left(\delta_{n=1} + g\sum_{i+j=n}\frac{C_{i+1}}{i!}\frac{C_{j+1}}{j!} + \frac{\hbar g}{2}\frac{C_{n+2}}{n!}\right)$$

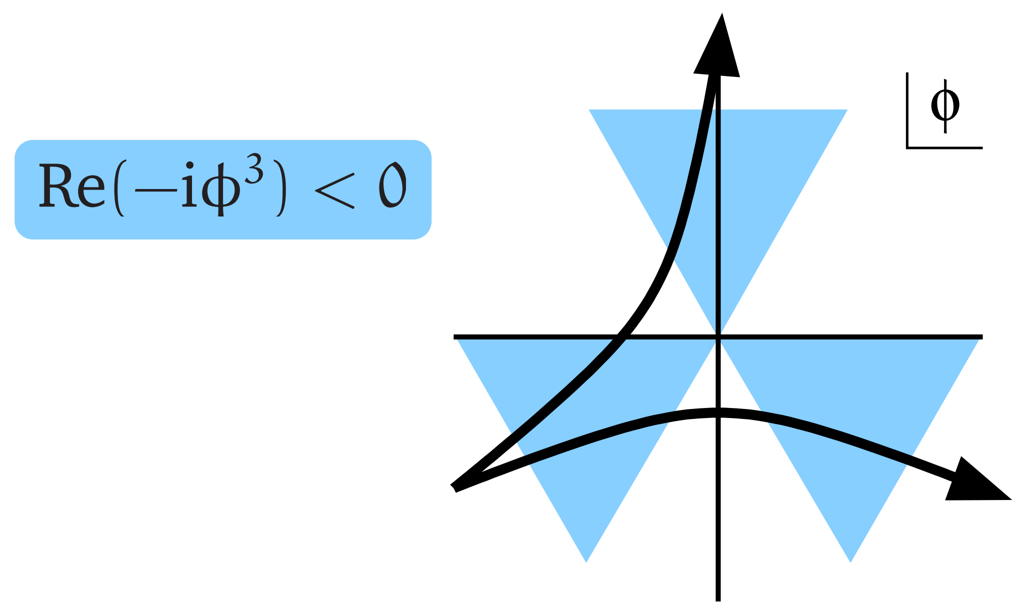
We may cast the equation into a graphical form



Solutions for tadpole

$$C_1^{\mathrm{pert}} \propto \frac{\hbar g^2}{m^4} \qquad C_1^{\mathrm{non-pert}} \propto \frac{m^2}{g}$$

Non-perturbative solution corresponds to other integration contour in the complex ϕ-plane in the definition of $Z[J]$.



Zero–dimensional QFT

Introduce more zero-dimensional points

$$S(\phi) = -\sum_{k,l} \frac{1}{2} A_{k,l}\phi_k\phi_l - \sum \frac{g}{6}\phi_l^3 \quad , \quad \mathrm{Im}(A_{k,k} < 0)$$

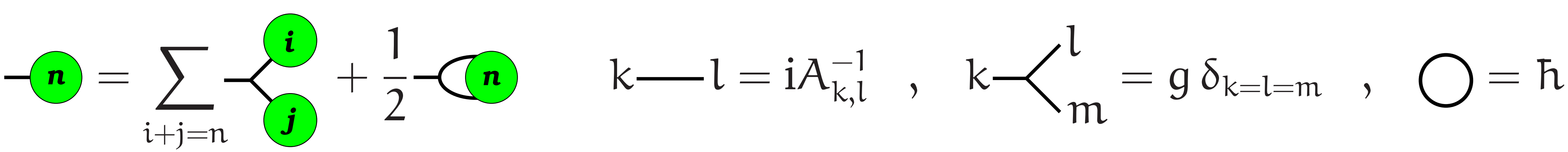
Dyson-Schwinger equation

$$0 = J_k + i\sum_l A_{k,l}\frac{\partial W[J]}{\partial J_l} + \frac{g}{2}\left[\hbar\frac{\partial^2 W[J]}{\partial J_k^2} + \left(\frac{\partial W[J]}{\partial J_k}\right)^2\right]$$

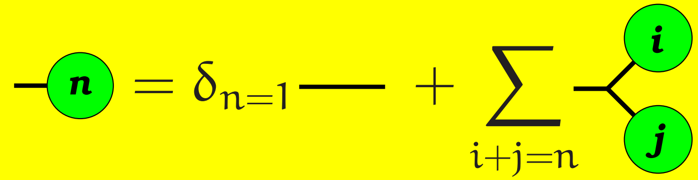
Expand generating function in terms of Green functions

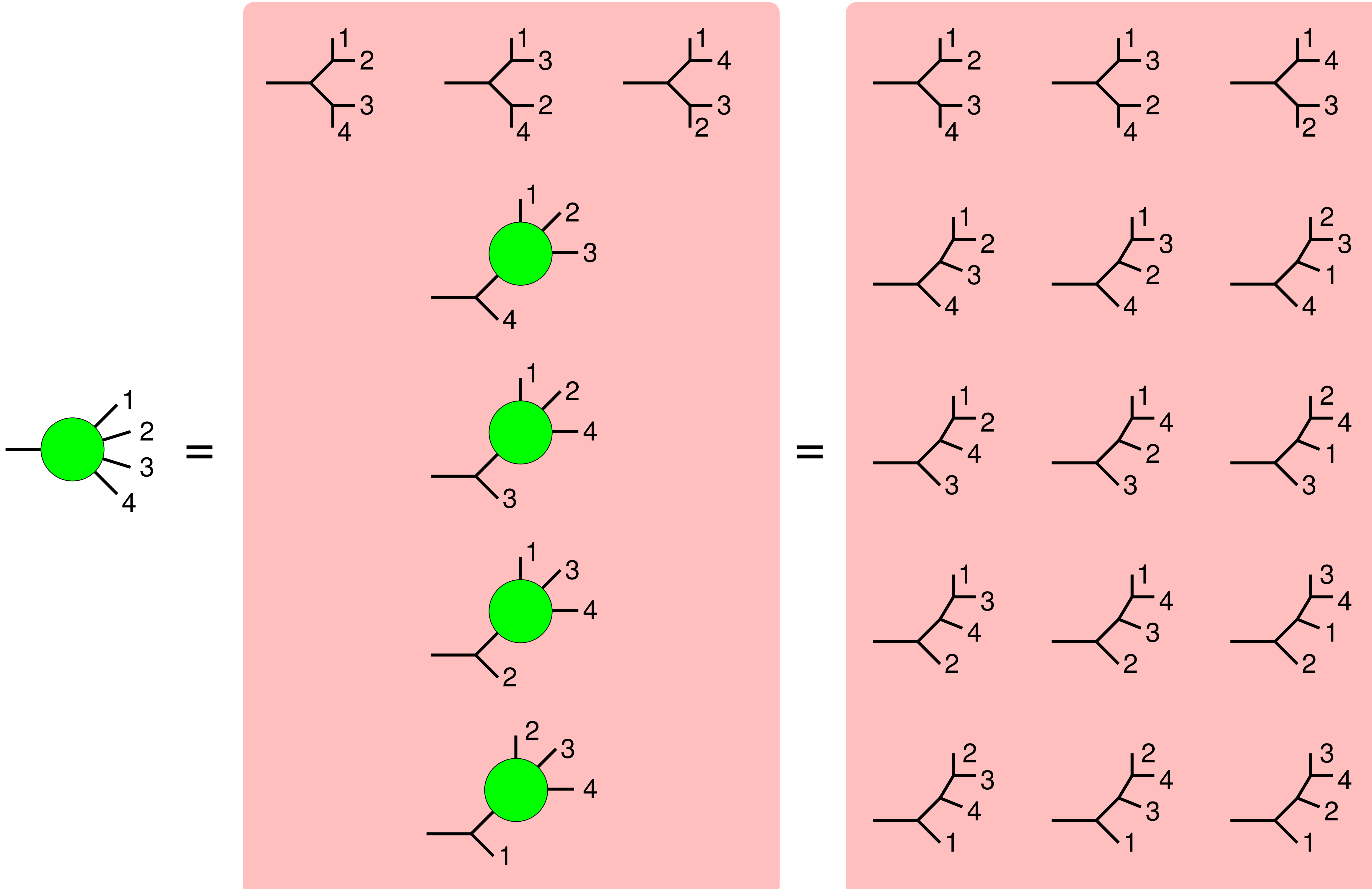
$$\frac{\partial W[J]}{\partial J_l} = \sum_{i_1+i_2+\cdots+i_k=n} C_{l;i_1 i_2\cdots i_k}\,\frac{J_1^{i_1}}{i_1!}\frac{J_2^{i_2}}{i_2!}\cdots\frac{J_k^{i_k}}{i_k!}$$

Graphical interpretation

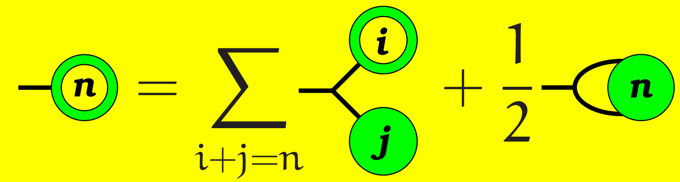


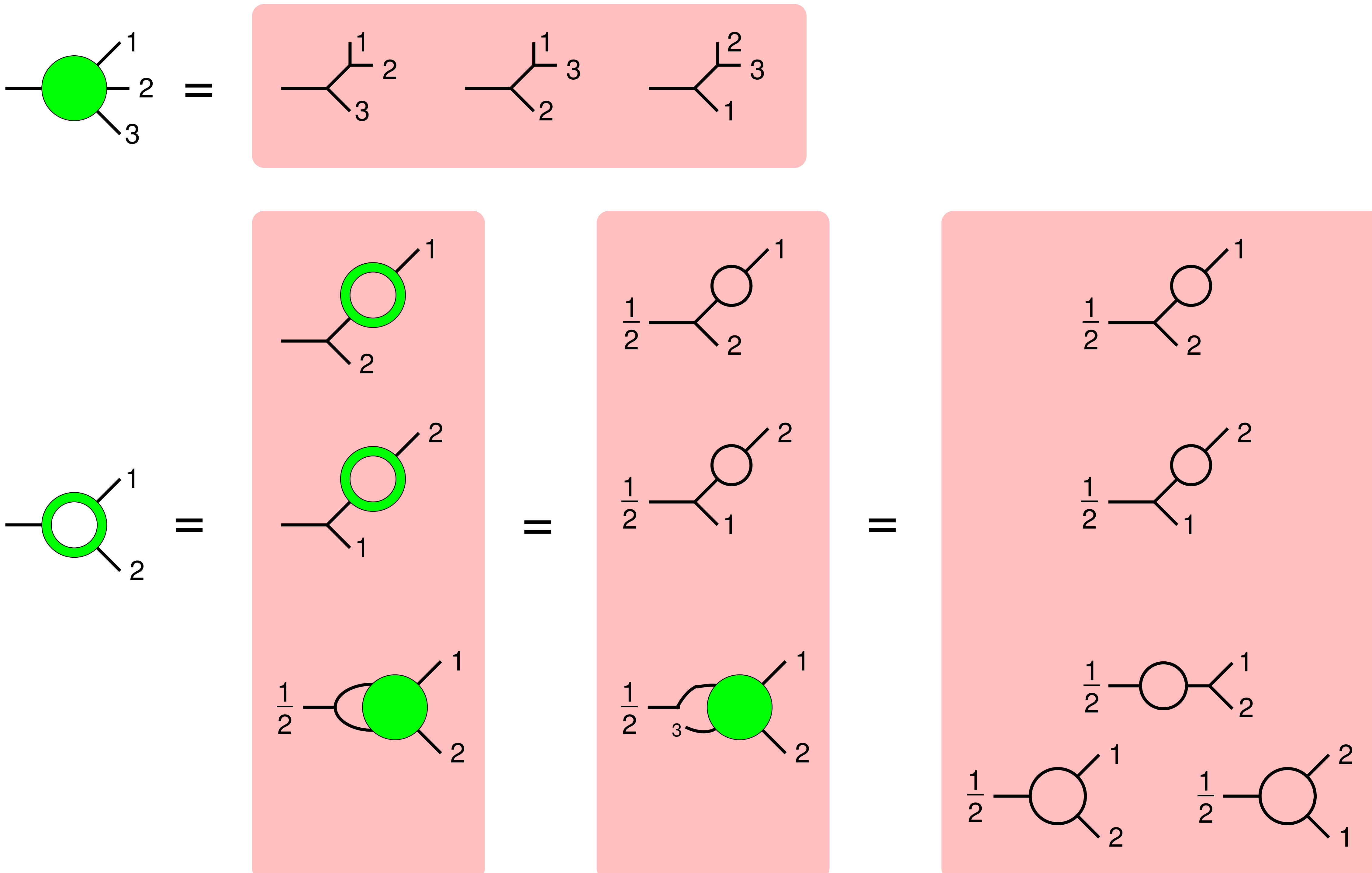
Tree–level recursion



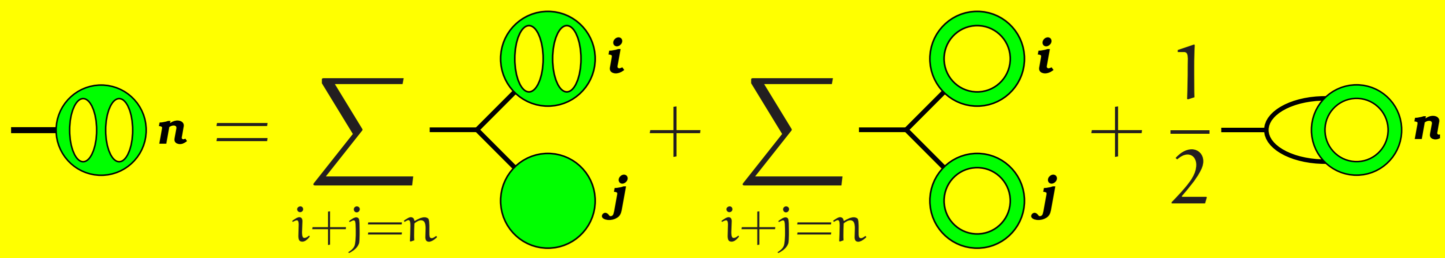


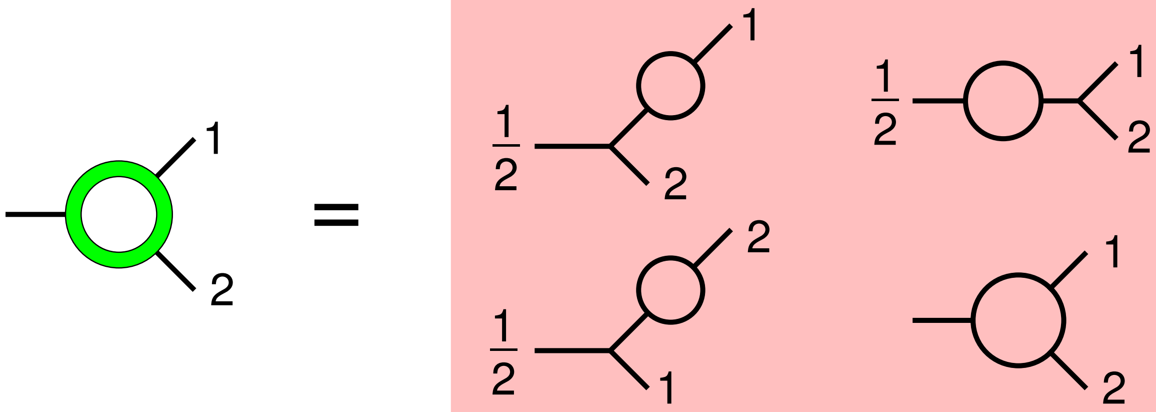
One–loop recursion

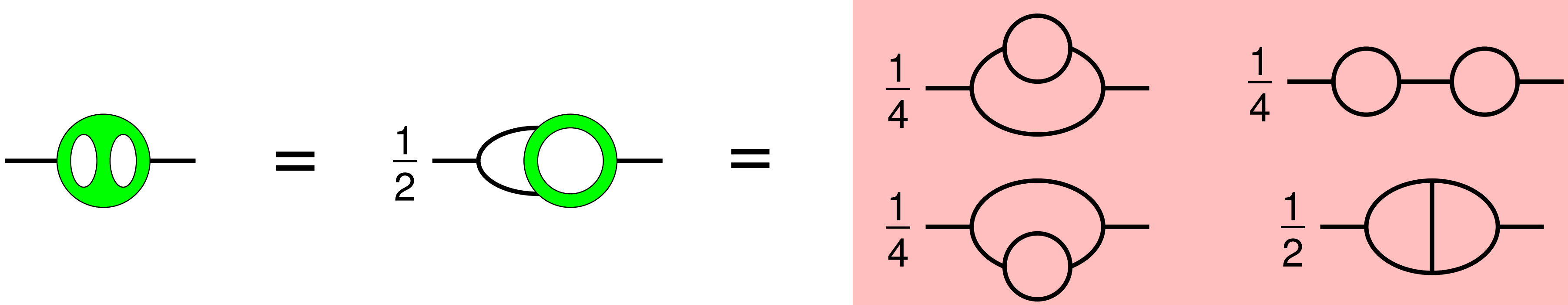




Two–loop recursion

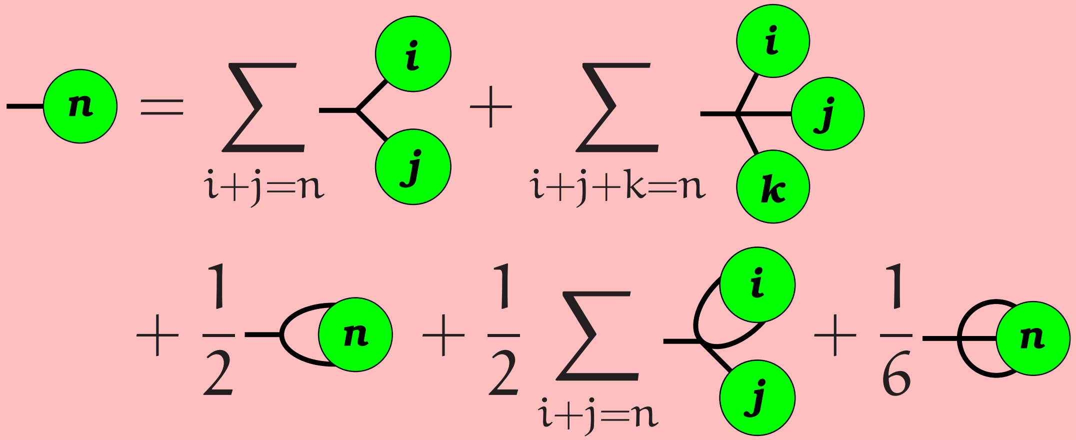




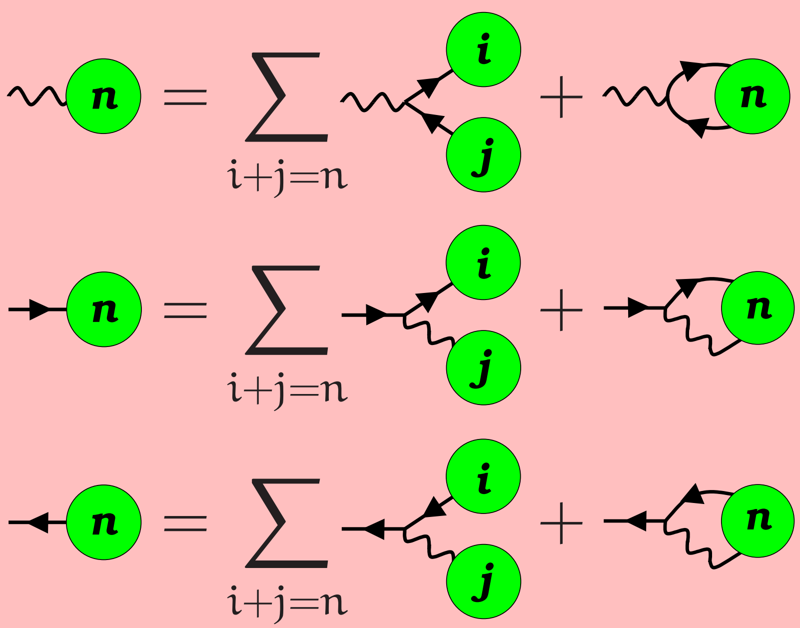


Generalization to real QFT

Theories with four-point vertices:



Theories with more types of currents:



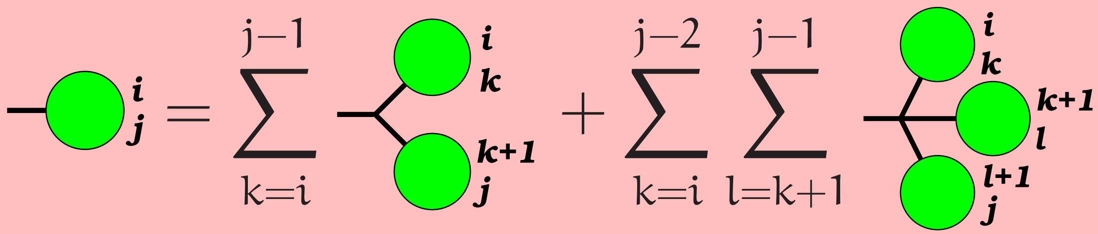
Currents may have several components.

- distinguishable external lines correspond to on-shell particles
  $\Longrightarrow$ polarization vectors, spinors, 1
- sum of momenta of on-shell lines is equal to momentum of off-shell line
- vertices directly from Feynman rules in momentum space
- off-shell line carries propagator from Feynman rules, in any gauge
- on-shell $(n+1)$-leg amplitude
  - from current with n on-shell legs
  - by omitting the final propagator
  - and contracting with pol.vec. or spinor instead

Expressions from Berends–Giele

For planar multi-gluon tree-amplitudes:

$$p_{i,j} = p_i + p_{i+1} + \cdots + p_j$$



$$A^{\mu}_{i,j} = \frac{-i}{p^2_{i,j}}\left[\sum_{k=i}^{j-1} V^{\mu}_{\nu\rho}(p_{i,k}, p_{k+1,j})\, A^{\nu}_{i,k} A^{\rho}_{k+1,j} + \sum_{k=i}^{j-2}\sum_{l=k+1}^{j-1} W^{\mu}_{\nu\rho\sigma}\, A^{\nu}_{i,k} A^{\rho}_{k+1,l} A^{\sigma}_{l+1,j}\right]$$

$$V^{\mu}_{\nu\rho}(p,q) = \frac{i}{\sqrt{2}}\left[(p-q)^{\mu} g_{\nu\rho} + 2g^{\mu}_{\rho}\, q_{\nu} - 2g^{\mu}_{\nu}\, p_{\rho}\right]$$

$$W^{\mu}_{\nu\rho\sigma} = \frac{i}{2}\left[2g^{\mu}_{\rho} g_{\nu\sigma} - g^{\mu}_{\nu} g_{\rho\sigma} - g^{\mu}_{\sigma} g_{\rho\nu}\right]$$

$$A^{\mu}_{i,i} = \varepsilon^{\mu}(p_i)$$

For special helicity configurations, compact expressions can be derived, *eg.* for $\{1,j\}$ negative, rest positive helicity

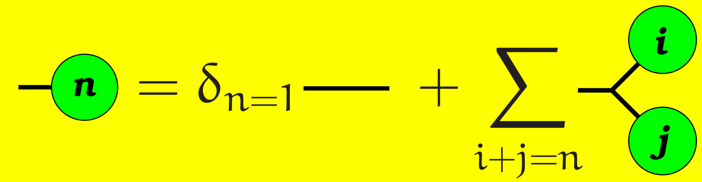
$$\mathcal{A}_n = \frac{\langle 1\, j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle\cdots\langle (n-1)\, n\rangle\langle n\, 1\rangle}$$

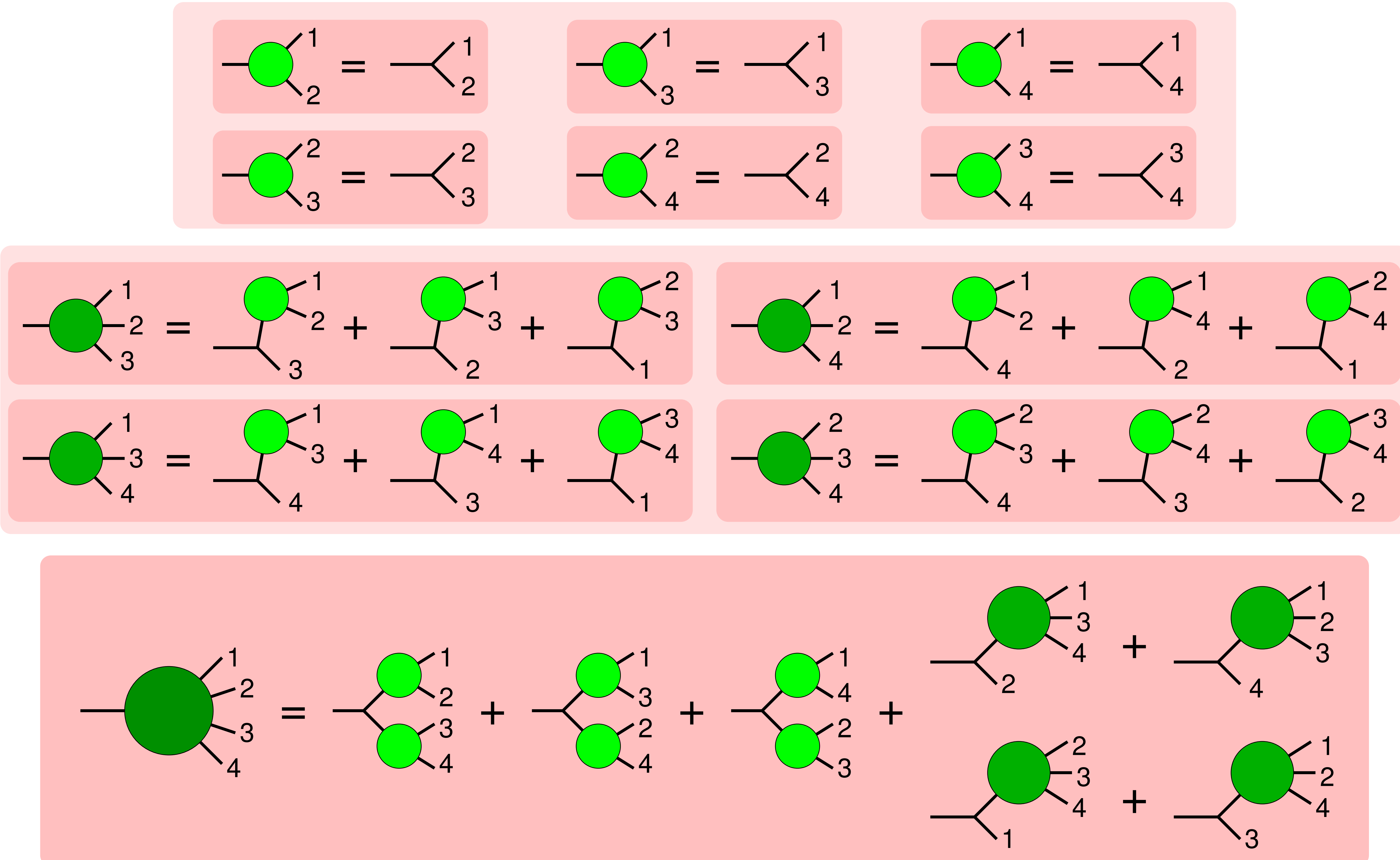
$\langle k\, l\rangle$ = product of Weyl spinors for external momentum k and l.

- what about non-planar amplitudes?
- what if theory is less symmetric?
- what if there are several mass scales?

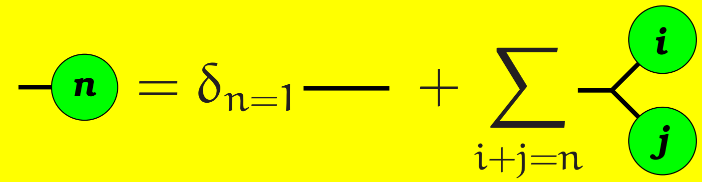
Direct numerical evaluation of Dyson-Schwinger relations.

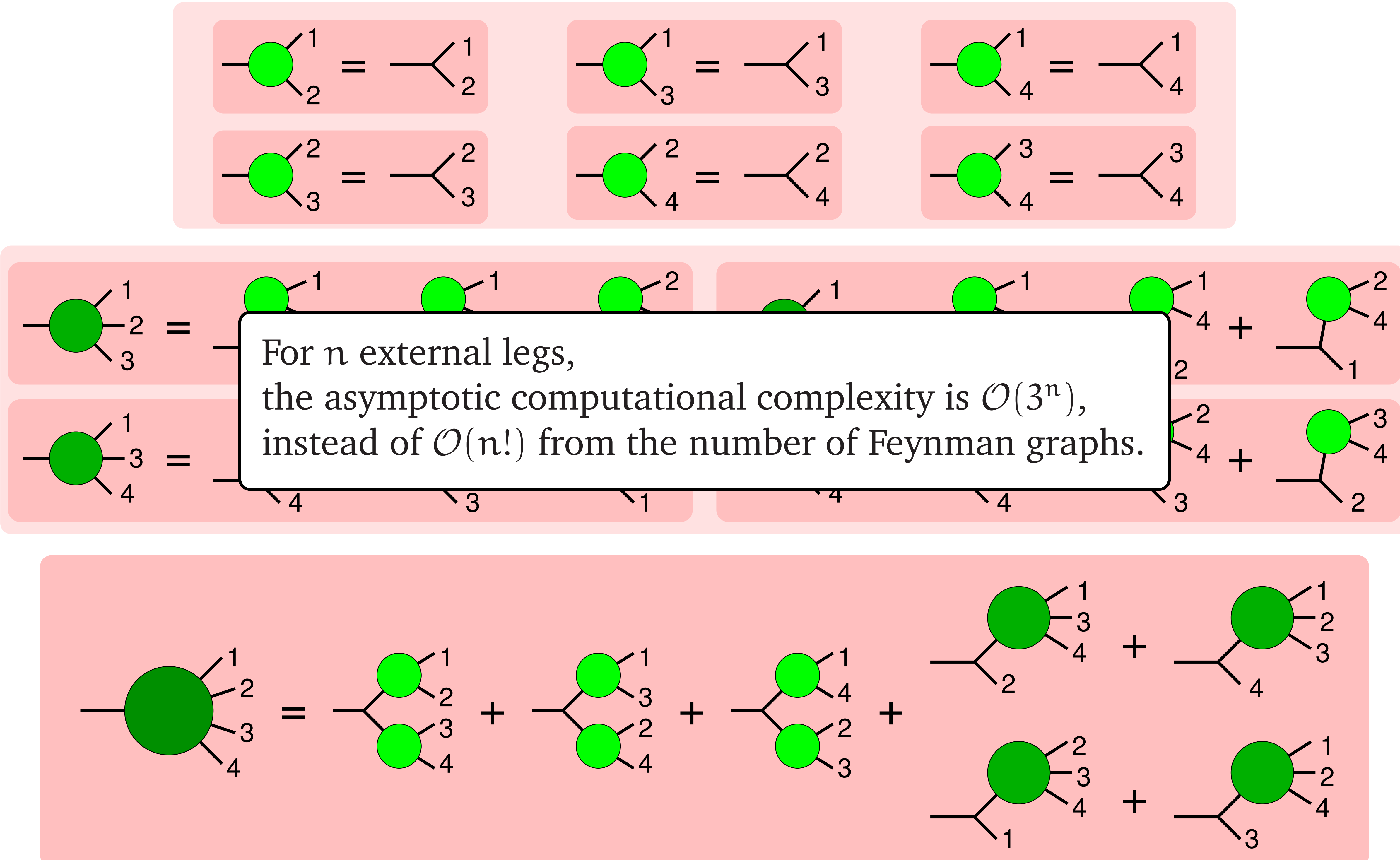
Recursive computation





Recursive computation

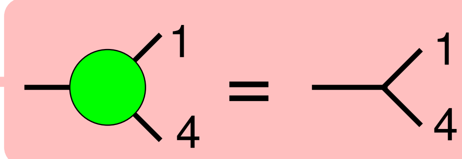


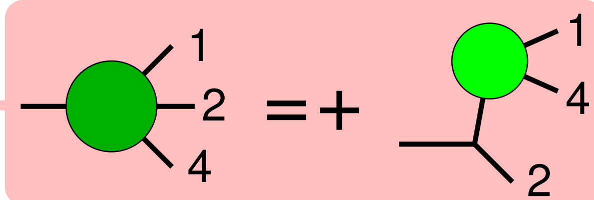


For n external legs,
the asymptotic computational complexity is $\mathcal{O}(3^n)$,
instead of $\mathcal{O}(n!)$ from the number of Feynman graphs.

DS skeleton for 0–>hhhhh

```
 1:     5[   3 h ] <--     2[   2 h ]     1[   1 h ]
 2:     6[   5 h ] <--     3[   4 h ]     1[   1 h ]
 3:     7[   9 h ] <--     4[   8 h ]     1[   1 h ]
 4:     8[   6 h ] <--     3[   4 h ]     2[   2 h ]
 5:     9[  10 h ] <--     4[   8 h ]     2[   2 h ]
 6:    10[  12 h ] <--     4[   8 h ]     3[   4 h ]
 7:    11[   7 h ] <--     3[   4 h ]     5[   3 h ]
 8:    11[   7 h ] <--     2[   2 h ]     6[   5 h ]
 9:    11[   7 h ] <--     1[   1 h ]     8[   6 h ]
10:    12[  11 h ] <--     4[   8 h ]     5[   3 h ]
11:    12[  11 h ] <--     2[   2 h ]     7[   9 h ]
12:    12[  11 h ] <--     1[   1 h ]     9[  10 h ]
13:    13[  13 h ] <--     4[   8 h ]     6[   5 h ]
14:    13[  13 h ] <--     3[   4 h ]     7[   9 h ]
15:    13[  13 h ] <--     1[   1 h ]    10[  12 h ]
16:    14[  14 h ] <--     4[   8 h ]     8[   6 h ]
17:    14[  14 h ] <--     3[   4 h ]     9[  10 h ]
18:    14[  14 h ] <--     2[   2 h ]    10[  12 h ]
19:    15[  15 h ] <--    10[  12 h ]     5[   3 h ]
20:    15[  15 h ] <--     9[  10 h ]     6[   5 h ]
21:    15[  15 h ] <--     8[   6 h ]     7[   9 h ]
22:    15[  15 h ] <--     4[   8 h ]    11[   7 h ]
23:    15[  15 h ] <--     3[   4 h ]    12[  11 h ]
24:    15[  15 h ] <--     2[   2 h ]    13[  13 h ]
25:    15[  15 h ] <--     1[   1 h ]    14[  14 h ]
```





particle identifier for off-shell leg

$p_{13} = p_4 + p_9$

Binary representation of momenta: external momenta are labeled by powers of 2, and

$$p_{2^{n-1}-1} = p_1 + p_2 + p_4 + \cdots + p_{2^{n-2}}$$
$$= -p_{2^{n-1}}$$

eg. for $n = 5$ we have $p_{15} = -p_{16}$

DS skeleton for 0 –> e+ e– e+ e– A

```
 1: -1,   5[  5 A ] <--   3[  4 E-]   1[  1 E+]
 2: -1,   6[  5 Z ] <--   3[  4 E-]   1[  1 E+]
 3:  1,   7[  9 E+] <--   4[  8 A ]   1[  1 E+]
 4: -1,   8[  6 A ] <--   3[  4 E-]   2[  2 E+]
 5: -1,   9[  6 Z ] <--   3[  4 E-]   2[  2 E+]
 6:  1,  10[ 10 E+] <--   4[  8 A ]   2[  2 E+]
 7:  1,  11[ 12 E-] <--   4[  8 A ]   3[  4 E-]
 8: -1,  12[  7 E+] <--   2[  2 E+]   5[  5 A ]
 9: -1,  12[  7 E+] <--   2[  2 E+]   6[  5 Z ]
10:  1,  12[  7 E+] <--   1[  1 E+]   8[  6 A ]
11:  1,  12[  7 E+] <--   1[  1 E+]   9[  6 Z ]
12: -1,  13[ 13 A ] <--   3[  4 E-]   7[  9 E+]
13: -1,  14[ 13 Z ] <--   3[  4 E-]   7[  9 E+]
14: -1,  13[ 13 A ] <--   1[  1 E+]  11[ 12 E-]
15: -1,  14[ 13 Z ] <--   1[  1 E+]  11[ 12 E-]
16: -1,  15[ 14 A ] <--   3[  4 E-]  10[ 10 E+]
17: -1,  16[ 14 Z ] <--   3[  4 E-]  10[ 10 E+]
18: -1,  15[ 14 A ] <--   2[  2 E+]  11[ 12 E-]
19: -1,  16[ 14 Z ] <--   2[  2 E+]  11[ 12 E-]
20: -1,  17[ 15 E+] <--  10[ 10 E+]   5[  5 A ]
21: -1,  17[ 15 E+] <--  10[ 10 E+]   6[  5 Z ]
22:  1,  17[ 15 E+] <--   8[  6 A ]   7[  9 E+]
23:  1,  17[ 15 E+] <--   9[  6 Z ]   7[  9 E+]
24:  1,  17[ 15 E+] <--   4[  8 A ]  12[  7 E+]
25: -1,  17[ 15 E+] <--   2[  2 E+]  13[ 13 A ]
26: -1,  17[ 15 E+] <--   2[  2 E+]  14[ 13 Z ]
27:  1,  17[ 15 E+] <--   1[  1 E+]  15[ 14 A ]
28:  1,  17[ 15 E+] <--   1[  1 E+]  16[ 14 Z ].
```

same momentum, different particle

$$\overline{\Psi}_{11} =+ \overline{\Psi}_3 \not{A}_4(-ie)\frac{i}{\not{p}_{12} - m}$$

$$\Psi_{12} =+ \frac{i}{-\not{p}_7 - m}(-ie)\not{A}_8 \Psi_1$$

$$A^{\mu}_{13} =+ \frac{-i}{p^2_{13}}(-ie)\overline{\Psi}_{11}\gamma^{\mu}\Psi_1$$

fermi sign

$$(-1)^{\chi(p,q)} \quad , \quad \chi(p,q) = \sum_{i=n}^{2} \hat{p}_i \sum_{j=1}^{i-1} \hat{q}_j$$

$\hat{p}_i = 1$ if external particle i is a fermion and is present in p, else $\hat{p}_i = 0$

Cross sections from Monte Carlo

Calculation of a cross section requires phase space integration and summation over spins and colors.

$$\sigma = \int d\Phi \sum_{\text{spin}} \sum_{\text{color}} |\mathcal{M}(\Phi, \text{spin}, \text{color})|^2 \ \mathcal{O}(\Phi)$$

- Phase space must we dealt with within a Monte Carlo approach (that's why we need to be able to evaluate scattering amplitudes numerically efficiently)

- Spin may be dealt with within a Monte Carlo approach:

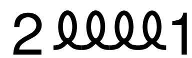
$$\sum_{+,-} \Rightarrow \int_0^1 d\rho \quad , \quad \varepsilon^\mu(\rho) = u_+(\rho)\varepsilon_+^\mu + u_-(\rho)\varepsilon_-^\mu \quad , \quad \int_0^1 u_i(\rho)u_j(\rho)^* = \delta_{i,j}$$

  - random helicities: $u_\pm(\rho) = \sqrt{2}\,\theta(\pm(\frac{1}{2} - \rho))$
  - random polarizations: $u_\pm(\rho) = e^{\pm i\pi\rho}$

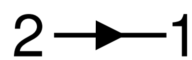
- Color may be dealt with also within a Monte Carlo approach

What color representation to use?

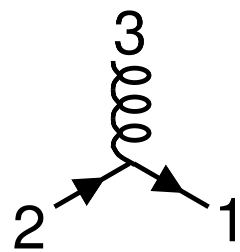
QCD Feynman rules



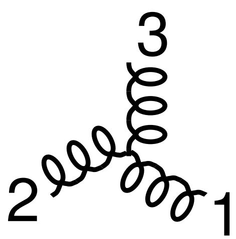
$$= \frac{-i}{p^2}\,\eta^{\mu_1\mu_2}\,\delta^{a_1a_2}$$



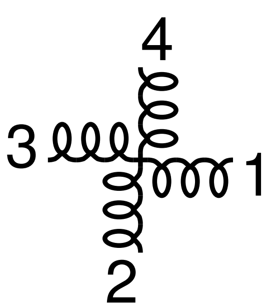
$$= \frac{i}{\not{p} - m}\,\delta_{i_1i_2}$$



$$= -ig\,T^{a_3}_{i_1i_2}\,\gamma^{\mu_3}$$



$$= g\,f^{a_1a_2a_3}\left[(p_1-p_2)^{\mu_3}\eta^{\mu_1\mu_2} + (p_2-p_3)^{\mu_1}\eta^{\mu_2\mu_3} + (p_3-p_1)^{\mu_2}\eta^{\mu_3\mu_1}\right]$$



$$\begin{aligned} = ig^2\Big[\; &(f^{a_1a_3b}f^{a_2a_4b} - f^{a_1a_4b}f^{a_3a_2b})\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} \\ +&(f^{a_1a_2b}f^{a_3a_4b} - f^{a_1a_4b}f^{a_2a_3b})\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} \\ +&(f^{a_1a_3b}f^{a_4a_2b} - f^{a_1a_2b}f^{a_3a_4b})\eta^{\mu_1\mu_4}\eta^{\mu_3\mu_2}\Big] \end{aligned}$$

Color representation

- Represent gluons as 8-times higher-dim vectors A_μ^a
  increases the number of operations per vertex unacceptably

- Treat gluons with different color as different particles

$$f^{abc} \neq 0 \quad \Rightarrow \quad abc \in \{123, 147, 156, 246, 257, 345, 367, 458, 678\}$$

all possible fusions unique, except $(4,5) \to \{3,8\}$ and $(6,7) \to \{3,8\}$

```
1:    5[   3 g 3 ] <--    2[   2 g 2 ]    1[   1 g 1 ]
2:    6[   5 g 7 ] <--    3[   4 g 4 ]    1[   1 g 1 ]
3:    7[   9 g 6 ] <--    4[   8 g 5 ]    1[   1 g 1 ]
4:    8[   6 g 6 ] <--    3[   4 g 4 ]    2[   2 g 2 ]
5:    9[  10 g 7 ] <--    4[   8 g 5 ]    2[   2 g 2 ]
6:   10[  12 g 3 ] <--    4[   8 g 5 ]    3[   4 g 4 ]
7:   10[  12 g 8 ] <--    4[   8 g 5 ]    3[   4 g 4 ]
```

skeleton depends on external color configuration

Color flow representation

$$\sum_a |\mathcal{A}^a|^2 = \sum_{a,b} \delta^{ab}\mathcal{A}^a\mathcal{A}^{b*} = \sum_{a,b} 2\mathrm{Tr}\{T^aT^b\}\mathcal{A}^a\mathcal{A}^{b*} = \sum_{i,j} |\mathcal{A}^i_j|^2 \quad , \quad \mathcal{A}^i_j = \sqrt{2}(T^a)^i_j\mathcal{A}^a$$

Contract all external gluons with $\sqrt{2}(T^a)^i_j$
and replace in all gluon propagators $\delta^{ab} = 2\mathrm{Tr}\{T^aT^b\}$
Color structure of the vertices become

$$\text{3-gluon:} \quad 2^{3/2} f^{abc}(T^a)^{i_1}_{j_1}(T^b)^{i_2}_{j_2}(T^c)^{i_3}_{j_3} = \frac{-i}{\sqrt{2}}\left(\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}\right)$$

$$\begin{aligned}\text{4-gluon:} \quad & 4(f^{abe}f^{cde} - f^{ade}f^{bce})(T^a)^{i_1}_{j_1}(T^b)^{i_2}_{j_2}(T^c)^{i_3}_{j_3}(T^d)^{i_4}_{j_4} \\ &= \frac{-1}{2}\Big(2\delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_4}\delta^{i_4}_{j_1} + 2\delta^{i_1}_{j_4}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}\delta^{i_4}_{j_3} \\ &\quad - \delta^{i_1}_{j_2}\delta^{i_2}_{j_4}\delta^{i_3}_{j_1}\delta^{i_4}_{j_3} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_4}\delta^{i_4}_{j_2} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_4}\delta^{i_3}_{j_2}\delta^{i_4}_{j_1} - \delta^{i_1}_{j_4}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1}\delta^{i_4}_{j_2}\Big)\end{aligned}$$

$$\text{quark-gluon:} \quad \sqrt{2}\,(T^a)^{i_1}_{j_1}(T^b)^{i_2}_{j_2} = \frac{1}{\sqrt{2}}\left(\delta^{i_1}_{j_2}\delta^{i_2}_{j_1} - \frac{1}{N_c}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\right)$$

$1/N_c$ contribution in quark-gluon vertex, but trivial gluon propagator: $\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}$

DS skeleton for 0 –> ggggg

```
color configuration: (2,3) (3,1) (1,1) (3,3) (1,2)

  1:    6[  9 g (2,3)] <--  4[  8 g (3,3)]    1[  1 g (2,3)]
  2:    7[  6 g (3,1)] <--  3[  4 g (1,1)]    2[  2 g (3,1)]
  3:    8[ 10 g (3,1)] <--  4[  8 g (3,3)]    2[  2 g (3,1)]
  4:   10[ 11 g (2,1)] <--  4[  8 g (3,3)]    2[  2 g (3,1)]    1[  1 g (2,3)]
  5:   10[ 11 g (2,1)] <--  2[  2 g (3,1)]    6[  9 g (2,3)]
  6:   10[ 11 g (2,1)] <--  1[  1 g (2,3)]    8[ 10 g (3,1)]
  7:   11[ 14 g (3,1)] <--  4[  8 g (3,3)]    7[  6 g (3,1)]
  8:   11[ 14 g (3,1)] <--  4[  8 g (3,3)]    3[  4 g (1,1)]    2[  2 g (3,1)]
  9:   11[ 14 g (3,1)] <--  3[  4 g (1,1)]    8[ 10 g (3,1)]
 10:   12[ 15 g (2,1)] <--  7[  6 g (3,1)]    6[  9 g (2,3)]
 11:   12[ 15 g (2,1)] <--  4[  8 g (3,3)]    1[  1 g (2,3)]    7[  6 g (3,1)]
 12:   12[ 15 g (2,1)] <--  3[  4 g (1,1)]   10[ 11 g (2,1)]
 13:   12[ 15 g (2,1)] <--  3[  4 g (1,1)]    2[  2 g (3,1)]    6[  9 g (2,3)]
 14:   12[ 15 g (2,1)] <--  3[  4 g (1,1)]    1[  1 g (2,3)]    8[ 10 g (3,1)]
 15:   12[ 15 g (2,1)] <--  1[  1 g (2,3)]   11[ 14 g (3,1)]
```

DS skeleton for 0 –> ggggg

```
color configuration: (1,3) (2,1) (1,2) (3,2) (2,1)

 1:    5[  3 g (2,3)] <--  2[  2 g (2,1)]    1[  1 g (1,3)]
 2:    6[  9 g (1,2)] <--  4[  8 g (3,2)]    1[  1 g (1,3)]
 3:    7[  6 g (1,1)] <--  3[  4 g (1,2)]    2[  2 g (2,1)]
 4:    8[  6 g (2,2)] <--  3[  4 g (1,2)]    2[  2 g (2,1)]
 5:    9[ 10 g (3,1)] <--  4[  8 g (3,2)]    2[  2 g (2,1)]
 6:   10[  7 g (1,3)] <--  3[  4 g (1,2)]    5[  3 g (2,3)]
 7:   10[  7 g (1,3)] <--  3[  4 g (1,2)]    2[  2 g (2,1)]    1[  1 g (1,3)]
 8:   10[  7 g (1,3)] <--  1[  1 g (1,3)]    7[  6 g (1,1)]
 9:   11[ 11 g (2,2)] <--  4[  8 g (3,2)]    5[  3 g (2,3)]
10:   13[ 11 g (1,1)] <--  4[  8 g (3,2)]    2[  2 g (2,1)]    1[  1 g (1,3)]
11:   11[ 11 g (2,2)] <--  4[  8 g (3,2)]    2[  2 g (2,1)]    1[  1 g (1,3)]
12:   13[ 11 g (1,1)] <--  2[  2 g (2,1)]    6[  9 g (1,2)]
13:   11[ 11 g (2,2)] <--  2[  2 g (2,1)]    6[  9 g (1,2)]
14:   13[ 11 g (1,1)] <--  1[  1 g (1,3)]    9[ 10 g (3,1)]
15:   14[ 14 g (3,2)] <--  4[  8 g (3,2)]    8[  6 g (2,2)]
16:   14[ 14 g (3,2)] <--  4[  8 g (3,2)]    3[  4 g (1,2)]    2[  2 g (2,1)]
17:   14[ 14 g (3,2)] <--  3[  4 g (1,2)]    9[ 10 g (3,1)]
18:   15[ 15 g (1,2)] <--  7[  6 g (1,1)]    6[  9 g (1,2)]
19:   15[ 15 g (1,2)] <--  8[  6 g (2,2)]    6[  9 g (1,2)]
20:   15[ 15 g (1,2)] <--  4[  8 g (3,2)]   10[  7 g (1,3)]
21:   15[ 15 g (1,2)] <--  4[  8 g (3,2)]    3[  4 g (1,2)]    5[  3 g (2,3)]
22:   15[ 15 g (1,2)] <--  4[  8 g (3,2)]    1[  1 g (1,3)]    7[  6 g (1,1)]
23:   15[ 15 g (1,2)] <--  4[  8 g (3,2)]    1[  1 g (1,3)]    8[  6 g (2,2)]
24:   15[ 15 g (1,2)] <--  3[  4 g (1,2)]   11[ 11 g (2,2)]
25:   15[ 15 g (1,2)] <--  3[  4 g (1,2)]   13[ 11 g (1,1)]
26:   15[ 15 g (1,2)] <--  3[  4 g (1,2)]    2[  2 g (2,1)]    6[  9 g (1,2)]
27:   15[ 15 g (1,2)] <--  3[  4 g (1,2)]    1[  1 g (1,3)]    9[ 10 g (3,1)]
28:   15[ 15 g (1,2)] <--  1[  1 g (1,3)]   14[ 14 g (3,2)]
```

Color connected amplitudes

Scattering amplitude with n color pairs can be expressed as

$$\mathcal{M}^{i_1 i_2 \cdots i_n}_{j_1 j_2 \cdots j_n} = \sum_{\text{all perm.}} \delta^{i_1}_{j_{\sigma(1)}} \delta^{i_2}_{j_{\sigma(2)}} \ldots \delta^{i_n}_{j_{\sigma(n)}} \mathcal{A}_\sigma(1, 2, \ldots, n)$$

where $\mathcal{A}_\sigma(1, 2, \ldots, n)$ does not depend on the external color, but may depend on N_c. For small n, the explicit color sum is more efficient than color sampling

$$\sum_{\text{color}} |\mathcal{M}|^2 = \sum_{\sigma,\sigma'} N_c^{y(\sigma,\sigma')} \mathcal{A}_\sigma \mathcal{A}^*_{\sigma'}$$

where $y(\sigma, \sigma')$ is the number of common cycles in σ and σ'.

The DS skeleton for $\mathcal{A}_\sigma$ can be found from $\mathcal{M}$, by imagining that $N_c = n$, and assigning the external color configuration

$$(1, \sigma(1)) \;\; (2, \sigma(2)) \;\; \cdots \;\; (n, \sigma(n))$$

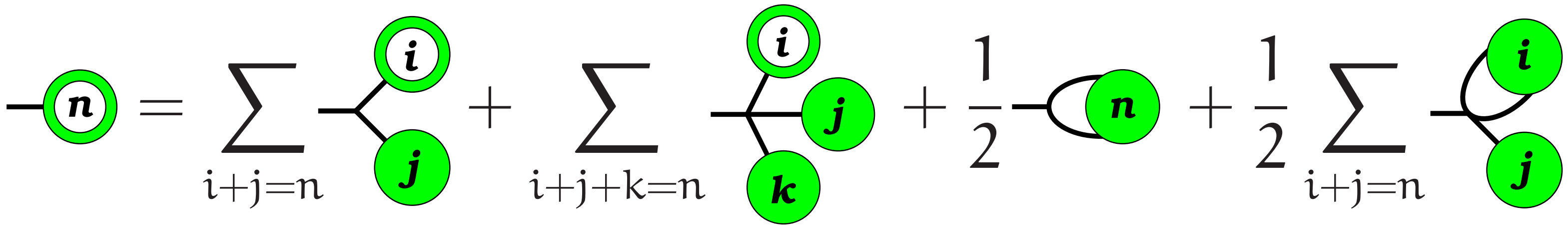
and multiplying quark-gluon vertices by $-i\sqrt{N_c}$ if they involve an internal gluon with $i = j$.

Color connected for 0 –> g g $\bar{u}$ d $\mu^+\nu_\mu$

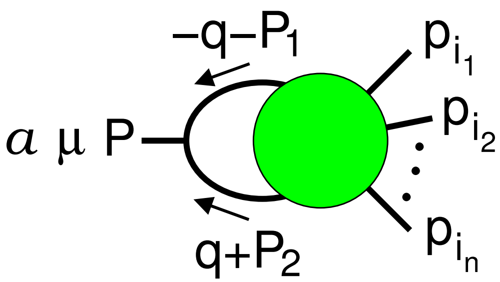
```
Tree:  1, Label:1
  1:  1   6[  3 u~] <--  2[  2 u~]    1[  1 g ]
  2:  1   7[  5 d ] <--  3[  4 d ]    1[  1 g ]
  3: -1   9[ 24 W+] <--  5[ 16 Mn]    4[  8 M+]
  4:  1  13[ 26 d~] <--  2[  2 u~]    9[ 24 W+]
  5:  1  14[ 28 u ] <--  3[  4 d ]    9[ 24 W+]
  6:  1  17[ 27 d~] <--  9[ 24 W+]    6[  3 u~]
  7:  1  17[ 27 d~] <--  1[  1 g ]   13[ 26 d~]
  8:  1  18[ 29 u ] <--  9[ 24 W+]    7[  5 d ]
  9:  1  18[ 29 u ] <--  1[  1 g ]   14[ 28 u ]
 10: -1  21[ 31 g ] <--  7[  5 d ]   13[ 26 d~]
 11: -1  21[ 31 g ] <--  6[  3 u~]   14[ 28 u ]
 12: -1  21[ 31 g ] <--  3[  4 d ]   17[ 27 d~]
 13: -1  21[ 31 g ] <--  2[  2 u~]   18[ 29 u ]

Tree:  2, Label:2
  1:  1   6[  3 u~] <--  2[  2 u~]    1[  1 g ]
  2:  1   7[  5 d ] <--  3[  4 d ]    1[  1 g ]
  3: -1   9[ 24 W+] <--  5[ 16 Mn]    4[  8 M+]
  4:  1  13[ 26 d~] <--  2[  2 u~]    9[ 24 W+]
  5:  1  14[ 28 u ] <--  3[  4 d ]    9[ 24 W+]
  6:  1  17[ 27 d~] <--  9[ 24 W+]    6[  3 u~]
  7:  1  17[ 27 d~] <--  1[  1 g ]   13[ 26 d~]
  8:  1  18[ 29 u ] <--  9[ 24 W+]    7[  5 d ]
  9:  1  18[ 29 u ] <--  1[  1 g ]   14[ 28 u ]
 10: -1  21[ 31 g ] <--  7[  5 d ]   13[ 26 d~]
 11: -1  21[ 31 g ] <--  6[  3 u~]   14[ 28 u ]
 12: -1  21[ 31 g ] <--  3[  4 d ]   17[ 27 d~]
 13: -1  21[ 31 g ] <--  2[  2 u~]   18[ 29 u ].

Tree:  3, Label:3
  1:  1   7[  5 d ] <--  3[  4 d ]    1[  1 g ]
  2: -1   9[ 24 W+] <--  5[ 16 Mn]    4[  8 M+]
  3:  1  13[ 26 d~] <--  2[  2 u~]    9[ 24 W+]
  4:  1  14[ 28 u ] <--  3[  4 d ]    9[ 24 W+]
  5:  1  18[ 29 u ] <--  9[ 24 W+]    7[  5 d ]
  6:  1  18[ 29 u ] <--  1[  1 g ]   14[ 28 u ]
  7: -1  20[ 30 g ] <--  3[  4 d ]   13[ 26 d~]
  8: -1  20[ 30 g ] <--  2[  2 u~]   14[ 28 u ]
  9: -1  21[ 31 g ] <--  7[  5 d ]   13[ 26 d~]
 10: -1  21[ 31 g ] <--  2[  2 u~]   18[ 29 u ]
 11:  1  21[ 31 g ] <--  1[  1 g ]   20[ 30 g ]

Tree:  4, Label:5
  1:  1   6[  3 u~] <--  2[  2 u~]    1[  1 g ]
  2: -1   9[ 24 W+] <--  5[ 16 Mn]    4[  8 M+]
  3:  1  13[ 26 d~] <--  2[  2 u~]    9[ 24 W+]
  4:  1  14[ 28 u ] <--  3[  4 d ]    9[ 24 W+]
  5:  1  17[ 27 d~] <--  9[ 24 W+]    6[  3 u~]
  6:  1  17[ 27 d~] <--  1[  1 g ]   13[ 26 d~]
  7: -1  20[ 30 g ] <--  3[  4 d ]   13[ 26 d~]
  8: -1  20[ 30 g ] <--  2[  2 u~]   14[ 28 u ]
  9: -1  21[ 31 g ] <--  6[  3 u~]   14[ 28 u ]
 10: -1  21[ 31 g ] <--  3[  4 d ]   17[ 27 d~]
 11:  1  21[ 31 g ] <--  1[  1 g ]   20[ 30 g ].

Tree:  5, Label:6
  1:  1   6[  3 u~] <--  2[  2 u~]    1[  1 g ]
  2:  1   7[  5 d ] <--  3[  4 d ]    1[  1 g ]
  3: -1   9[ 24 W+] <--  5[ 16 Mn]    4[  8 M+]
  4:  1  13[ 26 d~] <--  2[  2 u~]    9[ 24 W+]
  5:  1  14[ 28 u ] <--  3[  4 d ]    9[ 24 W+]
  6:  1  17[ 27 d~] <--  9[ 24 W+]    6[  3 u~]
  7:  1  17[ 27 d~] <--  1[  1 g ]   13[ 26 d~]
  8:  1  18[ 29 u ] <--  9[ 24 W+]    7[  5 d ]
  9:  1  18[ 29 u ] <--  1[  1 g ]   14[ 28 u ]
 10: -1  21[ 31 g ] <--  7[  5 d ]   13[ 26 d~]
 11: -1  21[ 31 g ] <--  6[  3 u~]   14[ 28 u ]
 12: -1  21[ 31 g ] <--  3[  4 d ]   17[ 27 d~]
 13: -1  21[ 31 g ] <--  2[  2 u~]   18[ 29 u ].
```

Recursive DS approach at one loop

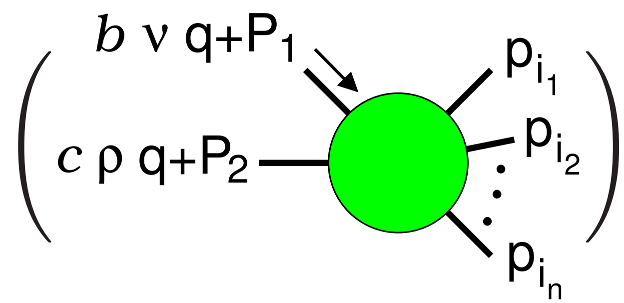
One-loop recursion:



Actual loops generated by last two terms. Third term in more detail:



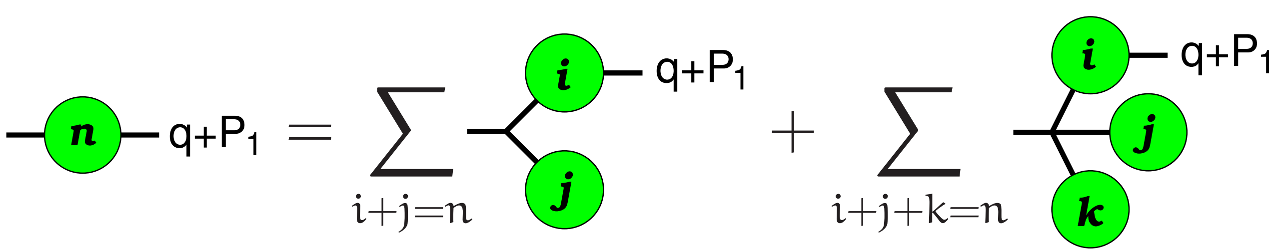
$$= \frac{-i}{P^2}\int \frac{d^{\omega}q}{i\pi^2}\, V^{\mu\ abc}_{\nu\rho}(-q-P_1, q+P_2)$$



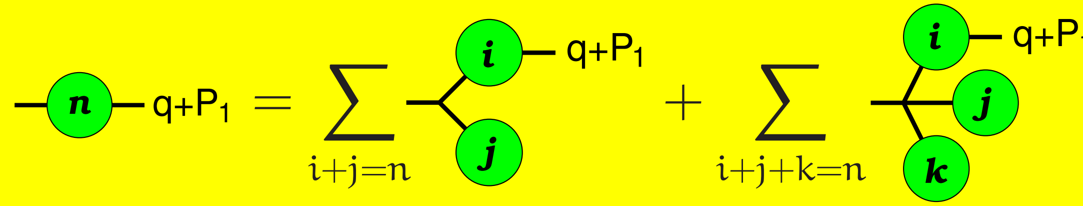
Momentum conservation:

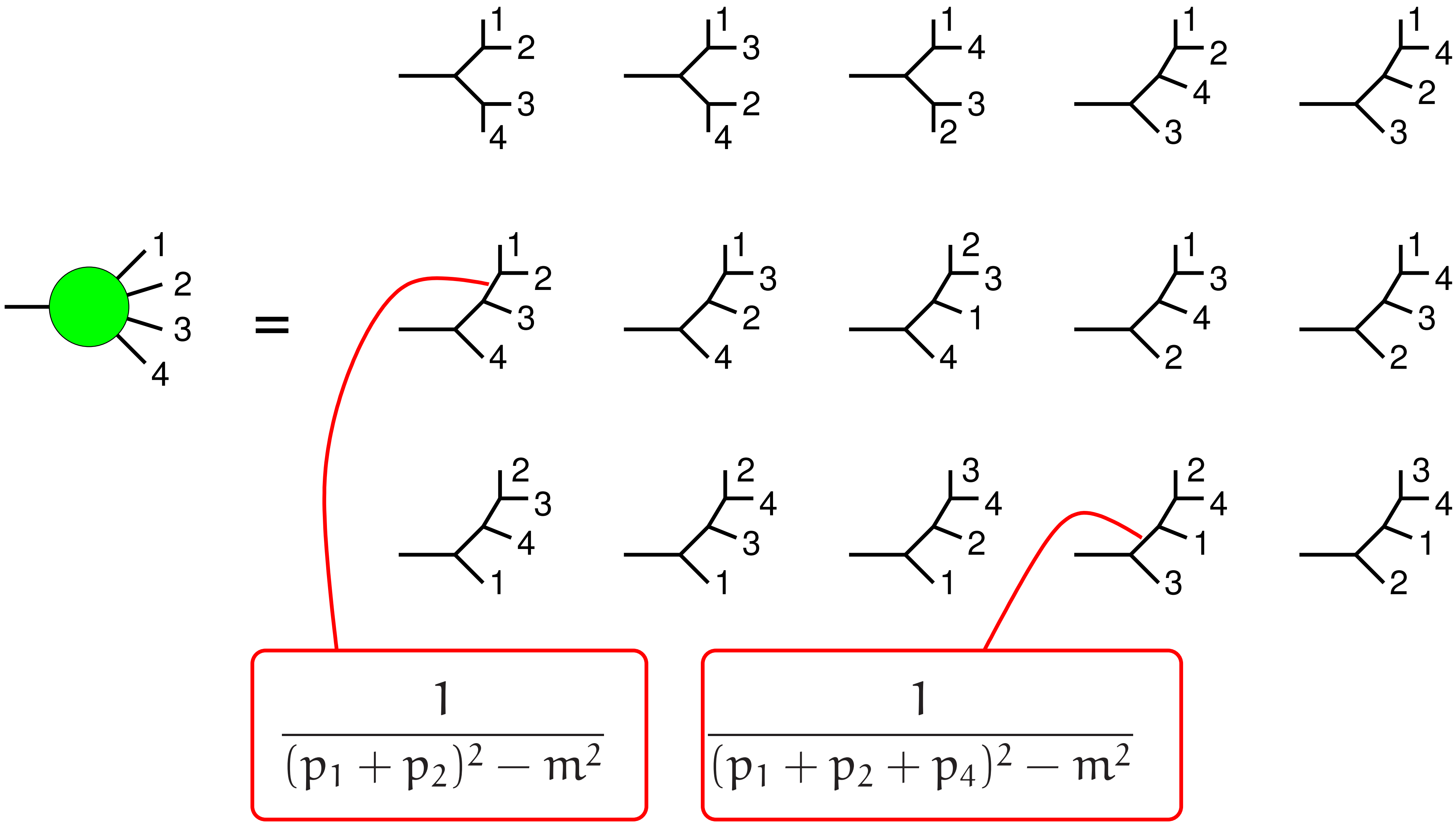
$$P_2 - P_1 = P = p_{i_1} + p_{i_2} + \cdots + p_{i_n}$$

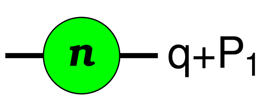
Integrand satisfies tree-level recursion:



Tree–level recursion







$$= \sum_{|\mathcal{D}|\leq n+1} \sum_{r=0}^{|\mathcal{D}|-1} \mathcal{G}^{bc\,\nu\rho}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D}) \frac{q^{\nu_1}q^{\nu_2}\cdots q^{\nu_r}}{\prod_{j\in\mathcal{D}}[(q+p_j)^2-m_j^2]}$$

Recursive nature of tensor integrals

Consider the integrals

$$\mathcal{F}_{n,0} = \int \frac{d^\omega q}{i\pi^{\omega/2}} \frac{f(q)}{\prod_{j=1}^{n}[(q+p_j)^2 - m_j^2]} \quad , \quad \mathcal{F}_{n,1}^{\nu} = \int \frac{d^\omega q}{i\pi^{\omega/2}} \frac{f(q)\, q_4^{\nu}}{\prod_{j=1}^{n}[(q+p_j)^2 - m_j^2]}$$

Using the relation

$$2(p_j - p_n)\cdot q = [(q+p_j)^2 - m_j^2] - [(q+p_n)^2 - m_n^2] + m_j^2 - p_j^2 - m_n^2 + p_n^2$$

we can write

$$2(p_j - p_n)_\nu \mathcal{F}_{n,1}^{\nu} = \mathcal{F}_{n-1,0}(j) - \mathcal{F}_{n-1,0}(n) + (m_j^2 - p_j^2 - m_n^2 + p_n^2)\,\mathcal{F}_{n,0}$$

where $\mathcal{F}_{n-1,0}(j)$ is obtained from $\mathcal{F}_{n,0}$ by removing the j-th denominator. Choosing 4 different vectors p_j appearing in the denominators, we get 4 relations, enough to determine the 4 integrals $\mathcal{F}_{n,1}^{\nu}$.

Take $f(q) = q^{\nu_1} q^{\nu_2} \cdots q^{\nu_r}$ and we have recursive tensor integral calculation.

For $n \leq 4$ express tensor integrals in terms of PV coefficient functions and use PV reduction.

Tensor symmetrization: $4^n \to n^4$

Tensor integrals are symmetric in the tensor indices, so the we only need to contract them with the symmetrized coefficients.

$$\mathcal{T}_{n,r}^{\nu_1\nu_2\cdots\nu_r} = \int \frac{d^\omega q}{i\pi^{\omega/2}} \frac{q_4^{\nu_1} q_4^{\nu_2} \cdots q_4^{\nu_r}}{\prod_{j=1}^{n}[(q+p_j)^2 - m_j^2]}$$

$$C_{r=2}^{\{1,2\}} = C_{r=2}^{1,2} + C_{r=2}^{2,1} \quad , \quad C_{r=3}^{\{1,2,2\}} = C_{r=3}^{1,2,2} + C_{r=3}^{2,1,2} + C_{r=3}^{2,2,1}$$
$$C_{r=3}^{\{1,2,3\}} = C_{r=3}^{1,2,3} + C_{r=3}^{2,3,1} + C_{r=3}^{3,1,2} + C_{r=3}^{3,2,1} + C_{r=3}^{2,1,3} + C_{r=3}^{1,3,2}$$

A symmetric tensor $C_r^{\{\nu_1\nu_2\cdots\nu_r\}}$ with 4-dimensional indices can be represented as

$$C_r^{\{\nu_1\nu_2\cdots\nu_r\}} = S_{n_0,n_1,n_2,n_3}^r \qquad n_0 + n_1 + n_2 + n_3 = r$$

where n_μ is the number of indices referring to dimension μ. Suppose we have a linear recursive relation between tensors of the type

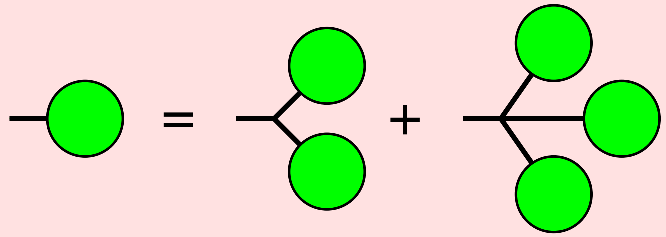
$$C_r^{\nu_1\nu_2\cdots\nu_r} = C_{r-1}^{\nu_1\nu_2\cdots\nu_{r-1}} * K_r^{\nu_r}$$

with $C_1^\nu = K_1^\nu$. The symmetrized relation is

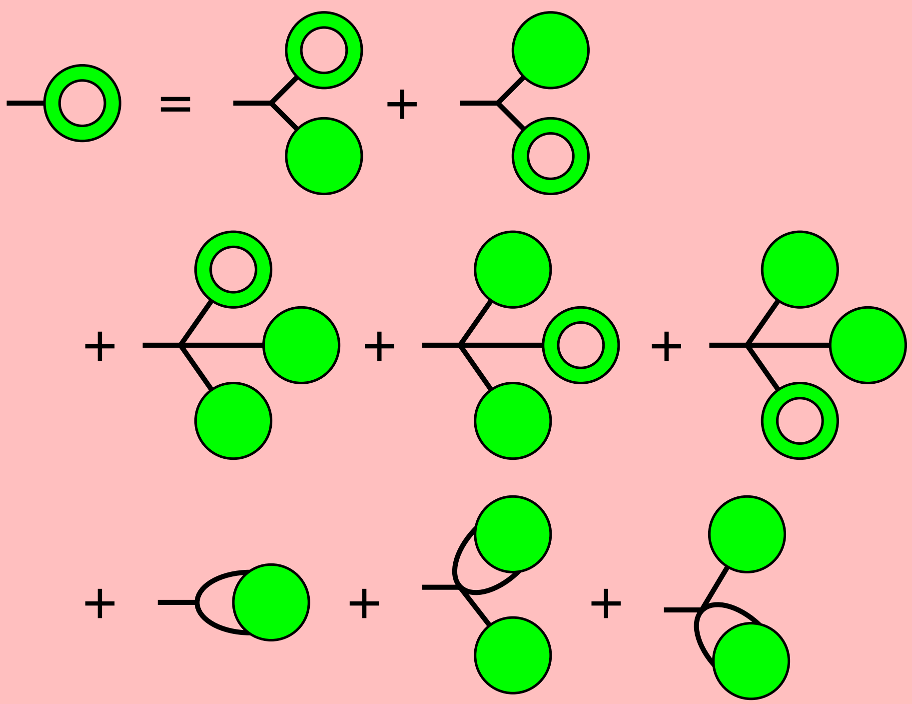
$$\begin{aligned} S_{n_0,n_1,n_2,n_3}^r &= S_{n_0-1,n_1,n_2,n_3}^{r-1} * K_r^0 + S_{n_0,n_1-1,n_2,n_3}^{r-1} * K_r^1 \\ &+ S_{n_0,n_1,n_2-1,n_3}^{r-1} * K_r^2 + S_{n_0,n_1,n_2,n_3-1}^{r-1} * K_r^3 \end{aligned}$$

with $S_{n_0,n_1,n_2,n_3}^r = 0$ whenever any of the indices is negative.

Recursion for primitive amplitudes



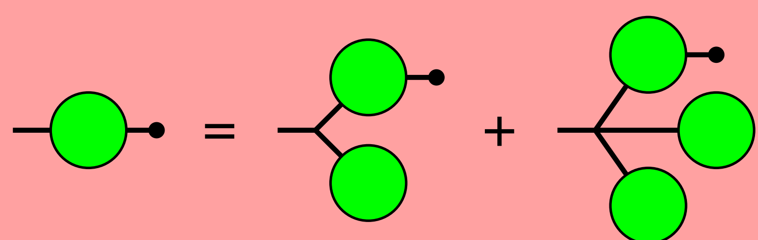
$$A^{\mu}_{i,j} = \frac{-i}{p^2_{i,j}}\left[\sum_{k=i}^{j-1} V^{\mu}_{\nu\rho}(p_{i,k},p_{k+1,j})A^{\nu}_{i,k}A^{\rho}_{k+1,j} + \sum_{k=i}^{j-2}\sum_{l=k+1}^{j-1} W^{\mu}_{\nu\rho\sigma}A^{\nu}_{i,k}A^{\rho}_{k+1,l}A^{\sigma}_{l+1,j}\right]$$



$$G^{\lambda\mu}_{i,j}(q) = \frac{-i}{(q+p_{1,j})^2}\left[\sum_{k=i-1}^{j-1} V^{\mu}_{\nu\rho}(q+p_{1,k},p_{k+1,j})G^{\lambda\nu}_{i,k}(q)A^{\rho}_{k+1,j} + \sum_{k=i-1}^{j-2}\sum_{l=k+1}^{j-1} W^{\mu}_{\nu\rho\sigma}G^{\lambda\nu}_{i,k}(q)A^{\rho}_{k+1,l}A^{\sigma}_{l+1,j}\right]$$

$$V^{\mu}_{\nu\rho}(q+p_{1,k},p_{k+1,j}) = V^{\mu}_{\nu\rho}(p_{1,k},p_{k+1,j}) + X^{\mu}_{\sigma\nu\rho}q^{\sigma}$$

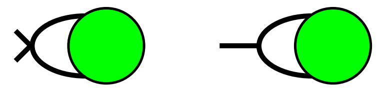
$$G^{\lambda\mu}_{i,j}(q) = \sum_{\mathcal{D}\subset\{i-1,i,\dots,j\}}\sum_{r=0}^{|\mathcal{D}|-1}\mathcal{G}^{\lambda\mu}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D})\,\frac{q^{\nu_1}q^{\nu_2}\cdots q^{\nu_r}}{\prod_{j\in\mathcal{D}}(q+p_{1,j})^2}$$

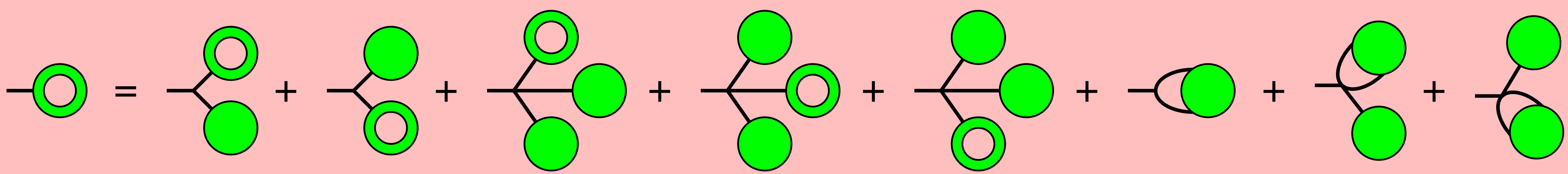


$$i\mathcal{G}^{\lambda\mu}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D}) = \mathcal{G}^{\lambda\nu}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D}',k)\,Y^{\mu}_{\nu}(k,j) + \mathcal{G}^{\lambda\nu}_{\nu_1\nu_2\cdots\nu_{r-1}}(\mathcal{D}',k)\,X^{\mu}_{\nu_r\nu\rho}A^{\rho}_{k+1,j}$$

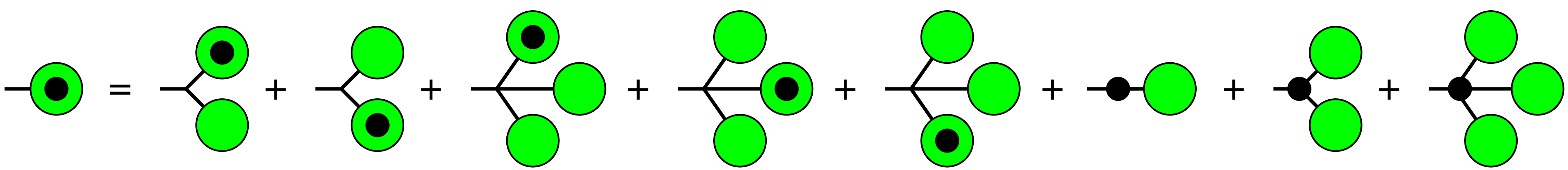
$$Y^{\mu}_{\nu}(k,j) = V^{\mu}_{\nu\rho}(p_{1,k},p_{k+1,j})A^{\rho}_{k+1,j} + \sum_{l=k+1}^{j-1} W^{\mu}_{\nu\rho\sigma}A^{\rho}_{k+1,l}A^{\sigma}_{l+1,j}$$

Recursion for primitive amplitudes

1. calculate the tensor integrals $\mathcal{T}^{\nu_1\nu_2\cdots\nu_r}(\mathcal{D})$
2. calculate the tree-level off-shell currents $A^{\mu}_{i,j}$
3. calculate the tensors $\mathcal{G}^{\rho\mu}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D})$
4. calculate the currents 
5. calculate the one-loop currents via



6. the R_2-terms, which is still missing because of 4-dim tensor integrals



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